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NUMBER THEORY

When a constant subsequence implies ultimate periodicity

by

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Summary. We show a curious property of sequences given by the recurrence $a_0 = h_1(0)$, $a_n = f(n)a_{n-1} + h_1(n)h_2(n)^n$, $n > 0$, where $f, h_1, h_2 \in \mathbb{Z}[X]$. Namely, if the sequence $(a_{kn+l})_{n\in\mathbb{N}}$ is constant for some $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$, then either $(a_{2n+1})_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$ and $(a_{2n})_{n\in\mathbb{N}}$ is a geometric progression, or $(a_n)_{n\in\mathbb{N}}$ is ultimately periodic with period dividing 2.

1. Introduction. It is obvious that any ultimately periodic sequence of integers is bounded. On the other hand, in general, boundedness of a sequence does not imply its ultimate periodicity. Of course, boundedness forces constancy for polynomial sequences or geometric progressions with ratio $\notin \{0, -1\}$ (for ratio 0 we can have an ultimately zero sequence with nonzero initial term, and for ratio -1 we can have a sequence with basic period 2).

In this paper we will focus on a special class of sequences, denoted by $\mathcal{R}$ and defined as the set of all sequences $\mathbf{a} = \mathbf{a}(f, h_1, h_2) = (a_n)_{n\in\mathbb{N}}$ satisfying a recurrence of the form

$$(1.1)\qquad a_0 = h_1(0), \quad a_n = f(n)a_{n-1} + h_1(n)h_2(n)^n, \quad n > 0,$$

where $f, h_1, h_2 \in \mathbb{Z}[X]$ are given. Note that $\mathcal{R}$ contains the following well known sequences:

- if $f = h_2 = 1$, $h_1 = c \in \mathbb{Z}$, then $(a_n)_{n\in\mathbb{N}} = (c(n + 1))_{n\in\mathbb{N}}$ is an arithmetic progression;

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- if $f = q \in \mathbb{Z}$, $h_1 = c \in \mathbb{Z}$, $h_2 = 0$, then $(a_n)_{n\in\mathbb{N}} = (cq^n)_{n\in\mathbb{N}}$ is a geometric progression;
- if $f = 1$, $h_1 = c \in \mathbb{Z}$, $h_2 = q \in \mathbb{Z}$, then $(a_n)_{n\in\mathbb{N}} = (\sum_{j=0}^{n} cq^j)_{n\in\mathbb{N}}$ is the sequence of partial sums of a geometric progression;
- if $f = X$, $h_1 = 1$, $h_2 = 0$, then $(a_n)_{n\in\mathbb{N}} = (n!)_{n\in\mathbb{N}}$ is the sequence of factorials;
- if $f = 2X + l$, $l \in \{0,1\}$, $h_1 = 1$, $h_2 = 0$, then $(a_n)_{n\in\mathbb{N}} = ((2n+l)!!)_{n\in\mathbb{N}}$ is the sequence of double factorials;
- if $f = X$, $h_1 = 1$, $h_2 = -1$, then $(a_n)_{n\in\mathbb{N}} = (D_n)_{n\in\mathbb{N}}$ is the sequence of numbers of *derangements* in S_n, i.e. permutations of an n-element set without fixed points; its arithmetic properties were the subject of [2].

Arithmetic properties of sequences from the class $\mathcal{R}$ were studied in [4]. In Section 5 there, we showed an upper bound $a_n = O(e^{Cn\ln n})$, where $C = \max\{\deg f, \deg h_2, 1\}$. On the other hand, taking $h_1 = a(1-f)$, $h_2 = 1$ we get a constant sequence $\mathbf{a}$, while the degree of f can be arbitrarily large.

The aim of this paper is to show that if the sequence $(a_{kn+l})_{n\in\mathbb{N}}$ is constant for some $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$, then $h_1 = 0$, or $h_2 \in \{-1,0,1\}$, or $\max\{\deg f, \deg h_1, \deg h_2\} \le 0$ with $h_2 = -f$, and we will give the form of the corresponding sequence. In particular, such a sequence is ultimately constant, or ultimately periodic with period 2, or contains a geometric progression as a subsequence.

The motivation to consider the above property is its application in an elementary proof of infinitude of the set $\mathbb{P}_{\mathbf{a}} = \{p \in \mathbb{P} : \exists_{n\in\mathbb{N}}\ a_n \neq 0 \text{ and } p \mid a_n\}$, where $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ is a given sequence of integers (not necessarily in $\mathcal{R}$). If we assume that $\mathbf{a}$ is unbounded with no constant subsequence of the form $(a_{kn+l})_{n\in\mathbb{N}}$ for some $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$, and moreover the sequence $(a_n \pmod{d})_{n\in\mathbb{N}}$ is periodic for each $d \in \mathbb{N}_+$, then the set $\mathbb{P}_{\mathbf{a}}$ is infinite. Indeed, suppose that $\mathbb{P}_{\mathbf{a}} = \{p_1, \dots, p_s\}$ and take the smallest $l \in \mathbb{N}$ such that $a_l \neq 0$. Then $a_l = p_1^{\alpha_1} \cdot \dots \cdot p_s^{\alpha_s}$ for some $\alpha_1, \dots, \alpha_s \in \mathbb{N}$. We know that the sequence $(a_n \pmod{p_1^{\alpha_1+2} \cdot \dots \cdot p_s^{\alpha_s+2}})_{n\in\mathbb{N}}$ is periodic with some period k. Thus $a_{kn+l} \equiv a_l \pmod{p_1^{\alpha_1+2} \cdot \dots \cdot p_s^{\alpha_s+2}}$ for each $n \in \mathbb{N}$, which together with the assumption $\mathbb{P}_{\mathbf{a}} = \{p_1, \dots, p_s\}$ implies that $a_{kn+l} = a_l$, $n \in \mathbb{N}$—a contradiction. Notice that if $\mathbf{a} = (n)_{n\in\mathbb{N}}$, then the above reasoning is Euclid's proof of infinitude of $\mathbb{P}$. Knowing that for each sequence $\mathbf{a} \in \mathcal{R}$ and positive integer d the sequence $(a_n \pmod{d})_{n\in\mathbb{N}}$ is ultimately periodic, we may try to prove infinitude of $\mathbb{P}_{\mathbf{a}}$ in a similar way (using some additional assumptions, if necessary).

2. Boundedness and periodicity of $\mathbf{a} \in \mathcal{R}$. First, we prove that if there is a constant subsequence of the form $(a_{kn+l})_{n\in\mathbb{N}}$ for some $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$ then $h_1 = 0$, or $h_2 \in \{-1,0,1\}$, or f, h_1, h_2 are constant and $h_2 = -f$.

Then, assuming that $h_1 = 0$ or $h_2 \in \{-1, 0, 1\}$, we will show that $(a_n)_{n\in\mathbb{N}}$ is ultimately constant or ultimately periodic with period 2. Next, assuming the boundedness of $(a_n)_{n\in\mathbb{N}}$, we will use the periodicity of $(a_n \pmod p))_{n\in\mathbb{N}}$ for a sufficiently large prime number p to deduce the ultimate periodicity of $(a_n)_{n\in\mathbb{N}}$.

THEOREM 1. *Let* $\mathbf{a} = \mathbf{a}(f, h_1, h_2)$, $k \in \mathbb{N}_+$ *and* $l \in \mathbb{N}$. *If the sequence* $(a_{kn+l})_{n\in\mathbb{N}}$ *is constant then one of the following conditions holds:*

- $h_1 = 0$ (*and then* $(a_n)_{n\in\mathbb{N}}$ *is constantly* 0),
- $h_2 \in \{-1, 0, 1\}$,
- $h_1 = c \in \mathbb{Z}$ *and* $h_2 = -f = b \in \mathbb{Z}$ (*then* $a_{2n} = b^{2n}c$ *and* $a_{2n+1} = 0$ *for all* $n \in \mathbb{N}$).

Proof. Assume that $h_1 \neq 0$. If $f = 0$ then $a_n = h_1(n)h_2(n)^n$ for all $n \in \mathbb{N}$ and thus the assumption can be satisfied only if $h_2 \in \{-1, 0, 1\}$. Hence, we can assume that $f \neq 0$.

Assume that $\deg h_2 > 0$. Let a be the value attained by $(a_{kn+l})_{n\in\mathbb{N}}$. We choose a nonnegative integer n so large that $h_2(kn + l) \neq 0$. Then, applying the recurrence definition of $\mathbf{a}$, we obtain

$$
\begin{aligned}
(2.1)\qquad a &= a_{k(n+1)+l} \\
&= a_{kn+l}\prod_{i=1}^{k} f(kn+l+i) \\
&\quad + \sum_{j=1}^{k} h_1(kn+l+j)h_2(kn+l+j)^{kn+l+j} \prod_{i=j+1}^{k} f(kn+l+i) \\
&= a\prod_{i=1}^{k} f(kn+l+i) + h_2(kn+l)^{kn+l}\sum_{j=1}^{k} h_1(kn+l+j) \\
&\quad \times h_2(kn+l+j)^j \left(\frac{h_2(kn+l+j)}{h_2(kn+l)}\right)^{kn+l} \prod_{i=j+1}^{k} f(kn+l+i).
\end{aligned}
$$

Let $d = \deg h_2 > 0$ and write $h_2 = \sum_{i=0}^{d} w_i X^i$. Then for each $j \in \mathbb{N}$,

$$
\begin{aligned}
&h_2(kn+l+j) - h_2(kn+l) \\
&\quad = \sum_{i=0}^{d} w_i(kn+l+j)^i - \sum_{i=0}^{d} w_i(kn+l)^i \\
&\quad = w_d(kn+l)^d + dw_dj(kn+l)^{d-1} + w_{d-1}(kn+l)^{d-1} + O((kn+l)^{d-2}) \\
&\qquad - w_d(kn+l)^d - w_{d-1}(kn+l)^{d-1} + O((kn+l)^{d-2}) \\
&\quad = dw_dj(kn+l)^{d-1} + O((kn+l)^{d-2})
\end{aligned}
$$

as $n \to +\infty$. Since

$$\lim_{n\to+\infty} \frac{h_2(kn+l+j)-h_2(kn+l)}{h_2(kn+l)} = 0,$$
$$\lim_{n\to+\infty} \frac{(kn+l)^d}{h_2(kn+l)} = \frac{1}{w_d},$$
$$\lim_{n\to+\infty} \frac{O((kn+l)^{d-1})}{h_2(kn+l)} = 0,$$

we have

$$\lim_{n\to+\infty} \left(\frac{h_2(kn+l+j)}{h_2(kn+l)}\right)^{kn+l} = e^{dj}. \tag{2.2}$$

Let us define

$$F(x,k,j) = h_2(x)^x h_1(x+j) h_2(x+j)^j \left(\frac{h_2(x+j)}{h_2(x)}\right)^x \prod_{i=j+1}^{k} f(x+i)$$

for $x \in \mathbb{N}$, $k \in \mathbb{N}_+$ and $j \in \{1,\dots,k\}$, where we assume that $0^0 = 1$ and $\prod_{i=k+1}^{k} f(x+i) = 1$.

If $\deg f > \deg h_2$ and $j > 1$, then we can easily compute the following limits:

$$\lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i)}{h_2(kn+l)^{kn+l} h_1(kn+l+1) h_2(kn+l+1) e^d \prod_{i=2}^{k} f(kn+l+i)} = 0,$$
$$\lim_{n\to+\infty} \frac{F(kn+l,k,1)}{h_2(kn+l)^{kn+l} h_1(kn+l+1) h_2(kn+l+1) e^d \prod_{i=2}^{k} f(kn+l+i)} = 1,$$
$$\lim_{n\to+\infty} \frac{F(kn+l,k,j)}{h_2(kn+l)^{kn+l} h_1(kn+l+1) h_2(kn+l+1) e^d \prod_{i=2}^{k} f(kn+l+i)} = 0.$$

The first limit is 0 because the numerator grows polynomially while the denominator grows exponentially. Adding these limits yields

$$\lim_{n\to+\infty} \frac{a}{h_2(kn+l)^{kn+l} h_1(kn+l+1) h_2(kn+l+1) e^d \prod_{i=2}^{k} f(kn+l+i)} =$$
$$\lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i) + \sum_{j=1}^{k} F(kn+l,k,j)}{h_2(kn+l)^{kn+l} h_1(kn+l+1) h_2(kn+l+1) e^d \prod_{i=2}^{k} f(kn+l+i)} = 1.$$

This is in contradiction with the fact that the left-hand limit is clearly zero.

Similarly, if $\deg f < \deg h_2$ and $1 \le j < k$, then

$$\lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k e^{dk}} = 0,$$
$$\lim_{n\to+\infty} \frac{F(kn+l,k,k)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k e^{dk}} = 1,$$
$$\lim_{n\to+\infty} \frac{F(kn+l,k,j)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k e^{dk}} = 0.$$

We add these limits to obtain

$$\begin{aligned}&\lim_{n\to+\infty} \frac{a}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k e^{dk}}\\ &\qquad = \lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i) + \sum_{j=1}^{k} F(kn+l,k,j)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k e^{dk}} = 1.\end{aligned}$$

This again contradicts the fact that the left-hand limit is zero.

Consider finally the case when $\deg f = \deg h_2$. Let $f = \sum_{i=0}^{d} u_i X^i$. Then

$$\lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k} = 0,$$
$$\lim_{n\to+\infty} \frac{F(kn+l,k,j)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k} = \left(\frac{u_d}{w_d}\right)^{k-j} e^{dj},$$

as $1 \le j \le k$. By adding these limits we obtain

$$\begin{aligned}&\lim_{n\to+\infty} \frac{a}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k}\\ &= \lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i) + \sum_{j=1}^{k} F(kn+l,k,j)}{h_2(kn+l)^{kn+l}h_1(kn+l+k)h_2(kn+l+k)^k} = \sum_{j=1}^{k}\left(\frac{u_d}{w_d}\right)^{k-j} e^{dj}.\end{aligned}$$

However, the left-hand limit is zero, while $\sum_{j=1}^{k}(u_d/w_d)^{k-j}e^{dj} \neq 0$ because e is a transcendental number (see [1]). This is a contradiction again.

We have proved that if the sequence $(a_{kn+l})_{n\in\mathbb{N}}$ is constant and $h_1 \neq 0$, then $h_2 \in \mathbb{Z}$. When $h_2 = b$ the equality (2.1) takes the form

$$a = a\prod_{i=1}^{k} f(kn+l+i) + b^{kn+l}\sum_{j=1}^{k} h_1(kn+l+j)b^j \prod_{i=j+1}^{k} f(kn+l+i). \tag{2.3}$$

Assume that $|b| > 1$ and define the polynomial

$$G = \sum_{j=1}^{k} h_1(kX+l+j)b^j \prod_{i=j+1}^{k} f(kX+l+i) \in \mathbb{Z}[X].$$

If $G \neq 0$, then by (2.3) we have

$$\frac{a}{b^{kn+l}G(n)} = \frac{a\prod_{i=1}^{k} f(kn+l+i)}{b^{kn+l}G(n)} + 1.$$

Since

$$\lim_{n\to+\infty} \frac{a\prod_{i=1}^{k} f(kn+l+i)}{b^{kn+l}} = 0$$

we deduce that

$$\lim_{n\to+\infty} \frac{a}{b^{kn+l}G(n)} = 1.$$

We get a contradiction because this limit is 0.

If $G = 0$, then $h_1 = 0$ or $f \in \mathbb{Z}$. Indeed, if $h_1 \neq 0$ and $\deg f > 0$, then

$$\deg\Big[h_1(kX+l+j)b^j \prod_{i=j+1}^{k} f(kX+l+i)\Big] = (k-j)\deg f + \deg h_1$$

for $j \in \{1, \ldots, k\}$ and as a result $\deg G = k \deg f + \deg h_1 > 0$. Assume that $f = c \in \mathbb{Z}$. Then

$$\begin{aligned}
0 = \lim_{n\to+\infty} \frac{G(n)}{bh_1(kn+l)} &= \lim_{n\to+\infty} \frac{\sum_{j=1}^{k} h_1(kn+l+j)b^{j-1}c^{k-j}}{h_1(kn+l)} \\
&= \sum_{j=1}^{k} b^{j-1}c^{k-j} = \begin{cases} \frac{b^k-c^k}{b-c} & \text{if } b \neq c, \\ kb^{k-1} & \text{if } b = c, \end{cases}
\end{aligned}$$

which means that either $b = c = 0$, or $c = -b$ and $2 \mid k$. The case $b = c = 0$ contradicts the assumption that $|b| > 1$. If $c = -b$, then by induction we obtain

$$a_{l+n} = (-b)^n a_l + \sum_{j=1}^{n} (-1)^{n-j} b^n h_1(l+j) = (-b)^n a_l + (-b)^n \sum_{j=1}^{n} (-1)^j h_1(l+j)$$

for $n \in \mathbb{N}$.

Define

$$\begin{aligned}
H(n) &= \sum_{j=1}^{2n} (-1)^j h_1(l+j) = \sum_{j=1}^{n} \big(h_1(l+2j) - h_1(l+2j-1)\big) \\
&= \sum_{j=1}^{n} \Delta h_1(l+2j-1), \qquad n \in \mathbb{N},
\end{aligned}$$

where $\Delta h_1 = h_1(X+1) - h_1(X)$. The function H can be seen as a polynomial in n and its degree is equal to

$$\deg H(X) = 1 + \deg \Delta h_1(l+2X-1) = 1 + \deg \Delta h_1(X) = \deg h_1(X).$$

Since

$$a = a_l = a_{l+kn} = (-b)^{kn} a + (-b)^{kn} H\left(\frac{k}{2}n\right) = (-b)^{kn}\left(a + H\left(\frac{k}{2}n\right)\right)$$

for all $n \in \mathbb{N}$ and $|b| > 1$, we deduce that the polynomial H must be constant. This implies that h_1 is constant.

Summing up: we have shown that either $h_1 = 0$, or $h_2 \in \{-1, 0, 1\}$, or f, h_1, h_2 are constant and $f = -h_2$. ■

THEOREM 2. *Let* $\mathbf{a} = \mathbf{a}(f, h_1, 1)$, $k \in \mathbb{N}_+$ *and* $l \in \mathbb{N}$. *If the sequence* $(a_{kn+l})_{n\in\mathbb{N}}$ *is constant, then* $(a_n)_{n\in\mathbb{N}}$ *is either ultimately constant or of the form* $(c, 0, c, 0, c, 0, \ldots)$ *for some integer* c.

Proof. For $k = 1$ the statement is obvious. Hence, assume that $k \geq 2$. Let a be the value attained by $(a_{kn+l})_{n\in\mathbb{N}}$. Then

$$\begin{aligned} a_{k(n+1)+l} &= a_{kn+l} \prod_{i=1}^{k} f(kn+l+i) + \sum_{j=1}^{k} h_1(kn+l+j) \prod_{i=j+1}^{k} f(kn+l+i) \\ &= a \prod_{i=1}^{k} f(kn+l+i) + \sum_{j=1}^{k} h_1(kn+l+j) \prod_{i=j+1}^{k} f(kn+l+i). \end{aligned} \tag{2.4}$$

Let

$$G(X) = a \prod_{i=1}^{k} f(kX+l+i) + \sum_{j=1}^{k} h_1(kX+l+j) \prod_{i=j+1}^{k} f(kX+l+i) \in \mathbb{Z}[X].$$

From (2.4) we know that $G = a$. If $h_1 = 0$, then $a_n = 0$ for all $n \in \mathbb{N}$, so we can assume that $h_1 \neq 0$.

If $\deg f > 0$, then $\deg \prod_{i=1}^{k} f(kX+l+i) = k \deg f$ and

$$\deg h_1(kX+l+j) \prod_{i=j+1}^{k} f(kX+l+i) = (k-j)\deg f + \deg h_1$$

for $j \in \{1, \ldots, k\}$. Since $\deg G \leq 0$ we get

$$\deg \prod_{i=1}^{k} f(kX+l+i) = \deg h_1(kX+l+1) \prod_{i=2}^{k} f(kX+l+i),$$

which implies that $\deg f = \deg h_1$. Moreover, we have the following chain of equivalences:

$$\begin{aligned}
&\deg\Big(a\prod_{i=1}^{k} f(kX+l+i) + h_1(kX+l+1)\prod_{i=2}^{k} f(kX+l+i)\Big)\\
&\qquad\qquad\qquad = \deg h_1(kX+l+2)\prod_{i=3}^{k} f(kX+l+i)\\
&\qquad\qquad \iff \deg(af(kX+l+1)+h_1(kX+l+1))\prod_{i=2}^{k} f(kX+l+i)\\
&\qquad\qquad\qquad = \deg h_1(kX+l+2)\prod_{i=3}^{k} f(kX+l+i)\\
&\qquad\qquad \iff \deg(af(kX+l+1)+h_1(kX+l+1)) + (k-1)\deg f\\
&\qquad\qquad\qquad = \deg h_1 + (k-2)\deg f\\
&\qquad\qquad \iff \deg(af(kX+l+1)+h_1(kX+l+1)) = 0.
\end{aligned}$$

Hence, $af+h_1=b$ for some integer b. Therefore,

$$\begin{aligned}
G &= (af(kX+l+1)+h_1(kX+l+1))\prod_{i=2}^{k} f(kX+l+i)\\
&\qquad + \sum_{j=2}^{k} h_1(kX+l+j)\prod_{i=j+1}^{k} f(kX+l+i)\\
&= b\prod_{i=2}^{k} f(kX+l+i) + \sum_{j=2}^{k} h_1(kX+l+j)\prod_{i=j+1}^{k} f(kX+l+i)\\
&= \big(bf(kX+l+2)+h_1(kX+l+2)\big)\prod_{i=3}^{k} f(kX+l+i)\\
&\qquad + \sum_{j=3}^{k} h_1(kX+l+j)\prod_{i=j+1}^{k} f(kX+l+i).
\end{aligned}$$

Similarly, from $\deg G \le 0$ we get the equivalences

$$\begin{aligned}
&\deg\big(bf(kX+l+2)+h_1(kX+l+2)\big)\prod_{i=3}^{k} f(kX+l+i)\\
&\qquad\qquad\qquad = \deg h_1(kX+l+3)\prod_{i=4}^{k} f(kX+l+i)\\
&\qquad\qquad \iff \deg\big(bf(kX+l+2)+h_1(kX+l+2)\big) + (k-2)\deg f\\
&\qquad\qquad\qquad = \deg h_1 + (k-3)\deg f\\
&\qquad\qquad \iff \deg(bf(kX+l+2)+h_1(kX+l+2)) = 0
\end{aligned}$$

provided that $k \geq 3$. If $k = 2$, then $\deg(bf(kX+l+2)+h_1(kX+l+2) \leq 0$. Since

$$\deg\bigl(af(kX+l+2)+h_1(kX+l+2)\bigr), \deg\bigl(bf(kX+l+2)+h_1(kX+l+2)\bigr) \leq 0,$$

we also have

$$\begin{aligned}\deg(a-b)f(kX+l+2) = {} & \deg\bigl[\bigl(af(kX+l+2)+h_1(kX+l+2)\bigr) \\ & - \bigl(bf(kX+l+2)+h_1(kX+l+2)\bigr)\bigr] \leq 0.\end{aligned}$$

From the assumption $\deg f > 0$ we get $a = b$. From this we obtain the equality $af + h_1 = a$, which by a simple induction yields $a_n = a$ for all $n \geq l$.

Assume now that $f = b$ for some integer b. Then

$$G = a = ab^k + \sum_{j=1}^{k} b^{k-j} h_1(kX+l+j) \in \mathbb{Z}[X]. \tag{2.5}$$

If $\deg h_1 = d > 0$ and $h_1 = \sum_{i=0}^{d} w_i X^i$, then the coefficient of X^d in G is 0 (since $\deg G \leq 0$). On the other hand, this coefficient is equal to

$$k^d w_d \sum_{j=1}^{k} b^{k-j} = \begin{cases} k^d w_d \frac{b^k-1}{b-1} & \text{if } b \neq 1, \\ k^{d+1} w_d & \text{if } b = 1, \end{cases}$$

which means that $2 \mid k$ and $b = -1$. Write $k' = k/2$ and $\Delta h_1 = h_1(X+1) - h_1(X)$. It is clear that $\deg \Delta h_1 = \deg h_1 - 1$. Now (2.5) takes the form

$$0 = \sum_{j=1}^{k'} h_1(kn+l+2j) - h_1(kn+l+2j-1) = \sum_{j=1}^{k'} \Delta h_1(kn+l+2j-1).$$

Let $H = \sum_{j=1}^{k'} \Delta h_1(kX+l+2j-1) \in \mathbb{Z}[X]$. Then $H = 0$. However, the coefficient of X^{d-1} in H is equal to k' times the leading coefficient of Δh_1, which is clearly a contradiction.

We are left with the case $h_1 = c \in \mathbb{Z} \setminus \{0\}$. By (2.5), we have

$$0 = a(b^k-1) + c\sum_{j=1}^{k} b^{k-j} = \begin{cases} a(b^k-1) + c\frac{b^k-1}{b-1} & \text{if } b \neq 1, \\ kc & \text{if } b = 1. \end{cases}$$

Since $c \neq 0$, we have $b \neq 1$ and $\bigl(a + \frac{c}{b-1}\bigr)(b^k-1) = 0$. Thus, $b = -1$ and $(a_n)_{n\in\mathbb{N}} = (c, 0, c, 0, c, 0, \ldots)$, or $c = a(1-b)$ which implies that $ba + c = a$ and $(a_n)_{n\in\mathbb{N}}$ is ultimately constant. ■

Example 1. Consider the sequence $\mathbf{a}(X-3, 28-7X, 1)$. Then $a_1 = -35$, $a_2 = 49$ and $a_n = 7$ for $n \geq 3$. This means that a sequence $\mathbf{a}$ satisfying the assumptions of Theorem 2 can be ultimately constant, but not constant.

Corollary 1. *Let* $\mathbf{a} = \mathbf{a}(f, h_1, -1)$. *If the sequence* $(a_{kn+l})_{n\in\mathbb{N}}$ *is constant, then there is an integer* c *such that either* $a_n = (-1)^n c$ *for almost all* $n \in \mathbb{N}$, *or* $a_{2n} = c$, $a_{2n+1} = 0$ *for all* $n \in \mathbb{N}$.

Proof. Consider the sequence $(\widetilde{a}_n)_{n\in\mathbb{N}} = \mathbf{a}(-f, h_1, 1)$. Since $\widetilde{a}_n = (-1)^n a_n$ for $n \in \mathbb{N}$, the sequence $(\widetilde{a}_{2kn+l})_{n\in\mathbb{N}}$ is constant, and by Theorem 2 there is an integer c such that either $\widetilde{a}_n = c$ for almost all $n \in \mathbb{N}$, or $\widetilde{a}_{2n} = c$, $\widetilde{a}_{2n+1} = 0$ for all $n \in \mathbb{N}$. ∎

Proposition 1. *Consider a sequence* $\mathbf{a}(f, h_1, 0)$. *Let* $k \in \mathbb{N}_+$ *and* $l \in \mathbb{N}$. *If the sequence* $(a_{kn+l})_{n\in\mathbb{N}}$ *is constant then one of the following conditions holds:*

- $a_n = 0$ *for almost all* $n \in \mathbb{N}$,
- $a_n = h_1(0)$ *for all* $n \in \mathbb{N}$,
- $a_n = (-1)^n h_1(0)$ *for all* $n \in \mathbb{N}$.

Proof. If $a_{n_0} = 0$ for some $n_0 \in \mathbb{N}$ then $(a_n)_{n\in\mathbb{N}}$ is ultimately constant and equal to 0, so we may assume that $a_n \neq 0$ for any $n \in \mathbb{N}$. Write $a = a_{kn+l}$, $n \in \mathbb{N}$. Then

$$a = a_{k(n+1)+l} = a_{kn+l}\prod_{i=1}^{k} f(kn+l+i) = a\prod_{i=1}^{k} f(kn+l+i),$$

and since $a \neq 0$, we get $\prod_{i=1}^{k} f(kn+l+i) = 1$ for all $n \in \mathbb{N}$. Hence, $|f(n)| = 1$ for all but finitely many $n \in \mathbb{N}$, which implies that $f = 1$ or $f = -1$. ∎

Theorem 3. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a bounded sequence given by the relation* $a_0 = h_1(0)$, $a_n = f(n)a_{n-1} + h_1(n)h_2(n)^n$, $n > 0$. *Then one of the following conditions is true:*

- $h_1 = 0$ (*and then the sequence* $(a_n)_{n\in\mathbb{N}}$ *is constantly* 0),
- $h_2 \in \{-1, 0, 1\}$.

Moreover,

- *if* $h_2 = 1$, *then there is an integer* c *such that either* $a_n = c$ *for almost all* $n \in \mathbb{N}$ *or* $a_{2n} = c$, $a_{2n+1} = 0$ *for all* $n \in \mathbb{N}$,
- *if* $h_2 = -1$, *then there is an integer* c *such that either* $a_n = (-1)^n c$ *for almost all* $n \in \mathbb{N}$, *or* $a_{2n} = c$, $a_{2n+1} = 0$ *for all* $n \in \mathbb{N}$,
- *if* $h_2 = 0$, *then either* $a_n = 0$ *for almost all* $n \in \mathbb{N}$, *or* $a_n = h_1(0)$ *for all* $n \in \mathbb{N}$, *or* $a_n = (-1)^n h_1(0)$ *for all* $n \in \mathbb{N}$.

Proof. By Theorems 1 and 2, Corollary 1 and Proposition 1, it suffices to show that there are $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$ such that the sequence $(a_{kn+l})_{n\in\mathbb{N}}$ is constant.

Let p be a prime number greater than $\max_{n\in\mathbb{N}} a_n - \min_{n\in\mathbb{N}} a_n$. Then the sequence of remainders $(a_n \pmod p)_{n\in\mathbb{N}}$ is periodic (see [3, Section 4.1]). Moreover, the values of this sequence and $\min_{n\in\mathbb{N}} a_n$ uniquely determine the values of $(a_n)_{n\in\mathbb{N}}$. Indeed, if $a_{n_1} \equiv a_{n_2} \pmod{p}$ then $a_{n_1} - \min_{n\in\mathbb{N}} a_n \equiv a_{n_2} - \min_{n\in\mathbb{N}} a_n \pmod{p}$ and since $a_{n_1} - \min_{n\in\mathbb{N}} a_n, a_{n_2} - \min_{n\in\mathbb{N}} a_n < p$, we have $a_{n_1} = a_{n_2}$. Therefore $(a_n)_{n\in\mathbb{N}}$ is periodic. This implies the existence of $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$ such that the sequence $(a_{kn+l})_{n\in\mathbb{N}}$ is constant. ∎

3. Concluding remarks. An analysis of the proofs shows that in fact the statements of our results are true if we assume there exists an increasing sequence $(n_m)_{m\in\mathbb{N}}$ such that $\liminf_{m\to+\infty}(n_{m+1}-n_m) < +\infty$ (equivalently, there exists a positive integer k such that $n_{m+1}-n_m = k$ for infinitely many $m \in \mathbb{N}$) and the sequence $(a_{n_m})_{m\in\mathbb{N}}$ is constant. Moreover, the statement of Proposition 1 is also true without the assumption $\liminf_{m\to+\infty}(n_{m+1}-n_m) < +\infty$.

On the other hand, we do not know if the statements of Theorems 1 and 2 remain true if the sequence $(a_{n_m})_{m\in\mathbb{N}}$ is constant and the increasing sequence $(n_m)_{m\in\mathbb{N}}$ is arbitrary.

QUESTION 1. Is there an unbounded sequence $\mathbf{a} \in \mathcal{R}$ such that the sequence $(a_{n_m})_{m\in\mathbb{N}}$ is constant for some increasing sequence $(n_m)_{m\in\mathbb{N}}$ and moreover $(a_{2n+1})_{n\in\mathbb{N}} \neq (0)_{n\in\mathbb{N}}$ or $(a_{2n})_{n\in\mathbb{N}}$ is not a geometric progression?

We expect that the answer to the above question is negative.

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