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Ioannis K. Argyros (Lawton, OK)
Santhosh George (Karnataka)

BALL CONVERGENCE FOR A SIXTH-ORDER MULTI-POINT METHOD IN BANACH SPACES UNDER WEAK CONDITIONS

Abstract. The aim of this paper is to extend the applicability of some high order iterative methods without using hypotheses on derivatives not appearing in those methods. Numerical examples are given where earlier convergence conditions are not satisfied but the new ones are satisfied.

1. Introduction. Consider the problem of approximating a locally unique solution x^* of a nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subseteq X \to Y$ is a Fréchet differentiable operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y. In earlier studies such as [2, 6, 7, 11, 12], higher order methods are considered for approximating the solution x^* of (1.1). But, for the convergence analysis of these methods, in addition to the assumptions on F' and F'', assumptions of the form (see [2, 6, 7, 11, 12])

$$\|F'''(x) - F'''(y)\| \le L\|x - y\|, \quad x, y \in \Omega,\ L \ge 0, \tag{1.2}$$

or

$$\|F'''(x) - F'''(y)\| \le w(\|x - y\|), \quad x, y \in \Omega, \tag{1.3}$$

are required where $w(z)$ is a nondecreasing continuous function for $z > 0$ and $w(0) = 0$ (see [11]).

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A motivational example of (1.1) that does not satisfy (1.2) or (1.3) is

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{1.4}$$

where $F : \Omega = [-5/2, 1/2] \to \mathbb{R}$. We have

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Obviously, F''' is unbounded on Ω. Hence, results requiring (1.3) or (1.4) cannot be used to solve the equation $F(x) = 0$, since there is no guarantee that the corresponding methods converge to x^* [1–13].

Since the computational cost of inversion is very large in general, many authors considered iterative methods with less computation of inversion [1–13].

In this paper we study the local convergence of the multi-step method defined for each $n = 0, 1, \dots$ [11, 13] by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ u_n &= y_n + \tfrac{2}{3} F'(x_n)^{-1} F(x_n), \\ z_n &= y_n - A_n F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= z_n - B_n^{-1} F'(x_n)^{-1} F(z_n), \end{aligned} \tag{1.5}$$

where x_0 is an initial point, $K_n = F'(x_n)^{-1} F''(u_n) F'(x_n)^{-1} F(x_n), A_n = \frac{1}{2} K_n (I - K_n)^{-1}$ and $B_n = F'(x_n)^{-1} F''(u_n)(I + A_n) F'(x_n)^{-1} F(x_n)$.

The derivation, motivation, usefulness and cost of method (1.5) were analyzed in detail in [11] (see also [13]), so we do not repeat these items in this study. Moreover, the almost sixth semilocal convergence order of method (1.5) was shown in [11, 13] using the preceding Lipschitz-type conditions. Some advantages of using this method over others using similar information were also reported in [11, 13]. However, as already mentioned, these results or other results using (1.2) or (1.3) cannot apply to solve (1.4).

The aim of this paper is to address this problem. That is, we extend the applicability of method (1.5) and show convergence using only hypotheses up to the second Fréchet derivatives. Notice also that only the first and second Fréchet derivatives appear in (1.5). In the main local convergence result (Theorem 2.1), we show linear convergence using the weak Lipschitz conditions. However, we can still obtain the order of convergence by avoiding Taylor series expansions or recurrence relations (which bring in hypotheses on (higher than second order) derivatives) [11, 13]. We use instead the computational order of convergence or the approximate computational order of convergence (see Remark 2.2(4)). Ball convergence results are important since they show the degree of difficulty in choosing initial points. Our technique can be applied to other iterative methods using (1.2) or (1.3) [1–9, 12].

The paper is structured as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Applications are given in Section 3.

2. Local convergence. The local convergence analysis is based on some scalar functions and parameters. Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function satisfying $w_0(0) = 0$. Define

$$r_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let $w, v, v_1 : [0, r_0) \to [0, +\infty)$ be continuous and nondecreasing functions satisfying $w(0) = 0$.

Moreover, define scalar functions g_1, g_2, h_1, h_2, p, h_p and p_1 on $[0, r_0)$ by

$$\begin{aligned}
g_1(t) &= \frac{\int_0^1 w((1-\theta)t)\, d\theta}{1 - w_0(t)},\\
h_1(t) &= g_1(t) - 1,\\
g_2(t) &= \frac{\int_0^1 w((1-\theta)t) d\theta + \frac{2}{3}\int_0^1 v(\theta t)\, d\theta}{1 - w_0(t)},\\
h_2(t) &= g_2(t) - 1,\\
p(t) &= \frac{\int_0^1 v(\theta t)\, d\theta\, v_1(g_2(t)t)t}{(1 - w_0(t))^2},\\
h_p(t) &= p(t) - 1,\\
p_1(t) &= \frac{1}{2}\,\frac{p(t)}{1 - p(t)}.
\end{aligned}$$

We have $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_0^-$. It follows from the intermediate value theorem that h_1 has zeros in $(0, r_0)$. Denote by r_1 the smallest such zero. Suppose that

$$v(0) < 3/2. \tag{2.2}$$

Then $h_2(0) = \frac{2}{3}v(0) - 1 < 0$ and $h_2(r_1) = \frac{\frac{2}{3}\int_0^1 v(\theta r_1)\, d\theta}{1 - w_0(r_1)} > 0$. Denote by r_2 the smallest zero of h_2 in $(0, r_1)$. Further, $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to r_0^-$. Denote by r_p the smallest zero of h_p in $(0, r_0)$. Furthermore, define functions g_3, h_3, p_2 and h_{p_2} on $[0, r_p)$ by

$$
\begin{aligned}
g_3(t) &= g_1(t) + \frac{p(t)\int_0^1 v(\theta t)d\theta}{(1-p(t))(1-w_0(t))},\\
h_3(t) &= g_3(t) - 1,\\
p_2(t) &= \frac{v_1(g_2(t)t)(1+p_1(t))\int_0^1 v(\theta t)\,d\theta\, t}{1-w_0(t)},\\
h_{p_2}(t) &= p_2(t) - 1.
\end{aligned}
$$

We have $h_3(0) = h_{p_2}(0) = -1 < 0$, $h_2(t) \to +\infty$ as $t \to r_p^-$ and $h_{p_2}(t) \to +\infty$ as $t \to r_p^-$. Denote by r_3 and r_{p_2} the smallest zeros of h_3 and h_{p_2} respectively in $(0, r_p)$. Finally, define functions g_4 and h_4 on $[0, r_{p_2})$ by

$$
g_4(t) = \left(1 + \frac{\int_0^1 v(\theta g_3(t)t)\,d\theta}{(1-p_2(t))(1-w_0(t))}\right) g_3(t), \qquad h_4(t) = g_4(t) - 1.
$$

We obtain $h_4(0) = -1 < 0$ and $h_4(t) \to +\infty$ as $t \to r_p^-$. Denote by r_4 the smallest zero of h_4 in $(0, r_{p_2})$.

Define

$$
r = \min\{r_i\}, \qquad i = 2, 3, 4. \tag{2.3}
$$

Then for each $t \in [0, r)$,

$$
0 \le g_i(t) < 1, \qquad i = 1, 2, 3, 4, \tag{2.4}
$$
$$
0 \le p(t) < 1, \tag{2.5}
$$
$$
0 \le p_1(t), \tag{2.6}
$$
$$
0 \le p_2(t) < 1. \tag{2.7}
$$

The reason why these scalar functions are defined this way is revealed in the proof of Theorem 2.1 below.

Let $U(y, \rho), \bar{U}(y, \rho)$ denote respectively the open and closed balls in X with center $y \in X$ and radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.5) using the preceding notation.

THEOREM 2.1. *Let $F : \Omega \subseteq X \to Y$ be a twice continuously Fréchet differentiable operator. Suppose there exist $x^* \in \Omega$ and a continuous nondecreasing function $w_0 : [0, +\infty) \to [0, +\infty)$ with $w_0(0) = 0$ such that for each $x \in \Omega$,*

$$
F(x^*) = 0, \qquad F'(x^*)^{-1} \in L(Y, X), \tag{2.8}
$$

and

$$
\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|). \tag{2.9}
$$

Moreover, suppose there exist continuous nondecreasing functions w, v, v_1 :

$[0, r_0) \to [0, +\infty)$ *with* $w(0) = 0$ *such that for each* $x, y \in \Omega_0 := \Omega \cap U(x^*, r_0)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|), \tag{2.10}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|), \tag{2.11}$$

$$\|F'(x^*)^{-1}F''(x)\| \leq v_1(\|x - x^*\|), \tag{2.12}$$

$$\bar{U}(x^*, r) \subseteq \Omega \tag{2.13}$$

and (2.2) *holds, where* r_0, r *are defined by* (2.1) *and* (2.3)*, respectively. Then the sequence* $\{x_n\}$ *generated for* $x_0 \in U(x^*, r) - \{x^*\}$ *by method* (1.5) *is well defined, remains in* $U(x^*, r)$ *for each* $n = 0, 1, 2, \dots$ *and converges to* x^*. *Moreover,*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.14}$$

$$\|u_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

$$\|z_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.16}$$

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.17}$$

where the functions g_i, $i = 1, 2, 3, 4$, *are as defined previously. Furthermore, if for* $R \in [r, r_0)$,

$$\int_0^1 w_0(\theta R)\, d\theta < 1, \tag{2.18}$$

then the limit point x^* *is the only solution of the equation* $F(x) = 0$ *in the set* $\Omega_1 = \Omega \cap U(x^*, r)$.

Proof. We shall show by induction that the sequence $\{x_n\}$ is well defined in $U(x^*, r)$ and converges to x^* so that estimates (2.14)–(2.17) are satisfied. By the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1), (2.3) and (2.9) we get

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) \leq w_0(r) \leq w_0(r_0) < 1. \tag{2.19}$$

It follows from (2.19) and the Banach lemma on invertible operators [9, 10] that $F'(x_0)^{-1} \in L(Y, X)$, y_0, u_0 are well defined by the first and second substep of method (1.5) for $n = 0$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.20}$$

We can write

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.21}$$

Using (2.3), (2.4) (for $i = 1$), (2.8), (2.10), (2.20) and (2.21), we get

$$
\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1}F(x^*)\| \\
&\quad \times \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)\, d\theta \right\| \\
&\le \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)\, d\theta\, \|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned} \tag{2.22}
$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*, r)$. By (2.8) we can write

$$
F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.23}
$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. Then, by (2.11) and (2.23), we have

$$
\|F'(x^*)^{-1}F(x_0)\| \le \int_0^1 v(\theta\|x_0 - x^*\|)\|x_0 - x^*\|\, d\theta. \tag{2.24}
$$

Using the second substep of method (1.5) for $n = 0$, (2.3), (2.4) (for $i = 2$), (2.20), (2.22) and (2.24), we get in turn

$$
\begin{aligned}
\|u_0 - x^*\| &\le \|y_0 - x^*\| + \tfrac{2}{3}\|F'(x_0)^{-1}F'(x^*)\|\, \|F'(x^*)^{-1}F(x_0)\| \\
&\le g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{2}{3}\, \frac{\int_0^1 v(\theta\|x_0 - x^*\|)\, d\theta\, \|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned} \tag{2.25}
$$

so (2.15) holds for $n = 0$ and $u_0 \in U(x^*, r)$. Next, we show that $(I - K_0)^{-1} \in L(Y, X)$. In view of (2.5), (2.20), (2.12), (2.24) and (2.25) we obtain

$$
\begin{aligned}
\|K_0\| &\le \|F'(x_0)^{-1}F'(x^*)\|\, \|F'(x^*)^{-1}F''(u_0)\| \\
&\quad \times \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\le \frac{\int_0^1 v(\theta\|x_0 - x^*\|)\, d\theta\, v_1(\|u_0 - x^*\|)\|x_0 - x^*\|}{(1 - w_0(\|x_0 - x^*\|))^2} \\
&\le p(\|x_0 - x^*\|) \le p(r_0) < 1,
\end{aligned} \tag{2.26}
$$

so

$$
\|(I - K_0)^{-1}\| \le \frac{1}{1 - p(\|x_0 - x^*\|)}, \tag{2.27}
$$

z_0 is well defined and

$$
\begin{aligned}
\|A_0\| &\le \tfrac{1}{2}\|K_0\|\, \|(I - K_0)^{-1}\| \\
&\le \frac{1}{2}\, \frac{p(\|x_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)} = p_1(\|x_0 - x^*\|).
\end{aligned} \tag{2.28}
$$

Using the third substep of method (1.5) for $n = 0$, (2.3), (2.4) (for $i = 3$), (2.20), (2.22), (2.24) and (2.29), we obtain

$$
\begin{aligned}
(2.29)\quad \|z_0 - x^*\| &\le \|y_0 - x^*\| + \|A_0\|\,\|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F(x_0)\| \\
&\le g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\quad + \frac{1}{2}\,\frac{p(\|x_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)\,d\theta\,\|x_0 - x^*\|}{(1 - p(\|x_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))} \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}
$$

which shows (2.16) for $n = 0$ and $z_0 \in U(x^*, r)$.

We need an estimate on B_0^{-1}. By the definition of B_0, (2.3), (2.7), (2.12), (2.20), (2.24) and (2.29) we have

$$
\begin{aligned}
(2.30)\quad &\|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F''(u_0)\| \\
&\qquad \times (\|I\| + \|A_0\|)\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\
&\le \frac{v_1(\|u_0 - x^*\|)(1 + p_1(\|x_0 - x^*\|))\int_0^1 v(\theta\|x_0 - x^*\|)\,d\theta\,\|x_0 - x^*\|}{(1 - w_0(\|x_0 - x^*\|))^2} \\
&= p_2(\|x_0 - x^*\|) \le p_2(r_0) < 1,
\end{aligned}
$$

so $B_0^{-1} \in L(Y, X)$, x_1 is well defined and

$$
(2.31)\qquad \|B_0^{-1}\| \le \frac{1}{1 - p_2(\|x_0 - x^*\|)}.
$$

Then, using (2.3), (2.4) (for $i = 4$), (2.20), (2.24) (for $x_0 = z_0$), (2.29) and (2.31) we get

$$
\begin{aligned}
(2.32)\quad \|x_1 - x^*\| &\le \|z_0 - x^*\| + \|B_0^{-1}\|\,\|F'(x_0)^{-1}F(x^*)\|\,\|F'(x^*)^{-1}F(z_0)\| \\
&\le \|z_0 - x^*\| + \frac{\int_0^1 v(\|z_0 - x^*\|)\,d\theta\,\|z_0 - x^*\|}{(1 - p_2(\|x_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))} \\
&\le \left(1 + \frac{\int_0^1 v(\|z_0 - x^*\|)\,d\theta}{(1 - p_2(\|x_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))}\right)\|z_0 - x^*\| \\
&\le g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}
$$

which shows (2.17) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, u_0, z_0, x_1 by $x_k, y_k, u_k, z_k, x_{k+1}$ in the preceding estimates, we arrive at (2.14)–(2.17). Then, from the estimate

$$
(2.33)\qquad \|x_{k+1} - x^*\| \le c\|x_k - x^*\| < r, \qquad c = g_4(\|x_0 - x^*\|) \in [0, 1),
$$

we deduce that $\lim x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in \Omega_1$ with $F(y^*) = 0$. Define a linear operator T

by $T = \int_0^1 F'(x^* + \theta(y^* - x^*))\, d\theta$. It follows from (2.9) and (2.18) that

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \le \int_0^1 w_0(\theta\|x^* - y^*\|)\, d\theta \le \int_0^1 w_0(\theta R)\, d\theta < 1, \tag{2.34}$$

so $T^{-1} \in L(Y, X)$. Using the identity

$$0 = F(y^*) - F(x^*) = T(y^* - x^*),$$

we conclude that $x^* = y^*$. ■

Remark 2.2. (1) In view of (2.9) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.11) can be dropped and v can be replaced by

$$v(t) = 1 + w_0(t).$$

(2) The results obtained here can be used for operators F satisfying autonomous differential equations [6] of the form

$$F'(x) = P(F(x))$$

where $P : Y \to Y$ is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

(3) The radius r_1 was shown by us to be the convergence radius of Newton's method [4, 5]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for each } n = 0, 1, 2, \cdots \tag{2.35}$$

under the conditions (2.5)–(2.8). It follows from the definition of r that the convergence radius r of method (1.5) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.35). Let $w_0(t) = L_0 t$, $w(t) = Lt$ for some $L_0, L > 0$. As already noted in [4, 5], r_1 is at least as large as the convergence ball given by Rheinboldt [9]

$$r_R = \frac{2}{3L_1}, \tag{2.36}$$

where L_1 is the Lipschitz constant on Ω. Notice that $r_1 = \frac{2}{2L_0 + L}$ and the ball given in [4, 5] is given by $\bar{r}_1 = 2/(2L_0 + L_1)$. We have $L_0 \le L_1$ and $L \le L_1$. If $L_0 < L$ and $L < L_1$, we have

$$r_R < \bar{r}_1 < r_1$$

and

$$r_R/r_1 \to 1/3 \quad \text{as } L_0/L_1 \to 0.$$

That is, our convergence ball r_1 is at least three times larger than Rheinboldt's. The same value for r_R was given by Traub [10]. Looking at the example listed in Remark 2.2(2) above, we find that for $\Omega = U(0,1)$, $x^* = 0$, $L_0 = e - 1, L_1 = e$ and $L = e^{1/L_0}$. Therefore, we obtain

$$r_R = 0.2453 < \bar{r}_1 = 0.3249 < r_1 = 0.3827.$$

(4) It is worth noticing that method (1.5) does not change when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [2, 7, 11–13]. Moreover, we can compute the *computational order of convergence* (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the *approximate computational order of convergence*

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids estimating higher than second Fréchet derivatives of F. The computation of ξ_1 does not require knowledge of x^*.

3. Applications

Example 3.1. Let $X = Y = \mathbb{R}^3$, $\Omega = \bar{U}(0,1)$, $x^* = (0,0,1)^T$. Define a function F on Ω for $w = (x,y,z)^T$ by

$$F(w) = (\sin x, y^2/5 + y, z)^T.$$

Then the Fréchet derivatives are given by

$$F'(v) = \begin{bmatrix} \cos x & 0 & 0 \\ 0 & 2y/5+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$F''(v) = \left[\begin{array}{ccc|ccc|ccc} -\sin x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using conditions (2.6)–(2.11), we find that $w_0(t) = w(t) = t$ and $v(t) = 7/5$ and $v_1(t) = 2/5$. Notice that since $v(0) = 7/5 < 3/2$, condition (2.2) is satisfied. Then the parameters are

$$r_1 = 0.0667, \quad r_2 = 0.0222 = r, \quad r_3 = 0.0544, \quad r_4 = 0.1031.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$, equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define a function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3\, d\theta. \tag{3.1}$$

We have

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)\, d\theta \quad \text{for each } \xi \in \Omega.$$

Then for $x^* = 0$, we get $w_0(t) = 7.5t$, $w(t) = 15t$, $v(t) = 1 + 7.5t$ and $v_1(t) = 1 + 30t$. Then the parameters are

$$r_1 = 0.0667, \quad r = r_2 = 0.0190, \quad r_3 = 0.1775, \quad r_4 = 0.1197.$$

EXAMPLE 3.3. Returning to the motivational example of the introduction, we have $w_0(t) = w(t) = 96.6629073t$, $v(t) = \sup \|F'(x^*)^{-1}F'(x)\| = 0.7272$ and $v_1(t) = \sup \|F'(x^*)^{-1}F''(x)\| = 0.3411$. Then the parameters are

$$r_1 = 0.0069, \quad r = r_2 = 0.0036, \quad r_3 = 0.0068, \quad r_4 = 0.01.$$

EXAMPLE 3.4. Let us consider the isothermal continuous stirred tank reactor (CSTR) problem [5]. Components A and R are fed to the reactor at rates of Q and $q - Q$, respectively. Then we obtain the following reaction scheme in the reactor:

$$\begin{aligned} A + R &\to B, \\ B + R &\to C, \\ C + R &\to D, \\ D + R &\to E. \end{aligned}$$

The problem was analysed by Douglas [4] in order to design simple feedback control systems. He presented the following expression for the transfer function of the reactor:

$$K_C \frac{2.98(x + 2.25)}{(s + 1.45)(s + 2.85)^2(s + 4.35)} = -1,$$

where K_C is the gain of the proportional controller. The control system is stable for values of K_C that yield roots of the transfer function having negative real part. If we choose $K_C = 0$, we get the poles of the open-loop transfer function as roots of the polynomial

$$f_1(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{3.2}$$

The function f_1 has four zeros $x^* = -1.45, -2.85, -2.85, -4.35$. Let $\Omega = [-4.5, -4]$. Then $w_0(t) = 1.2547945t, w(t) = 29.610958t, v(t) = 1 + w_0(t)$ and $v_1(t) = 29.610958$. Hence, the radii are

$$r_1 = 0.0623, \quad r_2 = 0.0313, \quad r = r_3 = 0.0043, \quad r_4 = 0.0147.$$

EXAMPLE 3.5. In this example, we consider one of the famous applied science problem which is known as the Hammerstein integral equation [1, 2, 6]:

$$x(s) = T(x(s)) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3\,dt \tag{3.3}$$

where $x \in C[0,1], s, t \in [0,1]$ and the kernel G is

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

Set

$$F(x(s)) = 0 \tag{3.4}$$

where $F(x(s)) = x(s) - T(x(s))$. These equations arise in electric-magnetic fluid dynamics. Moreover, these equations appeared in the 1930s as special models for studying boundary value problems, where the kernel is Green's function [1, 2, 6]. The method converges towards the root

$$x^* = (1.002096\ldots, 1.009900\ldots, 1.019727\ldots, 1.026436\ldots, 1.026436\ldots, \\ 1.019727\ldots, 1.009900\ldots, 1.002096\ldots)^T.$$

Then for $\Omega = U(x^*, 0.11)$ we get $w_0(t) = w(t) = \frac{3}{40}t$, $v(t) = 1 + w_0(t)$ and $v_1(t) = 3/40$, and the radii are

$$r_1 = 8.8889, \quad r_2 = 2.5185, \quad r_3 = 0.3843, \quad r_4 = 0.8126,$$

so we must choose $r = 0.11$ by the choice of Ω.

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Ioannis K. Argyros
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, U.S.A.
ORCID: 0000-0003-1609-3195
E-mail: iargyros@cameron.edu

Santhosh George
Department of Mathematical
and Computational Sciences
National Institute of Technology Karnataka
Karnataka 757 025, India
ORCID: 0000-0002-3530-5539
E-mail: sgeorge@nitk.edu.in