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A family of weakly holomorphic modular forms for $\Gamma_0(2)$ with all zeros on a certain geodesic

by

SoYoung Choi (Jinju) and Bo-Hae Im (Daejeon)

1. Introduction and preliminaries. The zeros of special modular forms and their locations have been studied actively, and there are some results which show that the zeros of some modular forms lie on the boundary of the canonical fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. In particular, Rankin and Swinnerton-Dyer [19] showed that the zeros of the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ lie on the lower boundary of the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. In [11, 9, 10, 13, 17, 1], it is proved that all zeros or almost all zeros of various weakly holomorphic modular forms for the groups $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(N)$ $(N = 2, 3, 4)$ and $\Gamma_0^+(p)$ $(p = 2, 3)$ lie on the lower boundary of the canonical fundamental domains for those groups. Here, $\Gamma_0^+(p)$ is the Fricke group of level p generated by $\Gamma_0(p)$ and the Fricke involution $W_p = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$.

In this paper, we consider zeros of some weakly holomorphic modular forms of weight k for $\Gamma_0(2)$. Let $M_k^!(\Gamma_0(2))$ be the space of weakly holomorphic modular forms of weight k for $\Gamma_0(2)$, and set

$$M_k^{!+}(\Gamma_0(2)) = \{f \in M_k^!(\Gamma_0(2)) : f|_k W_p = f\},$$
$$M_k^{!-}(\Gamma_0(2)) = \{f \in M_k^!(\Gamma_0(2)) : f|_k W_p = -f\}.$$

Then

$$M_k^!(\Gamma_0(2)) = M_k^{!+}(\Gamma_0(2)) \oplus M_k^{!-}(\Gamma_0(2)).$$

We also note that $M_k^{!+}(\Gamma_0(2))$ is the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(2)$. The location of zeros of almost all of the basis elements for $M_k^{!+}(\Gamma_0(2))$ has been studied by the present authors [1, 2]. More precisely, it has been proved in [1] that the zeros of each basis element $f_{k,m}$ of $M_k^{!+}(\Gamma_0(2))$ in the fundamental domain $\mathfrak{F}^+$ for $\Gamma_0^+(2)$ lie on the lower

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boundary of $\mathfrak{F}^+$, and it has been proved in [2] that the zeros of each $f_{k,m}$ interlace with the zeros of another such form.

In this paper we consider the zeros of the basis elements for $M_k^{!-}(\Gamma_0(2))$ and find the location of zeros of almost all elements in a certain basis of $M_k^{!-}(\Gamma_0(2))$. More precisely, we prove that the zeros lie on a certain geodesic inside the standard fundamental domain $\mathfrak{F}$ for $\Gamma_0(2)$ but not on the boundary of $\mathfrak{F}$. Also we show that the zeros of each basis element interlace with the zeros of another element.

We let $\mathfrak{F}$ be the standard fundamental domain for $\Gamma_0(2)$ given by

$$\begin{aligned}\mathfrak{F} := \{z \in \mathbb{C} : |z + 1/2| \geq 1/2,\ -1/2 \leq \mathrm{Re}(z) \leq 0\} \\ \cup \{z \in \mathbb{C} : |z - 1/2| > 1/2,\ 0 \leq \mathrm{Re}(z) < 1/2\}.\end{aligned}$$

Let

$$S := \left\{ \frac{1}{\sqrt{2}} e^{i\theta} : \theta \in (\pi/2, 3\pi/4) \right\},$$

which is a part of a geodesic inside $\mathfrak{F}$ but not on the boundary of $\mathfrak{F}$. We define the usual slash operator as follows:

$$(f|_k\gamma)(z) = (cz + d)^{-k} f(\gamma z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

For a given $k \in 2\mathbb{Z}$, we can write

$$k = 8\ell_k + r_k$$

for unique $\ell_k \in \mathbb{Z}$ and $r_k \in \{2, 4, 6, 8\}$.

Let $\rho := -\frac{1}{2} + \frac{1}{2}i \in \mathfrak{H}$. Then ρ is an elliptic point of order 2 for $\Gamma_0(2)$, since $\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)\rho = \rho$ for $\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$.

By the valence formula (see for example [15, (1)]), we have the following.

Lemma 1.1 ([15]). *For $f \in M_k^!(\Gamma_0(2))$,*

$$\sum_{\rho \neq \tau \in \Gamma_0(2)\backslash\mathfrak{H}^*} \mathrm{ord}_\tau f + \frac{1}{2}\,\mathrm{ord}_\rho f = \frac{k}{4}. \ \blacksquare$$

Proposition 1.2. *For each $k = 2, 4, 6, 8$ and $f \in M_k^{!-}(\Gamma_0(2))$, the zeros of $f(z)$ in $\mathfrak{F}$ belong to $\left\{-\frac{1}{2} + \frac{1}{2}i, \frac{1}{\sqrt{2}}i\right\} \subseteq \overline{S}$.*

Proof. First, if $k = 2$, then by Lemma 1.1, $\mathrm{ord}_\rho f = 1$ and $\mathrm{ord}_\tau f = 0$ for all $\tau \neq \rho$. Hence ρ is the only zero of f in $\mathfrak{F}$.

Let $z = \frac{1}{\sqrt{2}}i$. Since $f \in M_k^{!-}(\Gamma_0(2))$, we have

$$(f|_k W_2)(z) = -f(z),$$

and by the definition of the slash operator,

$$(f|_k W_2)(z) = (\sqrt{2}\,z)^{-k} f(W_2(z)) = f(z)(i)^{-k}.$$

Hence if $k \not\equiv 2 \bmod 4$, then $f(z) = 0$, and so $z = \frac{1}{\sqrt{2}}i$ is a zero of f in $\mathfrak{F}$.

If $k=4$, since $\operatorname{ord}_z f \geq 1$ by the above, Lemma 1.1 implies $\operatorname{ord}_z f = 1$, so $z = \frac{1}{\sqrt{2}} i$ is the only zero of f in $\mathfrak{F}$.

If $k=6$, Lemma 1.1 implies $\sum_{\rho \neq \tau \in \Gamma_0(2)\backslash \mathfrak{H}^*} \operatorname{ord}_\tau f + \frac{1}{2} \operatorname{ord}_\rho f = 1 + \frac{1}{2}$, hence $\operatorname{ord}_\rho f \geq 1$. Let $V = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$. Then $(2\tau-1)^{-6} f(V\tau) = f(\tau)$ gives

$$f'(\tau) = (2\tau-1)^{-8} f'(V\tau) - 12(2\tau-1)^{-7} f(V\tau).$$

Letting $\tau=\rho$, we have $f'(\rho) = (2\rho-1)^{-8} f'(\rho)$, which implies that $f'(\rho)=0$. So $\operatorname{ord}_\rho f = 3$ and ρ is the only zero of f in $\mathfrak{F}$.

If $k=8$, then since $(\sqrt{2}\,\tau)^{-8} f\left(-\frac{1}{2\tau}\right) = f(\tau)$, this implies that

$$f'(\tau) = (\sqrt{2}\,\tau)^{-8} f\left(-\frac{1}{2\tau}\right)\left(\frac{1}{2\tau^2}\right) + f\left(-\frac{1}{2\tau}\right)(-8)\sqrt{2}^{-8}\tau^{-9}.$$

By letting $\tau = z$, we get

$$f'(z) = f'(z)\left(\frac{1}{2\cdot\frac{-1}{2}}\right) i^{-8} + f(z)(-8)\sqrt{2}^{-8} z^{-9}.$$

Since $f(z)=0$, we have $f'(z) = -f'(z)$, which implies $f'(z)=0$. Hence $\operatorname{ord}_z f \geq 2$. Then by Lemma 1.1, $\sum_{\rho, z \neq \tau \in \Gamma_0(2)\backslash \mathfrak{H}^*} \operatorname{ord}_\tau f + \operatorname{ord}_z f + \frac{1}{2}\operatorname{ord}_\rho f = 2$, which implies that $\operatorname{ord}_z f = 2$ and $z = \frac{1}{\sqrt{2}} i$ is the only zero of f in $\mathfrak{F}$. ∎

PROPOSITION 1.3. *For $f \in M_k^{!-}(\Gamma_0(2))$ with real Fourier coefficients, $e^{i(k\theta+\pi)/2} f\left(\frac{1}{\sqrt{2}} e^{i\theta}\right)$ is real for $\theta \in (\pi/2, 3\pi/4)$.*

Proof. Let $z = \frac{1}{\sqrt{2}} e^{i\theta} = a + bi$ with $a, b \in \mathbb{R}$ and let

$$f(z) = \sum_{n \geq n_f} a_n e^{2\pi i n(a+bi)} = \sum_{n\geq n_f} a_n e^{2\pi n(-b+ai)}.$$

Since $a^2+b^2 = |z|^2 = \frac{1}{2}$, we have $-\frac{1}{2z} = -a+bi$, and since $a_n \in \mathbb{R}$, we get

$$f\left(-\frac{1}{2z}\right) = \sum_{n\geq n_f} a_n e^{2\pi i n(-a+bi)} = \overline{f(z)}.$$

Thus,

$$\begin{aligned} e^{i(k\theta+\pi)/2} f(z) &= e^{ik\theta} e^{i(\pi-k\theta)/2} f(z) = e^{i(\pi-k\theta)/2}\left(\overline{e^{-ik\theta} f\left(-\frac{1}{2z}\right)}\right) \\ &= e^{i(\pi-k\theta)/2}\left(\overline{(\sqrt{2}\,z)^{-k} f\left(-\frac{1}{2z}\right)}\right) = -e^{i(\pi-k\theta)/2}\overline{f(z)}. \end{aligned}$$

Hence,

$$ie^{ik\theta/2} f(z) = -ie^{-ik\theta/2}\overline{f(z)} = -i\overline{e^{ik\theta/2} f(z)} = \overline{ie^{ik\theta/2} f(z)}. \ ∎$$

REMARK 1.4. Let

$$m_k^- := \ell_k + \dim S_{r_k}^-(\Gamma_0(2)) = \ell_k,$$

which is defined in [7]. Let

$$
\begin{cases}
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \\
\Delta_2^+(z) = (\eta(z)\eta(2z))^8 = q \prod_{n=1}^{\infty} (1-q^n)^8 \prod_{m=1}^{\infty} (1-q^{2m})^8, \\
t(z) := j_2^+(z) = \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 24 + 2^{12}\left(\frac{\eta(2z)}{\eta(z)}\right)^{24}, \\
E_{r_k}(z) = 1 - \frac{2r_k}{B_{r_k}} \sum_{n=1}^{\infty} \sigma_{r_k-1}(n) q^n, \\
E_{r_k}^-(z) := E_{2,r_k}^-(z) = \frac{1}{1-2^{r_k/2}} (E_{r_k}(z) - 2^{r_k/2} E_{r_k}(2z)), \\
\Delta_{2,r_k}^-(z) = E_{2,r_k}^-(z),
\end{cases} \tag{1}
$$

where B_{r_k} is the r_kth Bernoulli number and $\sigma_{r_k-1}(n)$ is the standard divisor sum of n.

For each integer $m \geq -m_k^-$, there exists a unique $f_{k,m}^- \in M_k^{!-}(\Gamma_0(2))$ with q-expansion of the form

$$f_{k,m}^-(z) = q^{-m} + \mathcal{O}(q^{m_k^-} + 1).$$

Moreover,

$$
\begin{aligned}
f_{k,m}^-(z) &= (\Delta_2^+(z))^{\ell_k} \Delta_{2,r_k}^-(z) P_D(j_2^+(z)) \\
&= (\Delta_2^+(z))^{\ell_k} E_{2,r_k}^-(z) P_D(j_2^+(z)),
\end{aligned} \tag{2}
$$

where P_D is a polynomial of degree $D := m + m_k^- = m + \ell_k$. Furthermore, $f_{k,m}$ has integral Fourier coefficients (see [7]).

We also note that for $k = 8\ell_k + r_k$ and $2-k = 8\ell_{2-k} + r_{2-k}$,

$$2-k = 8(-\ell_k - 1) + (10 - r_k) = 8\ell_{2-k} + r_{2-k},$$

and

$$\ell_{2-k} = m_{2-k}^- = -\ell_k - 1 \quad \text{and} \quad r_{2-k} = 10 - r_k. \tag{3}$$

Now we state our first main result.

THEOREM 1.5. *If $m \geq 2|\ell_k| - \ell_k + 8$, then all zeros of $f_{k,m}^-$ lie on a part of a certain geodesic inside $\mathfrak{F}$ but not on the boundary of $\mathfrak{F}$. In particular, they lie on $S \cup W_2 S$.*

Throughout, we let

$$F^- := f_{k,-\ell_k}^-(z) = q^{\ell_k} + \mathcal{O}(q^{-\ell_k+1}).$$

We prove the following theorem referring to [7, Lemma 4.1].

THEOREM 1.6. *For each $n \geq -\ell_k$,*

$$f_{k,n}^-(z) = F^-(z) \sum_{r+s=n} a_r (1/F^-) j_{2,s}^+.$$

Proof. Recall that $j_{2,s}^{+} = q^{-s} + \mathcal{O}(q)$. Let $F^{-} = \sum_{\alpha\geq \ell_k} a_\alpha(F^{-})q^\alpha$. Then

$$1 = F^{-}(z)\left(\frac{1}{F^{-1}}\right)(z) = \sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha \sum_{\beta\geq -\ell_k} a_\beta(1/F^{-})q^\beta.$$

Thus, $\sum_{\alpha+\beta=\gamma} a_\alpha(F^{-})a_\beta(1/F^{-}) = 0$ for all $\gamma\geq 1$.

Note that $F^{-}(z)\sum_{r+s=n} a_r(1/F^{-})j_{2,s}^{+}(z) \in M_k^{!-}(\Gamma_0(2))$ and

$$\begin{aligned}
&F^{-}(z)\sum_{r+s=n} a_r\left(\frac{1}{F^{-}}\right)j_{2,s}^{+}(z) = F^{-}(z)\left(\sum_{r+s=n} a_r\left(\frac{1}{F^{-}}\right)(q^{-s}+\mathcal{O}(q))\right)\\
&= F^{-}(z)\left(\sum_{r+s=n} a_r\left(\frac{1}{F^{-}}\right)q^{-s}+\mathcal{O}(q)\right)\\
&= \left(\sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha\right)\left(\sum_{r+s=n} a_r\left(\frac{1}{F^{-}}\right)q^{-s}+\mathcal{O}(q)\right)\\
&= \left(\sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha\right)\left(a_{-\ell_k}\left(\frac{1}{F^{-}}\right)q^{-\ell_k-n} + a_{-(\ell_k-1)}\left(\frac{1}{F^{-}}\right)q^{-(\ell_k-1)-n}\right.\\
&\quad \left. + a_{-(\ell_k-2)}\left(\frac{1}{F^{-}}\right)q^{-(\ell_k-2)-n} + \cdots + a_{n-1}\left(\frac{1}{F^{-}}\right)q^{-1} + a_n\left(\frac{1}{F^{-}}\right)q^{0} + \mathcal{O}(q)\right)\\
&\quad \text{(since } s\geq 0, -m\leq \ell_k, \text{ and } r\geq -\ell_k)\\
&= \left(\sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha\right)\left(\sum_{\beta\geq 0} a_{-\ell_k}+\beta\left(\frac{1}{F^{-}}\right)q^{-\ell_k+\beta}q^{-n}+\mathcal{O}(q)\right)\\
&= q^{-n}\left(\sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha\right)\left(\sum_{\beta=0}^{\ell_k+n} a_{-\ell_k}+\beta\left(\frac{1}{F^{-}}\right)q^{-\ell_k+\beta}\right)+\mathcal{O}(q^{\ell_k+1})\\
&= q^{-n}\left(\sum_{\alpha\geq\ell_k} a_\alpha(F^{-})q^\alpha\right)\left(\sum_{\beta=-\ell_k}^{n} a_\beta\left(\frac{1}{F^{-}}\right)q^{\beta}\right)+\mathcal{O}(q^{\ell_k+1})\\
&= q^{-n}\sum_{\gamma=0}^{\ell_k+n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha(F^{-})a_\beta\left(\frac{1}{F^{-}}\right)\right)q^\gamma+\mathcal{O}(q^{\ell_k+1}) = q^{-n}+\mathcal{O}(q^{\ell_k+1}).
\end{aligned}$$

By the uniqueness of $f_{k,n}^{-}$ in $M_k^{!-}(\Gamma_0(2))$, the conclusion follows. ∎

REMARK 1.7. Note that $f_{2,m}^{+}(z) = q^{-m}+\mathcal{O}(q)$ for all $m\geq 1$ and $j_{2,s}^{+}(z) = f_{0,s}^{+}(z)$ for all $s\geq 0$. Also we have the following:

(1) $\sum_{m\geq 0} f_{0,m}^{+}(\tau)e^{2\pi i m z} = \frac{f_{2,1}^{+}(z)}{j_2^{+}(z)-j_2^{+}(\tau)}$.

(2) $\Phi_{2,m}(z) = q^{-m}+\mathcal{O}(q)$ (see [8] fot the definition of $\Phi_{2,m}$). Hence, $f_{2,m}^{+}(z) = q^{-m}+\mathcal{O}(q)\in M_2^{!+}(\Gamma_0(2))$.

(3) $a_0(\Phi_{2,m}f^+_{0,s}) = 0$ for all m, s by the residue theorem since $\Phi_{2,m}f^+_{0,s} \in M_2^{!+}(\Gamma_0(2))$.

Applying [8, Theorem 1.2] to Remark 1.7 and Theorem 1.6, we obtain

THEOREM 1.8.

$$\sum_{n\geq -\ell_k} f^-_{k,n}(\tau)q^n = \frac{F^-(\tau)}{F^-(z)}\cdot\frac{f^+_{2,1}(z)}{j_2^+(z)-j_2^+(\tau)}.$$

Referring to [7, Lemma 4.1] again, we prove the following lemma.

LEMMA 1.9.

$$f^+_{2,1}(z) = f^-_{2-k,-\ell_{2-k}}(z)f^-_{k,-\ell_k}(z).$$

Proof. Note that

$$f^-_{2-k,-\ell_{2-k}}(z)f^-_{k,-\ell_k}(z) = q^{\ell_{2-k}+\ell_k} + \mathcal{O}(1) = q^{-1} + \mathcal{O}(1),$$

since $\ell_{2-k}+\ell_k = -1$ by (3) in Remark 1.4. Moreover, since $f^-_{2-k,-\ell_{2-k}}f^-_{k,-\ell_k} \in M_2^{!+}(\Gamma_0(2))$,

$$f^-_{2-k,-\ell_{2-k}}(z)f^-_{k,-\ell_k}(z) = q^{-1} + \mathcal{O}(q).$$

Hence the conclusion follows from the uniqueness of $f^+_{2,1}(z) = q^{-1}+\mathcal{O}(q)$ in $M_k^{!+}(\Gamma_0(2))$. ∎

LEMMA 1.10. *In the notations above,*

$$\frac{F^-(\tau)f^+_{2,1}(z)}{F^-(z)(j_2^+(z)-j_2^+(\tau))} = \frac{f^-_{k,-\ell_k}(\tau)f^-_{2-k,-\ell_{2-k}}(z)}{j_2^+(z)-j_2^+(\tau)}.$$

Proof. This follows from Lemma 1.9 and $f^-_{k,-\ell_k}(z) = F^-(z)$. ∎

We let $\mathfrak{F}^+$ be the standard fundamental domain for $\Gamma_0^+(2)$, given by

$$\begin{aligned}\mathfrak{F}^+ := \{z\in\mathbb{C} : |z|\geq 1/\sqrt{2}, -1/2\leq \mathrm{Re}(z)\leq 0\}\\ \cup\{z\in\mathbb{C} : |z|>1/\sqrt{2}, 0\leq \mathrm{Re}(z)<1/2\}\end{aligned}$$

(see [17, p. 694]). By Theorem 1.8 and Lemma 1.10, we have

$$f^-_{k,m}(z) = \oint_C \frac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)}e^{-2\pi i m\tau}e^{-2\pi i\tau}\,\mathrm{d}e^{2\pi i\tau},$$

where C is a (counterclockwise oriented) circle centered at 0 in the q-plane with a sufficiently small radius. (Here, $q = e^{2\pi i\tau}$.) Let

$$\begin{aligned}(4)\qquad G^-(\tau,z) &= \frac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)}e^{-2\pi i m\tau}\\ &= \frac{(\Delta_2^+(z))^{\ell_k}E^-_{r_k}(z)E^-_{10-r_k}(\tau)}{(\Delta_2^+(\tau))^{\ell_k+1}(j_2^+(\tau)-j_2^+(z))}e^{-2\pi i m\tau}.\end{aligned}$$

Now we compute the residues of $\dfrac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)}$ at $\tau=z$ and $\tau=-\frac{1}{2z}$:

$$
\begin{aligned}
(5)\qquad \operatorname{Res}_{\tau=z}\frac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)} &= \lim_{\tau\to z}(\tau-z)\frac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)}\\
&= \left(\frac{\mathrm{d}j_2^+}{\mathrm{d}\tau}\bigg|_{\tau=z}\right)^{-1}\cdot f^+_{2,1}(z)\quad\text{(by Lemma 1.9)}\\
&= \left(\frac{\mathrm{d}j_2^+}{\mathrm{d}\tau}\bigg|_{\tau=z}\right)^{-1}\cdot\frac{\mathrm{d}j_2^+(z)}{\mathrm{d}z}\cdot\frac{-1}{2\pi i}=-\frac{1}{2\pi i},
\end{aligned}
$$

$$
\begin{aligned}
(6)\qquad &\operatorname{Res}_{\tau=-\frac{1}{2z}}\frac{f^-_{k,-\ell_k}(z)f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(z)}\\
&= \operatorname{Res}_{\tau=-\frac{1}{2z}}\frac{-f^-_{k,-\ell_k}\left(-\frac{1}{2z}\right)(\sqrt{2}\,z)^{-k}f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+\left(-\frac{1}{2z}\right)}\\
&= \operatorname{Res}_{\tau=w}\frac{-f^-_{k,-\ell_k}(w)(\sqrt{2}\,z)^{-k}f^-_{2-k,-\ell_{2-k}}(\tau)}{j_2^+(\tau)-j_2^+(w)}\\
&\quad\big(\text{where } w=-\tfrac{1}{2z}\big)\\
&= \tfrac{1}{2\pi i}(\sqrt{2}\,z)^{-k}\qquad\text{by (5)}.
\end{aligned}
$$

Next, we consider when $\operatorname{Im}(z)\frac{1}{\sqrt{2}}\sin\theta>\frac{1}{2}$. Let A' be a real number such that $\frac{1}{2}<A'<\operatorname{Im}(z)$. Since $\mathfrak{F}^+$ is the fundamental domain for $\Gamma_0^+(2)$ and $j_2^+(\tau)$ is invariant under the action of $\Gamma_0^+(2)$, the residue theorem shows that for a real number $A>1/\sqrt{2}$,

$$
\begin{aligned}
(7)\qquad \int_{-1/2+iA'}^{1/2+iA'} G^-(\tau,z)\,\mathrm{d}\tau &= f^-_{k,m}(z)-\int_{1/2+iA'}^{1/2+iA} G^-(\tau,z)\,\mathrm{d}\tau-\int_{-1/2+iA}^{-1/2+iA'} G^-(\tau,z)\,\mathrm{d}\tau\\
&\quad+2\pi i\sum_{\tau=z,-\frac{1}{2z}}\operatorname{Res}_\tau G^-(\tau,z)\\
&= f^-_{k,m}(z)-e^{-2\pi imz}+\left(\frac{1}{\sqrt{2}\,z}\right)^k e^{-2\pi im(-\frac{1}{2z})},
\end{aligned}
$$

since

$$
\begin{aligned}
2\pi i\sum_{\tau=z,-\frac{1}{2z}}\operatorname{Res}_\tau G^-(\tau,z) &= \int_{1/2+iA'}^{1/2+iA} G^-(\tau,z)\,\mathrm{d}\tau+\int_{1/2+iA}^{-1/2+iA} G^-(\tau,z)\,\mathrm{d}\tau\\
&\quad+\int_{-1/2+iA}^{-1/2+iA'} G^-(\tau,z)\,\mathrm{d}\tau+\int_{-1/2+iA'}^{1/2+iA'} G^-(\tau,z)\,\mathrm{d}\tau,
\end{aligned}
$$

and

$$\int_{-1/2+iA}^{1/2+iA} G^{-}(\tau, z)\,\mathrm{d}\tau = f^{-}_{k,m}(z).$$

PROPOSITION 1.11. *We have*

$$(8)\quad ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta} f^{-}_{k,m}\left(\frac{1}{\sqrt{2}}e^{i\theta}\right) + 2\sin\left(\frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{2}}\cos\theta\right)$$
$$= ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\int_{-1/2+iA'}^{1/2+iA'} G^{-}\left(\tau, \frac{1}{\sqrt{2}}e^{i\theta}\right)\mathrm{d}\tau.$$

Proof. First, we get

$$(9)\quad ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\left(\frac{1}{\sqrt{2}}\,\frac{1}{\frac{1}{\sqrt{2}}e^{i\theta}}\right)^{k} e^{-2\pi i m\left(-\frac{1}{2(\frac{1}{\sqrt{2}}e^{i\theta})}\right)}$$
$$= ie^{-ik\theta/2}e^{\sqrt{2}\,\pi i m\cos\theta}$$
$$= i\left(\cos\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right) - \sin\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right)\right)$$
$$= i\cos\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right) + \sin\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right).$$

Also, at $z = \frac{1}{\sqrt{2}}e^{i\theta}$,

$$(10)\quad ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}e^{-2\pi i m z}$$
$$= i\cos\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right) - \sin\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right).$$

By subtracting (10) from (9), we get

$$ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\left(-e^{-2\pi i m z} + \left(\frac{1}{\sqrt{2}\,z}\right)^{k} e^{-2\pi i m\left(-\frac{1}{2z}\right)}\right)$$
$$= 2\sin\left(\frac{k\theta}{2} - \sqrt{2}\,\pi m\cos\theta\right).$$

Thus, we complete the proof by the formula (7) for the integral of G^{-}. ■

In order to prove Theorem 1.5, we need the following lemma.

LEMMA 1.12. *If $m \geq 2|\ell_k| - \ell_k + 8$, then for all $\theta \in (\pi/2, 3\pi/4)$,*

$$\left|ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta} f^{-}_{k,m}\left(\frac{1}{\sqrt{2}}e^{i\theta}\right) + 2\sin\left(\frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{2}}\cos\theta\right)\right| < 1.9457196.$$

Proof. We follow [1, p. 303, proof of Lemma 2.3] with the following replacement of the numerical quantities and computations:

$$
\begin{aligned}
&E^-_{r_k}\left(\frac{-1}{2z}\right) = -(\sqrt{2}\,z)^{r_k} E^-_{r_k}(z),\\
&E^-_{r_k}\left(\frac{-1}{2z\pm 2}\right) = -(\sqrt{2}\,z)^{r_k}(z\pm 1)^{r_k} E^-_{r_k}(z),\\
&B^-_k := iB_k, \qquad \text{where } B_k \text{ is defined in [1, p. 306]}.
\end{aligned}
$$

Then by Proposition 1.11,

$$
\begin{aligned}
ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta} f^-_{k,m}\left(\frac{1}{\sqrt{2}}e^{i\theta}\right) + 2\sin\left(\frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{2}}\cos\theta\right)\\
= ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\int_{-1/2+iA'}^{1/2+iA'} G^-\left(\tau, \frac{1}{\sqrt{2}}e^{i\theta}\right) d\tau,
\end{aligned}
$$

which equals

$$
\begin{cases}
-B^-_k + ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\displaystyle\int_{-1/2}^{1/2} G^-\left(x+0.3i, \frac{1}{\sqrt{2}}e^{i\theta}\right) dx\\
\qquad \text{if } 2\le\theta<3\pi/4 \text{ and } A' = 0.3 < \mathrm{Im}\left(\dfrac{-1}{2\frac{1}{\sqrt{2}}e^{i\theta}+2}\right),\\
ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta}\displaystyle\int_{-1/2}^{1/2} G^-\left(x+0.4i, \frac{1}{\sqrt{2}}e^{i\theta}\right) dx\\
\qquad \text{if } \pi/2\le\theta<2 \text{ and } A' = 0.4 > \mathrm{Im}\left(\dfrac{-1}{2\frac{1}{\sqrt{2}}e^{i\theta}+2}\right).
\end{cases}
$$

Then, we get the bounds in [1, Lemma 3.1] by replacing (1)(e) and (2)(d) there by (1)(e)′ and (2)(d)′ respectively, where

(1)(e)′ For $k = 2,4,6,8$, $|E^-_{r_k}(z)|\,|E^-_{10-r_k}(\tau)| \le 4613.738108$ if $\tau = x+0.3i$.
(2)(d)′ For $k = 2,4,6,8$, $|E^-_{r_k}(z)|\,|E^-_{10-r_k}(\tau)| \le 378.3098018$ if $\tau = x+0.4i$.

In fact, both upper bounds are smaller than or close enough to those of (1)(e) and (2)(d) in [1, Lemma 3.1], so we can proceed as in [1, proof of Lemma 3.1]. To be precise, we indicate some steps which contain different quantities from ones in that proof.

First, in order to prove (1)(e)$'$, we note that

$$
\begin{aligned}
|E_2^-(z)| &= |E_2(z) - 2E_2(2z)| \le \tfrac{1}{3}(|E_2(z)| + 2|E_2(2z)|) \\
&\le \left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-2\pi n \frac{1}{\sqrt{2}} \sin\theta}}{1 - e^{-2\pi n \frac{1}{\sqrt{2}} \sin\theta}} + 2\left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-4\pi n \frac{1}{\sqrt{2}} \sin\theta}}{1 - e^{-4\pi n \frac{1}{\sqrt{2}} \sin\theta}}\right)\right) \\
&\le \left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-\pi n}}{1 - e^{-\pi n}} + 2\left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-2\pi n}}{1 - e^{-2\pi n}}\right)\right) \le 4.270087259.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|E_2^-(\tau)| &\le \left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-2\pi n(0.3)}}{1 - e^{-2\pi n(0.3)}} + 2\left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-4\pi n(0.3)}}{1 - e^{-4\pi n(0.3)}}\right)\right) \\
&\le 9.930494810, \\
|E_4^-(z)| &\le \tfrac{1}{3}(|E_4(z)| + 4|E_4(2z)|) \le 7.278814458, \\
|E_4^-(\tau)| &\le \tfrac{1}{3}(|E_4(\tau)| + 4|E_4(2\tau)|) \le 51.51027855, \\
|E_6^-(z)| &\le \tfrac{1}{7}(|E_6(z)| + 8|E_6(2z)|) \le 11.69821417, \\
|E_6^-(\tau)| &\le \tfrac{1}{7}(|E_6(\tau)| + 8|E_6(2\tau)|) \le 222.6799583, \\
|E_8^-(z)| &\le \tfrac{1}{15}(|E_8(z)| + 16|E_8(2z)|) \le 19.35577645, \\
|E_8^-(\tau)| &\le \tfrac{1}{15}(|E_8(\tau)| + 16|E_8(2\tau)|) \le 1080.478648.
\end{aligned}
$$

Hence, for all $r_k \in \{2, 4, 6, 8\}$,

$$|E_{r_k}^-(z)|\,|E_{10-r_k}^-(\tau)| \le 4613.738108.$$

In order to prove (2)(d)$'$, we compute

$$
\begin{aligned}
|E_2^-(z)| &\le \left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-\sqrt{2}\,\pi n \sin(2)}}{1 - e^{-\sqrt{2}\,\pi n \sin(2)}} + 2\left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-2\sqrt{2}\,\pi n \sin(2)}}{1 - e^{-2\sqrt{2}\,\pi n \sin(2)}}\right)\right) \\
&\le 3.460102014, \\
|E_2^-(\tau)| &\le \left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-\sqrt{2}\,\pi n(0.4)}}{1 - e^{-\sqrt{2}\,\pi n(0.4)}} + 2\left(1 + 24\sum_{n=1}^{\infty} \frac{n^3 e^{-2\sqrt{2}\,\pi n(0.4)}}{1 - e^{-2\sqrt{2}\,\pi n(0.4)}}\right)\right) \\
&\le 5.796531643, \\
|E_4^-(z)| &\le \tfrac{1}{3}(|E_4(z)| + 4|E_4(2z)|) \le 3.409788137, \\
|E_4^-(\tau)| &\le \tfrac{1}{3}(|E_4(\tau)| + 4|E_4(2\tau)|) \le 16.58085726, \\
|E_6^-(z)| &\le \tfrac{1}{7}(|E_6(z)| + 8|E_6(2z)|) \le 3.572474692, \\
|E_6^-(\tau)| &\le \tfrac{1}{7}(|E_6(\tau)| + 8|E_6(2\tau)|) \le 40.94172134, \\
|E_8^-(z)| &\le \tfrac{1}{15}(|E_8(z)| + 16|E_8(2z)|) \le 3.576888371, \\
|E_8^-(\tau)| &\le \tfrac{1}{15}(|E_8(\tau)| + 16|E_8(2\tau)|) \le 109.3348694.
\end{aligned}
$$

Hence, for all $r_k \in \{2,4,6,8\}$,

$$|E_{r_k}^-(z)|\,|E_{10-r_k}^-(\tau)| \le 378.3098018.$$

Therefore, at $\tau = x + 0.3i$, if $k \ge 0$ and $m \ge \ell_k + 8$, then by [1, Lemma 3.1(1)(a–d)] together with (1)(e)′ above we have

$$\begin{aligned}
|G^-(\tau,z)| &\le \left|\frac{\Delta_2^+(z)}{\Delta_2^+(\tau)}\right|^{\ell_k} \frac{|E_{r_k}^-(z)|\,|E_{10-r_k}^-(\tau)|}{|\Delta_2^+(\tau)|\,|j_2^+(\tau)-j_2^+(z)|} e^{2\pi(0.3)m} \\
&\le (2.314348553)^{\ell_k} \underbrace{\frac{4613.738108}{0.02697058723 \cdot 9.145597363}}_{=:C_1} (e^{0.6\pi})^m.
\end{aligned}$$

Hence if $k \ge 0$, then by [1, Lemma 3.1(1)],

$$\begin{aligned}
&|B_k^-| + e^{-2\pi m \frac{1}{\sqrt{2}}\sin\theta} \int_{-1/2}^{1/2} \left| G\left(x+0.3i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\
&\le |B_k^-| + e^{-\pi m} \int_{-1/2}^{1/2} \left| G^-\left(x+0.3i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\
&\le 1.036225459 + (2.314348553)^{\ell_k} C_1 (e^{0.6\pi-\pi})^m \\
&\le 1.036225459 + (2.314348553)^{\ell_k} C_1 (0.2846095432)^m \\
&\le 1.036225459 + (2.314348553 \cdot 0.2846095432)^{\ell_k} (0.2846095432)^8 C_1 \\
&= 1.036225459 + 0.6586856845^{\ell_k} 0.8052785386 \\
&< 1.841503998 < 1.9457196.
\end{aligned}$$

If $k < 0$, then $\ell_k < 0$ and so if $m \ge -3\ell_k + 8$, then from (4), [1, Lemma 3.1(1)(a–d)] and (1)(e)′ above we have

$$\begin{aligned}
&|B_k^-| + e^{-2\pi m \frac{1}{\sqrt{2}}\sin\theta} \int_{-1/2}^{1/2} \left| G^-\left(x+0.3i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\
&\le |B_k^-| + e^{-\pi m} \int_{-1/2}^{1/2} \left| G^-\left(x+0.3i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\
&\le 1.8713088 + (4.008127019)^{-\ell_k} C_1 (e^{0.6\pi-\pi})^m \\
&\le 1.8713088 + (4.008127019)^{-\ell_k} C_1 (0.2846095432)^m \\
&\le 1.8713088 + (4.008127019 \cdot 0.2846095432^3)^{-\ell_k} (0.2846095432)^8 C_1 \\
&= 1.8713088 + 0.09240380412^{-\ell_k} 0.8052785386 \\
&< 1.9457196.
\end{aligned}$$

Next, at $\tau = x+0.3i$, if $k \geq 0$ and $m \geq \ell_k+8$, then by [1, Lemma 3.1(2)(a–c)] together with (2)(d)$'$ above we have

$$\begin{aligned}|G^-(\tau,z)| &\leq \left|\frac{\Delta_2^+(z)}{\Delta_2^+(\tau)}\right|^{\ell_k} \frac{|E_{r_k}^-(z)|\,|E_{10-r_k}^-(\tau)|}{|\Delta_2^+(\tau)|\,|j_2^+(\tau)-j_2^+(z)|} e^{2\pi(0.4)m} \\ &\leq (0.5509743592)^{\ell_k} \underbrace{\frac{378.3098018}{0.03690703328 \cdot 0.61658483}}_{=:C_2} (e^{0.8\pi})^m.\end{aligned}$$

Hence if $m \geq \ell_k + 6$, then

$$\begin{aligned}&e^{-2\pi m \frac{1}{\sqrt{2}}\sin\theta} \int_{-1/2}^{1/2} \left|G^-\left(x+0.4i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\ &\quad\leq e^{-\sqrt{2}\,\pi m \sin(2)} \int_{-1/2}^{1/2} \left|G^-\left(x+0.4i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\ &\quad\leq (0.5509743592)^{\ell_k} C_2 (e^{0.8\pi - \sqrt{2}\,\pi\sin(2)})^m \\ &\quad\leq (0.5509743592)^{\ell_k} C_2 (0.2172670797)^m \\ &\quad\leq (0.5509743592 \cdot 0.2172670797)^{\ell_k} (0.2172670797)^6 C_2 \\ &\quad= 0.1197085901^{\ell_k} 1.748675334 \leq 1.748675334 < 1.9457196.\end{aligned}$$

If $m \geq \ell_k + 8$, then $m \geq \ell_k + 6$, hence the above holds.

If $k < 0$, then by (4), [1, Lemma 3.1(2)(a–c)] and (2)(d)$'$ above again,

$$\begin{aligned}|G^-(\tau,z)| &\leq \left|\frac{\Delta_2^+(\tau)}{\Delta_2^+(z)}\right|^{-\ell_k} \frac{|E_{r_k}^-(z)|\,|E_{10-r_k}^-(\tau)|}{|\Delta_2^+(\tau)|\,|j_2^+(\tau)-j_2^+(z)|} e^{2\pi(0.4)m} \\ &\leq (3.850448548)^{-\ell_k} C_2 (e^{0.8\pi})^m.\end{aligned}$$

Hence if $m \geq -3\ell_k + 8$, then

$$\begin{aligned}&e^{-2\pi m \frac{1}{\sqrt{2}}\sin\theta} \int_{-1/2}^{1/2} \left|G^-\left(x+0.4i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\ &\quad\leq e^{-\sqrt{2}\,\pi m \sin(2)} \int_{-1/2}^{1/2} \left|G^-\left(x+0.4i, \frac{1}{\sqrt{2}}e^{i\theta}\right)\right| \mathrm{d}x \\ &\quad\leq (3.850448548)^{-\ell_k} C_2 (e^{0.8\pi - \sqrt{2}\,\pi\sin(2)})^m \\ &\quad\leq (3.850448548)^{-\ell_k} C_2 (0.2172670797)^m \\ &\quad\leq (3.850448548 \cdot 0.2172670797^3)^{-\ell_k} (0.2172670797)^8 C_2 \\ &\quad= 0.03949054300^{-\ell_k} 0.08254619103 \leq 1 < 1.9457196. \quad \blacksquare\end{aligned}$$

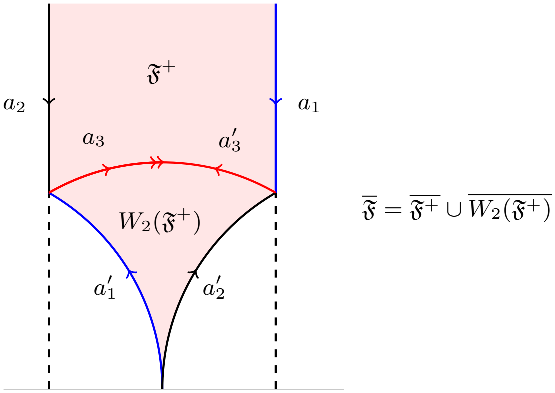

Fig. 1. Transformations of the boundary segments of $\mathfrak{F}^+$ in $\mathfrak{F}$ via W_2

Proof of Theorem 1.5. Since $f_{k,m}^-$ is the product of Δ_2^+, E_{2,r_k}^-, and a polynomial $P_D(j_2^+(z))$ in $t(z)$ of degree $D = m + \ell_k$ by (2), and since Δ_2^+ has no zero on $\mathfrak{H}$ and the zeros of E_{2,r_k}^- lie on $\overline{S}$ by Proposition 1.2, it is enough to consider the zeros of $P_D(j_2^+(z))$. We show that all zeros in $\mathfrak{F}^+$ of $P_D(j_2^+(z))$ lie on $\overline{S}$, and the number of zeros is D, the degree of P_D. We note that the arc S is transformed into $W_2(S) = \bigl\{\frac{1}{\sqrt{2}}e^{2\pi i\theta} : \pi/4 < \theta < \pi/2\bigr\} \subseteq \mathfrak{F}$ via W_2, and that the left and right vertical boundaries of $\mathfrak{F}^+$ are transformed into the lower left arc and the lower right arc of the boundary of $\mathfrak{F}$ via W_2, respectively, as shown in Fig. 1, and referring to [1, p. 319, Appendix A, (a)]. Therefore, all zeros in $\mathfrak{F}$ of $P_D(j_2^+(z))$ lie on $\overline{S} \cup W_2(\overline{S})$, which is a geodesic inside $\mathfrak{F}$.

We can prove that all zeros in $\mathfrak{F}^+$ of $P_D(j_2^+(z))$ lie on $\overline{S}$ by following [1, p. 303, proof of Theorem 1.2] with $2\cos\bigl(\frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{2}}\cos\theta\bigr)$ replaced by $2\sin\bigl(\frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{2}}\cos\theta\bigr)$ and using

$$ie^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta} f_{k,m}^-\left(\frac{1}{\sqrt{2}}e^{i\theta}\right)$$

instead of $e^{ik\theta/2}e^{-2\pi m\frac{1}{\sqrt{2}}\sin\theta} f_{k,m}^-\bigl(\frac{1}{\sqrt{2}}e^{i\theta}\bigr)$ in Proposition 1.3 as a real-valued function in [1] together with Lemma 1.12. ∎

2. Interlacing of zeros of $f_{k,m}^-$. In this section, we obtain the same result on the interlacing property of the zeros of $f_{k,m}^-$ with zeros of another such form as shown for $f_{k,m}^+$ in [2]. The following is our second main result.

Theorem 2.1. *Let $\epsilon > 0$. Then for large enough $k > 0$ and each fixed $m \geq \ell_k + 8$ (for large enough $m \geq \ell_k + 8$ when k is fixed, respectively), the zeros of $f_{k,m}^-(z)$ interlace with the zeros of $f_{k+12,m}(z)$ (resp. $f_{k,m+1}(z)$) on* *the arc* $\mathcal{A}_\epsilon$ *and the arc* $\mathcal{A}_\epsilon^{W_2}$ *respectively under the one-to-one correspondence between zeros* α *of* $f_{k,m}^-$ *in* $\mathcal{A}_\epsilon$ *and zeros* $W_2\alpha$ *in* $\mathcal{A}_\epsilon^{W_2}$*, where*

$$\mathcal{A}_\epsilon = \{e^{i\theta}/\sqrt{2} : \pi/2 < \theta < 3\pi/4 - \epsilon\} \subseteq S,$$
$$\mathcal{A}_\epsilon^{W_2} = \{e^{i\theta}/\sqrt{2} : \pi/4 - \epsilon < \theta < \pi/2\} \subseteq W_2(S).$$

We sketch the proof of Theorem 2.1 which uses the same argument as in [2] to show that the zeros of the sine functions (instead of the cosine functions in [2]) of the following functions interlace and then we show the interlacing of zeros of $f_{k,m}^-$ with those of $f_{k+8,m}^-$ and those of $f_{k,m+1}^-$. We note that it is enough to prove the interlacing on $\mathcal{A}_\epsilon$ by transforming $\mathcal{A}_\epsilon$ by the action of W_2 into $\mathcal{A}_\epsilon^{W_2}$.

As in [2], we define, for $\theta \in I = (\pi/2, 3\pi/4)$,

$$\begin{aligned} b(\theta) &= \frac{k\theta}{2} - 2\pi m \frac{1}{\sqrt{2}} \cos\theta = \frac{k\theta}{2} - \sqrt{2}\,\pi m \cos\theta, \\ b_{k+8}(\theta) &= \frac{(k+8)\theta}{2} - \sqrt{2}\,\pi m \cos\theta, \\ b_{m+1}(\theta) &= \frac{k\theta}{2} - \sqrt{2}\,\pi(m+1)\cos\theta. \end{aligned}$$

From now on, $b_*(\theta)$ denotes $b_{k+8}(\theta)$ or $b_{m+1}(\theta)$ unless otherwise specified.

Lemma 2.2. *If* $m \geq 2|\ell_k| - \ell_k + 8$*, then*

(a) *the first zero in* I *(of* $\sin(b(\theta))$ *or* $\sin(b_*(\theta))$*) is a zero of* $\sin(b_*(\theta))$*,*
(b) *the last zero in* I *is a zero of* $\sin(b_*(\theta))$*,*
(c) *the zeros of* $\sin(b_*(\theta))$ *and* $\sin(b(\theta))$ *in* I *are never equal, and*
(d) *between two consecutive zeros of* $\sin(b_*(\theta))$ *there is exactly one zero of* $\sin(b(\theta))$.

That is, the zeros of $\sin(b(\theta))$ *interlace on* I *with the zeros of* $\sin(b_{k+8}(\theta))$ *and with the zeros of* $\sin(b_{m+1}(\theta))$ *respectively.*

Proof. We can proceed as in [2, proof of Lemma 3.1] by replacing cosine functions by sine functions. ■

Throughout this section, we suppose that $k > 0$ to prove Theorem 2.1.

As in [14, Sec. 5] and [2], we estimate the zeros near $\theta = \pi/2$ and near $\theta = 3\pi/4$. The linear approximations by the Taylor series for b and b_* with error term $R_m(\theta)$ are given by

$$\begin{aligned} L_{k,m}(\theta) &= \frac{k\pi}{4} + \frac{k + 2\sqrt{2}\,m\pi}{2}\Big(\theta - \frac{\pi}{2}\Big), \\ L_{k+8,m}(\theta) &= \frac{(k+8)\pi}{4} + \frac{k + 8 + 2\sqrt{2}\,m\pi}{2}\Big(\theta - \frac{\pi}{2}\Big), \end{aligned}$$

$$L_{k,m+1}(\theta) = \frac{k\pi}{4} + \frac{k + 2\sqrt{2}(m+1)\pi}{2}\left(\theta - \frac{\pi}{2}\right).$$

Let α_1, α_2 and α_3 be the first zeros of $\sin(L_{k+8,m}(\theta))$, $\sin(L_{k,m}(\theta))$ and $\sin(L_{k,m+1}(\theta))$ in I respectively, and let β_1, β_2 and β_3 be the first zeros of $\sin(b_{k+8}(\theta))$, $\sin(b(\theta))$ and $\sin(b_{m+1}(\theta))$ in I respectively.

We note that by Lemma 2.2(a),

$$\beta_2 > \beta_1, \qquad \beta_2 > \beta_3.$$

Also we can get $\alpha_1, \alpha_2, \alpha_3$ explicitly from Lemma 2.3 below with

$$c = \begin{cases} \pi & \text{if } k \equiv 0 \bmod 4, \\ \pi/2 & \text{if } k \equiv 2 \bmod 4; \end{cases}$$

the proof of Lemma 2.3 is the same as that of [2, Lemma 4.1] with c defined above.

Lemma 2.3. *For $i = 1, 2, 3$,*

$$\alpha_i = \frac{\pi}{2} + \frac{2c}{g_i(k,m)}, \quad \textit{where} \quad g_i(k,m) = \begin{cases} k + 8 + 2\sqrt{2}\, m\pi & \textit{if } i = 1, \\ k + 2\sqrt{2}\, m\pi & \textit{if } i = 2, \\ k + 2\sqrt{2}\,(m+1)\pi & \textit{if } i = 3. \end{cases}$$

Hence $\alpha_2 > \alpha_1$.

Next we obtain the following by adapting the arguments for cosine functions in [2, proof of Lemma 4.2(a)] to sine functions:

Lemma 2.4. *For some integers n_1 and n_2, we have*

$$\begin{aligned} b_{k+8}(\beta_1) &= n_1\pi = L_{k+8,m}(\alpha_1), & b(\beta_2) &= n_2\pi = L_{k,m}(\alpha_2), \\ b_{k+8}(\alpha_1) &= n_1\pi - R_m(\alpha_1), & b(\alpha_2) &= n_2\pi - R_m(\alpha_2), \\ & & b_{m+1}(\beta_3) &= L_{k,m+1}(\alpha_3). \end{aligned}$$

In fact, $n_2 = n_1 - 2$.

Next, we find lower bounds near $\theta = 3\pi/4$ concretely.

The linear approximations by the Taylor series for b and b_* near $\theta = 3\pi/4$ are

$$U_{k,m}(\theta) = \frac{(3k + 8m)\pi}{8} + \frac{k + 2m\pi}{2}\left(\theta - \frac{3\pi}{4}\right),$$

$$U_{k+8,m}(\theta) = \frac{(3(k+8) + 8m)\pi}{8} + \frac{k + 8 + 2m\pi}{2}\left(\theta - \frac{3\pi}{4}\right),$$

$$U_{k,m+1}(\theta) = \frac{(3k + 8(m+1))\pi}{8} + \frac{k + 2(m+1)\pi}{2}\left(\theta - \frac{3\pi}{4}\right).$$

Now let γ_1, γ_2 and γ_3 be the last zeros of $\sin(U_{k+8,m}(\theta))$, $\sin(U_{k,m}(\theta))$ and $\sin(U_{k,m+1}(\theta))$ in I respectively, and let μ_1, μ_2 and μ_3 be the last zeros of $\sin(b_{k+8}(\theta))$, $\sin(b(\theta))$ and $\sin(b_{m+1}(\theta))$ in I respectively.

We note that by Lemma 2.2(b),

$$\mu_2 < \mu_1, \qquad \mu_2 < \mu_3.$$

Also we can get $\gamma_1, \gamma_2, \gamma_3$ explicitly from Lemma 2.5 below with

$$a = \begin{cases} \pi & \text{if } k \equiv 0 \bmod 8, \\ 3\pi/4 & \text{if } k \equiv 4 \bmod 8, \\ \pi/2 & \text{if } k \equiv 2 \bmod 8, \\ \pi/4 & \text{if } k \equiv 6 \bmod 8; \end{cases}$$

the proof of Lemma 2.5 is a modification of [2, proof of Lemma 4.3] with a defined above.

LEMMA 2.5. *For $i = 1, 2, 3$,*

$$\gamma_i = \frac{3\pi}{4} - \frac{2a}{g_i(k,m)}, \quad \textit{where} \quad g_i(k,m) = \begin{cases} k + 8 + 2m\pi & \textit{if } i = 1, \\ k + 2m\pi & \textit{if } i = 2, \\ k + 2(m+1)\pi & \textit{if } i = 3. \end{cases}$$

Hence $\gamma_1 > \gamma_2$.

Then, we can prove the following by adapting [2, proof of Lemma 4.4(a)] to sine functions.

LEMMA 2.6. *For some integers n_1 and n_2, we have*

$$b_{k+8}(\mu_1) = n_1\pi = U_{k+8,m}(\gamma_1), \qquad b(\mu_2) = n_2\pi = U_{k,m}(\gamma_2),$$
$$b_{k+8}(\gamma_1) = n_1\pi - R_m(\gamma_1), \qquad b(\gamma_2) = n_2\pi - R_m(\gamma_2).$$

In fact, $n_2 = n_1 - 3$.

Finally, Theorem 2.1 follows from the same argument as in [2, Section 4.3] by replacing cosine functions by sine functions and by using the bound $D < 1.9457196$ given in Lemma 1.12 and its proof in the previous section.

3. Future work. As shown in this paper as well as in [1] and [10], there are infinitely many weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(2)$ and $\Gamma_0^+(2)$ all of whose zeros lie on certain geodesics in the upper half-plane. In future work, we hope to find infinitely many modular forms for other arithmetic groups all of whose zeros lie on some geodesics in the upper half-plane.

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SoYoung Choi
Department of Mathematics Education
and RINS
Gyeongsang National University
501 Jinjudae-ro
Jinju, 52828, South Korea
E-mail: mathsoyoung@gnu.ac.kr

Bo-Hae Im
Department of Mathematical Sciences
KAIST
291 Daehak-ro, Yuseong-gu
Daejeon, 34141, South Korea
E-mail: bhim@kaist.ac.kr