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TWO-POINT METHODS FOR SOLVING EQUATIONS AND SYSTEMS OF EQUATIONS

Abstract. The aim of this study is to present a convergence analysis of a frozen secant-type method for solving nonlinear systems of equations defined on the k-dimensional Euclidean space. The novelty of the paper lies in the fact that the method is defined using a special divided difference which is well defined for distinct iterates making it suitable for solving systems involving a nondifferentiable mapping. The local and semi-local convergence analysis is based on generalized Lipschitz-type scalar functions that are only nondecreasing, whereas their continuity is not assumed as in earlier studies. Numerical examples involving systems of equations are provided to further validate the theoretical results.

1. Introduction. Let k be a positive integer and let $D \subseteq \mathbb{R}^k$ be an open set. Consider the system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where $F : D \to \mathbb{R}^k$ is a continuous mapping. Numerous problems in computational disciplines can be formulated as a system of nonlinear equations like (1.1) using mathematical modeling [1–18]. One desires the solution x_* of (1.1) to be determined explicitly. However, this can be achieved only in special situations. Hence, researchers resort to iterative methods for generating a sequence approximating x_*.

The study of convergence of iterative algorithms is usually centered around two tasks: semi-local and local convergence analysis. The semi-local conver-

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gence analysis is based on information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence analysis is based on information around a solution in order to find estimates of the computed radii of the convergence balls. Local results are important since they reveal the degree of difficulty in choosing initial points.

The convergence analysis of iterative methods that do not use derivatives requires the mapping associated with the divided difference to satisfy certain generalized Lipschitz-type conditions implying that the mapping F is differentiable [1, 2, 4, 6–12, 16]. Indeed for the 1-parameter family of secant-type methods [4, 9] defined for $x_{-1}, x_0 \in D$ by

$$\begin{aligned} y_n &= \lambda x_n + (1-\lambda)x_{n-1}, \qquad \lambda \in [0,1],\ n = 0,1,2,\ldots, \\ x_{n+1} &= x_n - [y_n, x_n; F]^{-1}F(x_n) \end{aligned}$$

and its special cases, Newton's method for $\lambda = 1$ and the secant method for $\lambda = 0$, we use the standard divided difference $[a,b;F] := ([a,b;F]_{ij})_{i,j=1}^{k} \in L(\mathbb{R}^k, \mathbb{R}^k)$ [1, 6, 10, 13] defined for $1 \leq i,j \leq k$ by

$$[a,b;F]_{i.j} = \frac{F_i(a_1,\ldots,a_j,b_{j+1},\ldots,b_k) - F_i(a_1,\ldots,a_{j-1},b_j,b_{j+1},\ldots,b_k)}{a_j - b_j},$$

where $a = (a_1,\ldots,a_k)^T$ and $b = (b_1,\ldots,b_k)^T$. This formula defines a bounded linear operator which satisfies $[b,a;F](b-a) = F(b) - F(a)$ for each $a \neq b$, $a,b \in D$.

If, for example, $k = 3$ (see the numerical examples) and

$$a = (a_1,a_2,a_3)^T, b = (b_1,b_2,b_3)^T \in \mathbb{R}^3$$

then

$$\begin{aligned} [a,b;F]_{i1} &= \frac{F_i(a_1,b_2,b_3) - F_i(b_1,b_2,b_3)}{a_1 - b_1}, \\ [a,b;F]_{i2} &= \frac{F_i(a_1,a_2,b_3) - F_i(a_1,b_2,b_3)}{a_2 - b_2}, \\ [a,b;F]_{i3} &= \frac{F_i(a_1,a_2,a_3) - F_i(a_1,a_2,b_3)}{a_3 - b_3}, \qquad i = 1,2,3. \end{aligned}$$

Clearly, by using the preceding definition of the divided difference, method (1.2) is suitable when the mapping F is not differentiable. Therefore, it is important to study the local as well as semi-local convergence of method (1.2) under generalized conditions.

The following conditions have been used for semi-local convergence:

- *Lipschitz conditions* [1–3, 8–11, 16, 17]:

  $$\|[x_{-1},x_0;F]^{-1}([x,y;F] - [u,v;F])\| \leq \mu(\|x-u\| + \|y-v\|)$$

  for some $\mu > 0$ and all $x,y,u,v \in D$;

- *Hölder conditions* [17]:

$$\|[x_{-1}, x_0; F]^{-1}([x, y; F] - [u, v; F])\| \leq \mu(\|x - u\|^{\lambda} + \|y - v\|^{\lambda})$$

  for some $\lambda \in [0, 1]$;
- *generalized conditions* [1, 3, 5, 9]:

$$\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [u, v; F])\| \leq \bar{\varphi}(\|x - y\|, \|y - v\|),$$

where $\bar{\varphi} : [0, +\infty)^2 \to [0, +\infty)$ is a continuous and nondecreasing function in both variables and $\tilde{z}$ is x_{-1} or some other point in D. The Lipschitz, Hölder and generalized conditions (for $\bar{\varphi}(0, 0) = 0$ [4, 7–11, 17]) imply that the mapping F is differentiable. This limits the applicability of these methods. As a motivational example consider the equation $F(x) = |x|$ for each $x \in (-\infty, +\infty)$. Clearly, F is not differentiable at $x = 0$. Therefore, the preceding methods cannot be used to solve the equation $F(x) = 0$ although for $x_{-1} = 0$ and $x_0 \in (-\infty, +\infty) - \{0\}$ the secant method gives $x_1 = x_* = 0$. It is worth noticing that the Fréchet derivative F' does not appear in these methods but it appears in the conditions for the convergence of these methods. That is why it is important to remove the requirement of the existence of F' from the convergence conditions. In [4], we assumed that the preceding generalized condition holds for $x \neq y$ and $u \neq v$ and $\bar{\varphi}(0, 0) > 0$. In the present study we drop the continuity of $\bar{\varphi}$.

Moreover, we use the generalized center condition (see Section 2)

$$\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [\tilde{z}, x_0; F])\| \leq \varphi_0(\|x - \tilde{z}\|, \|y - x_0\|),$$

where $\varphi_0 : (0, +\infty)^2 \to (0, +\infty)$ is a nondecreasing function in each variable and $x \neq y$, $\tilde{z} \neq x_0$. Notice that we do not require φ_0 to be a continuous function. Using this condition, we find a subset D_0 of D (which may be strict) containing the iterates (see, e.g., Theorem 2.2). That motivates us to introduce the condition

$$\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [u, v; F])\| \leq \varphi(\|x - \tilde{z}\|, \|y - x_0\|),$$

where $\varphi : (0, +\infty)^2 \to (0, +\infty)$ is nondecreasing in each variable and $x, y \in D_0$ with $x \neq y$, $\tilde{z} \neq x_0$. Then we have

$$\varphi_0(s, t) \leq \bar{\varphi}(s, t), \qquad \varphi(s, t) \leq \bar{\varphi}(s, t),$$

since $D_0 \subseteq D$ and $\bar{\varphi}/\varphi_0$ can be arbitrarily large [2–6, 16]. The $\bar{\varphi}$ condition implies the φ_0 and φ conditions. In practice, the computation of $\bar{\varphi}$ requires the computation of φ_0 and φ as special cases. Therefore, the latter are not additional conditions.

However, using tighter sequences, φ_0 and φ we obtain: tighter bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x_*\|$ and at least as precise information on the location of the solution (see also the Remarks that follow in Sections 2 and 3).

Similar conditions and advantages are obtained in the local convergence case (see Section 3).

Next, we shall introduce an iterative method suitable for generating a sequence approximating x_*, in cases when the mapping F is not necessarily differentiable, since many problems fall in that category (see also the numerical examples).

Description of the method. Define $x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})$, where $x_n^{(i)} \in \mathbb{R}$ for each $n = 0, 1, \ldots$ and $i = 1, \ldots, k$. Choose $x_0, x_{-1} \in D$ such that $x_{-1}^{(i)} \neq x_0^{(i)}$ for each $i = 1, \ldots, k$. Then $x_{-1} \neq x_0$. Define the frozen secant-type method $\{x_n\}$ for each $n = -1, 0, 1, \ldots$ by

$$x_{n+1} = x_n - [u_{n-1}, u_n; F]^{-1} F(x_n), \tag{1.2}$$

where $[u_{n-1}, u_n; F]$ denotes the last divided difference of order one such that $u_{n-1}^{(i)} \neq u_n^{(i)}$ for each $i = 1, \ldots, k$. Indeed, we have

$$x_1 = x_0 - [x_{-1}, x_0; F]^{-1} F(x_0)$$

by the definition of x_{-1} and x_0, so $u_{-1} = x_{-1}$ and $u_0 = x_0$ in this case. Then

$$x_2 = x_1 - [x_{-1}, x_0; F]^{-1} F(x_1),$$

if $x_0^{(i)} = x_1^{(i)}$ for some $i = 1, \ldots, k$ and $u_0 = x_{-1}, u_1 = x_0$ or

$$x_2 = x_1 - [x_0, x_1; F]^{-1} F(x_1),$$

if $x_0^{(i)} \neq x_1^{(i)}$ for some $i = 1, \ldots, k$ and $u_0 = x_0, u_1 = x_1$. Clearly, method (1.2) contains the modified secant method

$$y_{n+1} = y_n - [y_{n-1}, y_n; F]^{-1} F(y_n), \tag{1.3}$$

and may contain the secant method if $u_{n-1}^{(i)} \neq u_n^{(i)}$ for each $n = -1, 0, 1, \ldots$, $i = 1, \ldots, k$.

Therefore, depending on which subcase is used the convergence order is one (modified secant method) or $k+1$ (secant method), if the third derivative of F exists in a neighborhood of the solution [1]. One can also compute the order of convergence using (COC) or (ACOC) [5, 11] which avoid the use of derivatives. It is worth noticing that method (1.2) is very attractive for nondifferentiable mappings, since by construction the divided difference is well defined at distinct points.

The rest of the study is structured as follows. The semi-local and local convergence analysis for method (1.2) are presented in Sections 2 and 3, respectively, whereas the numerical examples are provided in Section 4.

2. Semi-local convergence analysis. We present the semi-local convergence analysis of method (1.2) based on the following *conditions* $(\mathcal{A})$:

($\mathcal{A}_1$) There exist $x_0 \in D$, $\gamma > 0$ and $\tilde{z} \in D$, with $\|x_0 - \tilde{z}\| = \gamma$, such that $[\tilde{z}, x_0; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$.

($\mathcal{A}_2$) $\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [\tilde{z}, x_0; F])\| \le \varphi_0(\|x - \tilde{z}\|, \|y - y_0\|)$ for all distinct $x, y \in D$, where $\varphi_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is nondecreasing in each variable. Suppose that there exists $r \in (0, +\infty)$ such that $\varphi_0(\gamma + r, r) < 1$. Let $r_0 = \sup\{r \in (0, +\infty) : \varphi_0(\gamma + r, r) < 1\}$. Set $D_0 = D \cap U(x_0, r_0)$.

($\mathcal{A}'_2$) $\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [u, v; F])\| \le \varphi(\|x - u\|, \|y - v\|)$ for all x, y, u, v in D_0 with $x \ne y$ and $u \ne v$, where $\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is nondecreasing in each variable.

We need an auxiliary result on inverses of divided differences.

LEMMA 2.1. *Under conditions* ($\mathcal{A}_1$) *and* ($\mathcal{A}'_2$), *if there exists* $r \in [0, +\infty)$ *with* $U(x_0, r) \subseteq D$ *and* $\varphi_0(\gamma + r, r) < 1$, *then* $[x, y; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ *and*

$$\begin{aligned}(2.1)\qquad &\|[x, y; F]^{-1}[\tilde{z}, x_0; F]\| \\ &\qquad \le \frac{1}{1 - \varphi_0(\|\tilde{z} - x\|, \|x_0 - y\|)} \le \frac{1}{1 - \varphi_0(\gamma + r, r)}\end{aligned}$$

for each pair $(x, y) \in U(x_0, r) \times U(x_0, r)$ *with* $x \ne y$.

Proof. We obtain, by ($\mathcal{A}'_2$),

$$\begin{aligned}\|[\tilde{z}, x_0; F]^{-1}([\tilde{z}, x_0; F] - [x, y; F])\| &\le \varphi_0(\|\tilde{z} - x\|, \|x_0 - y\|) \\ &\le \varphi_0(\|\tilde{z} - x_0\| + \|x_0 - x\|, \|x_0 - y\|) \le \varphi_0(\gamma + r, r) < 1.\end{aligned}$$

Then, by the Banach Lemma on invertible operators [3, 14], the operator $[x, y; F]^{-1}$ is in $L(\mathbb{R}^k, \mathbb{R}^k)$ so that (2.1) is satisfied. ■

Notice that if $x_{-1} \in D$ with $\|x_{-1} - x_0\| = \delta > 0$, then $x_{-1} \in D$, $x_{-1} \ne x_0$ and therefore $[x_{-1}, x; F] \in L(\mathbb{R}^k, \mathbb{R}^k)$. So, by Lemma 2.1, it follows that $[x_{-1}, x_0; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Suppose $\|[x_{-1}, x_0; F]^{-1} F(x_0)\| \le \eta$. If in addition $x_{-1} \in U(x_0, r)$, then $x_{-1} \in U(x_0, r)$.

We shall also use the following conditions:

($\mathcal{A}_3$) The equation

$$(2.2)\qquad \left(1 + \frac{g_0(t)}{1 - g(t)}\right)\eta - t = 0,$$

where $g_0(t) = \frac{\varphi(\eta + \delta, 0)}{1 - \varphi_0(\gamma + t, t)}$, $g(t) = \frac{\varphi(2t, 2t)}{1 - \varphi_0(\gamma + t, t)}$, has at least one positive zero. Denote by r the smallest such zero.

($\mathcal{A}_4$) $U(x_0, r) \subseteq D$ and $g_0(r) + g(r) < 1$.

Conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}'_2$), ($\mathcal{A}_3$) and ($\mathcal{A}_4$) constitute *conditions* ($\mathcal{A}$).

THEOREM 2.2. *Suppose that conditions* ($\mathcal{A}$) *hold and* $x_{-1}^{(i)} \ne x_0^{(i)}$ *for each* $i = 1, \ldots, k$. *Then the sequence* $\{x_n\}$ *generated for* x_{-1}, x_0 *with* $\|x_{-1} - x_0\| =$

$\delta > 0$ *by method* (1.2) *is well defined, remains in* $U(x_0, r)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x_* \in U(x_0, r)$ *of equation* (1.1).

Proof. By hypothesis we have $x_{-1} \neq x_0$, so $[x_{-1}, x; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ and x_1 is well defined. Moreover, as $\|x_1 - x_0\| \leq \eta < r$ by (2.2), we get $x_1 \in U(x_0, r)$.

Case 1: $x_0^{(i)} \neq x_1^{(i)}$, $i = 1, \ldots, k$. We can write

$$\begin{aligned} F(x_1) &= F(x_1) - F(x_0) - [x_{-1}, x_0; F](x_1 - x_0) \\ &= ([x_1, x_0; F] - [x_{-1}, x_0; F])(x_1 - x_0). \end{aligned} \tag{2.3}$$

Then we see that $x_0 \neq x_1$ and $[x_0, x_1; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Using (2.1) and (2.3) we obtain

$$\begin{aligned} &\|x_2 - x_1\| \\ &= \|([x_0, x_1; F]^{-1}[\tilde{z}, x_0; F])[\tilde{z}, x_0; F]^{-1}([x_1, x_0; F] - [x_{-1}, x_0; F])(x_1 - x_0)\| \\ &\leq \|[x_0, x_1; F]^{-1}[\tilde{z}, x_0; F]\| \|[\tilde{z}, x_0; F]^{-1}([x_1, x_0; F] - [x_{-1}, x_0; F])\| \, \|x_1 - x_0\| \\ &\leq \frac{\varphi(\|x_1 - x_{-1}\|, 0)}{1 - \varphi_0(\|\tilde{z} - x_0\|, \|x_0 - x_1\|)} \|x_1 - x_0\| \\ &\leq \frac{\varphi(\|x_1 - x_0\| + \|x_0 - x_{-1}\|, 0)}{1 - \varphi_0(\|\tilde{z} - x_0\|, \|x_0 - x_1\|)} \|x_1 - x_0\| \leq \frac{\varphi(\gamma + \delta, 0)}{1 - \varphi_0(\gamma, r)} \|x_1 - x_0\| \\ &= g_0(r)\|x_1 - x_0\|. \end{aligned} \tag{2.4}$$

Case 2: $x_0^{(i)} \neq x_1^{(i)}$ for some $i = 1, \ldots, k$. Then instead of (2.3) we write

$$\begin{aligned} F(x_1) &= F(x_1) - F(x_0) - [x_{-1}, x_0; F](x_1 - x_0) \\ &= ([x_1, x_0; F] - [x_{-1}, x_0; F])(x_1 - x_0). \end{aligned}$$

As in (2.4), we get

$$\begin{aligned} \|x_2 - x_1\| &= \|([x_1, x_0; F]^{-1}[\tilde{z}, x_0; F])[\tilde{z}, x_0; F]^{-1} \\ &\qquad \times ([x_1, x_0; F] - [x_{-1}, x_0; F])(x_1 - x_0)\| \\ &\leq \frac{\varphi(\eta + \alpha, 0)}{1 - \varphi_0(\|\tilde{z} - x_0\|, \|x_0 - x_1\|)} \|x_1 - x_0\| \\ &\leq \frac{\varphi(\gamma + \alpha, 0)}{1 - \varphi_0(\gamma + \alpha, r)} \|x_1 - x_0\| = g_0(r)\|x_1 - x_0\|. \end{aligned}$$

As $g_0(r) < 1$ by $(\mathcal{A}_4)$, from (2.2) and (2.4) we get

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| < (g_1(r) + 1)\|x_1 - x_0\| < r,$$

so $x_2 \in U(x_0, r)$. Therefore, in either case,

$$\|x_2 - x_1\| \leq \frac{\varphi(\gamma + \alpha, 0)}{1 - \varphi_0(\gamma + \alpha, r)} \|x_1 - x_0\| = g_0(r)\|x_1 - x_0\|.$$

CASE 3: $x_1^{(i)} \neq x_2^{(i)}$ for some $i = 1, \ldots, k$. Then again $x_1 \neq x_2$, $[x_1, x_2; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ and

$$\begin{aligned}
\|[x_1, x_2; F]^{-1}[\tilde{z}, x_0; F]\| &\le \frac{1}{1 - \varphi_0(\|\tilde{z} - x_1\|, \|x_2 - x_0\|)} \\
&\le \frac{1}{1 - \varphi_0(\|\tilde{z} - x_0\| + \|x_0 - x_1\|, \|x_2 - x_0\|)} \\
&\le \frac{1}{1 - \varphi_0(\gamma + r, r)}.
\end{aligned} \tag{2.5}$$

By method (1.2) we have

$$\begin{aligned}
F(x_2) &= F(x_2) - F(x_1) - [x_0, x_1; F](x_2 - x_1) \\
&= ([x_2, x_1; F] - [x_0, x_1; F])(x_2 - x_1).
\end{aligned} \tag{2.6}$$

Using (1.2), (2.5) and (2.6), we obtain

$$\begin{aligned}
\|x_3 - x_2\| &= \|([x_1, x_2; F]^{-1} F(x_2)\| \\
&\le \frac{\varphi(\|x_2 - x_0\|, 0)}{1 - \varphi_0(\gamma + r, r)} \|x_2 - x_1\| \\
&\le \frac{\varphi(r, 0)}{1 - \varphi_0(\gamma + r, r)} \|x_2 - x_1\|.
\end{aligned} \tag{2.7}$$

Using (2.2) and (2.4), we get

$$\begin{aligned}
\|x_3 - x_0\| &\le \|x_3 - x_2\| + \|x_2 - x_1\| + \|x_1 - x_0\| \\
&< (g(r) + 1)\|x_2 - x_1\| + \|x_1 - x_0\| \\
&< ((g(r) + 1)g_0(r) + 1)\,\eta < r,
\end{aligned}$$

so $x_3 \in U(x_0, r)$.

CASE 4: $x_1^{(i)} = x_2^{(i)}$ for some $i = 1, \ldots, k$. By method (1.2) we get

$$\begin{aligned}
F(x_2) &= F(x_2) - F(x_1) - [u_0, u_1; F](x_2 - x_1) \\
&= ([x_2, x_1; F] - [u_0, u_1; F])(x_2 - x_1),
\end{aligned} \tag{2.8}$$

where $u_0 = x_{-1}$, $u_1 = x_0$ or $u_0 = x_0$, $u_1 = x_1$. We shall choose the pair (u_0, u_1) corresponding to the largest $m = -1, 0$ such that for $u_0 = x_m, u_1 = x_{m+1}$, $x_m^{(i)} \neq x_{m+1}^{(i)}$. Denote that pair by (u_1, u_2). Notice that such a selection is always possible, since we can take $m = -1$ and then $x_{-1}^{(i)} \neq x_0^{(i)}$ by hypothesis. Then we have $[u_1, u_2; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ and

$$\begin{aligned}
\|[u_1, u_2; F]^{-1}[\tilde{z}, x_0; F]\| &\le \frac{1}{1 - \varphi_0(\|\tilde{z} - u_1\|, \|u_2 - x_0\|)} \\
&\le \frac{1}{1 - \varphi_0(\gamma + r, r)},
\end{aligned} \tag{2.9}$$

since $\|\tilde{z}-u_1\| \le \|\tilde{z}-x_0\| + \|x_0-u_2\| \le \gamma + r$ and $\|u_2 - x_0\| \le r$. Then we obtain

$$\begin{aligned}(2.10)\qquad \|x_3 - x_2\| &= \|([u_1,u_2;F]^{-1}F(x_2)\| \\ &\le \frac{\varphi(\|x_2-u_0\|,\|x_1-u_1\|)}{1-\varphi_0(\gamma+r,r)}\|x_2-x_1\| \\ &\le \frac{\varphi(\|x_2-x_0\|+\|x_0-u_0\|,\|x_1-x_0\|+\|x_0-u_1\|)}{1-\varphi_0(\gamma+r,r)}\|x_2-x_1\| \\ &< \frac{\varphi(2r,2r)}{1-\varphi_0(\gamma+r,r)}\|x_2-x_1\| = g(r)\|x_2-x_1\|.\end{aligned}$$

By (2.7) and (2.10), in either case we get

$$(2.11)\qquad \|x_3 - x_2\| \le g(r)\|x_2 - x_1\|.$$

Next, we shall show that the sequence $\{x_n\}$ satisfies the following items for $j \ge 2$.

(i) $F(x_j) = ([x_j, x_{j-1};F] - [x_{j-2}, x_{j-1};F])(x_j - x_{j-1})$ if $x_{j-2}^{(i)} \ne x_{j-1}^{(i)}$, $i = 1,\ldots,k$ or

(i)′ $F(x_j) = ([x_j, x_{j-1};F] - [u_{j-2}, u_{j-1};F])(x_j - x_{j-1})$ if $u_{j-2} = x_{-1}$, $u_{j-1} = x_0$ or $u_{j-2} = x_0$, $u_{j-1} = x_1 \ldots$ or $u_{j-2} = x_{j-2}$, $u_{j-1} = x_{j-1}$. Choose the pair (u_{j-2}, u_{j-1}) corresponding to the largest $m = -1, 0, \ldots, j-2$ such that for $u_{j-2} = x_m$, $u_{j-1} = x_{m+1}$, $x_m^{(i)} \ne x_{m+1}^{(i)}$, $i = 1,\ldots,k$. Denote that pair by (u_{j-1}, u_j). As noted previously, such a pair always exists.

(ii) $$\begin{aligned}\|x_{j+1} - x_j\| &\le g(r)\|x_j - x_{j-1}\| \\ &\le g(r)^{j-1}\|x_2-x_1\| \le g(r)^{j-1}g_0(r)\|x_1-x_0\| \le \eta.\end{aligned}$$

(iii) $$\begin{aligned}\|x_{j+1} - x_0\| &\le \sum_{p=0}^{j}\|x_{p+1}-x_p\| \\ &< (g(r)^{j-1} + \ldots + g(r) + 1)\|x_2 - x_1\| + \|x_1 - x_0\| \\ &< \frac{1}{1-g(r)}\|x_2-x_1\| + \|x_1-x_0\| < \left(\frac{g_0(r)}{1-g(r)}+1\right)\eta = r.\end{aligned}$$

(iv) $x_{j+1} \in U(x_0, r)$.

Items (i), (i)′–(iv) hold for $j = 2$ by method (1.2) and the preceding proof. Suppose that they hold for some $j = q$. We shall show that they hold for $j = q+1$. As in the case $j = 2$, we see that (i) and (i)′ hold for $j = q+1$. Concerning (ii), we have first, under Case 1,

$$\begin{aligned}
(2.12)\qquad \|x_{q+2}-x_{q+1}\| &= \|[x_q,x_{q+1};F]^{-1}F(x_{q+1})\| \\
&\le \|[x_q,x_{q+1};F]^{-1}[\tilde{z},x_0;F]\| \\
&\quad \times \|[\tilde{z},x_0;F]^{-1}([x_{q+1},x_q;F]-[x_{q-1},x_q;F])(x_{q+1}-x_q)\| \\
&\le \frac{\varphi(\|x_{q+1}-x_{q-1}\|,\|x_q-x_q\|)}{1-\varphi_0(\|x_{q+1}-\tilde{z}\|,\|x_{q+1}-x_0\|)}\|x_{q+1}-x_q\| \\
&\le \frac{\varphi(2r,0)}{1-\varphi_0(\gamma+r,r)}\|x_{q+1}-x_q\| \\
&< g(r)\|x_{q+1}-x_q\| \le g(r)^q g_0(r)\|x_1-x_0\|.
\end{aligned}$$

For Case 2 we have again, as before,

$$\begin{aligned}
(2.13)\qquad \|x_{q+2}-x_{q+1}\| &= \|[u_j,u_{j+1};F]^{-1}F(x_{q+1})\| \\
&\le \|[u_j,u_{j+1};F]^{-1}[\tilde{z},x_0;F]\|\,\|[\tilde{z},x_0;F]^{-1}F[x_{q+1})\| \\
&\le \frac{\varphi(\|x_{q+1}-u_{q-1}\|,\|x_q-u_q\|)}{1-\varphi_0(\|\tilde{z}-u_j\|,\|u_{j+1}-x_0\|)}\|x_{q+1}-x_q\| \\
&< g(r)\|x_{q+1}-x_q\|.
\end{aligned}$$

Hence, in either case (2.12) and (2.13),

$$(2.14)\qquad \|x_{q+2}-x_{q+1}\| < g(r)\|x_{q+1}-x_q\| \le g(r)^q g_0(r)\|x_1-x_0\|.$$

It follows from the definition of r, the induction hypotheses and the preceding inequality that

$$\begin{aligned}
(2.15)\qquad \|x_{q+2}-x_0\| &\le \sum_{p=0}^{q+2}\|x_{p+1}-x_p\| \\
&\le (1+g(r)+\cdots+g(r)^q)g_0(r)\|x_1-x_0\| \\
&< \left(\frac{g_0(r)}{1-g(r)}+1\right)\|x_1-x_0\| \le \left(\frac{g_0(r)}{1-g(r)}+1\right)\eta = r,
\end{aligned}$$

so $x_{q+2}\in U(x_0,r)$. By (2.14) and (ii) the sequence $\{x_n\}$ is Cauchy in $\mathbb{R}^k$ and so it converges to some $x_*\in \bar{U}(x_0,r)$. We also have the estimate

$$(2.16)\qquad \|[\tilde{z},x_0;F]^{-1}F(x_{q+2})\| \le \varphi(2r,2r)\|x_{q+2}-x_q\|,$$

which implies $F(x_*)=0$, as $q\to+\infty$. ∎

Next, we present a uniqueness result.

THEOREM 2.3. *Under conditions* $(\mathcal{A})$, *suppose further that there exists* $r_1\ge r$ *such that*

$$(2.17)\qquad \varphi_0(\gamma+r,r_1)<1.$$

Then the limit point x_* *is the only solution of equation* (1.1) *in* $D_1 := \bar{U}(x_0,r_1)\cap D$.

Proof. Let $y_* \in D_1$ be such that $F(y_*) = 0$. Define $M = [x_*, y_*; F]$. Using $(\mathcal{A}_2)$, we get

$$[\tilde{z}; x_*; F]^{-1}([x_*, y_*; F] - [\tilde{z}, x_0; F]) \\ \leq \varphi_0(\|x_* - \tilde{z}\|, \|y_* - x_0\|) \leq \varphi_0(\gamma + r, r_1) < 1.$$

Hence, $M^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Then, from the identity $0 = F(x_*) = F(y_*) = M(x_* - y_*)$, we conclude that $x_* = y_*$. ■

Remark 2.4. (a) Notice that usually in the studies involving secant-type methods instead of $(\mathcal{A}_2')$, one uses $(\mathcal{A}_2'')$ given by

$$\|[\tilde{z}, x_0; F]^{-1}([x, y; F] - [u, v; F])\| \leq \bar{\varphi}(\|x - u\|, \|y - v\|) \tag{2.18}$$

for $x, y, u, v \in D$, where $\bar{\varphi} : [0, +\infty)^2 \to [0, +\infty)$ is continuous and nondecreasing. Clearly,

$$\varphi(s_1, s_2) \leq \bar{\varphi}(s_1, s_2) \quad \text{for all } s_1, s_2 \geq 0, \tag{2.19}$$

$$\varphi_0(s_1, s_2) \leq \bar{\varphi}(s_1, s_2) \quad \text{for all } s_1, s_2 \geq 0, \tag{2.20}$$

since $D_0 \subseteq D$ and $\bar{\varphi}/\varphi_0$ can be arbitrarily large [3, 5]. Moreover, we do not require the functions φ_0 and φ to be continuous. This way, we can use conditions $(\mathcal{A})$ to study nondifferentiable equations. This is not possible using condition (2.18).

(b) Authors prefer to leave equations such as (2.2) as uncluttered as possible [8–11]. This equation determines the smallness of η (or the accuracy of the initial point to force the convergence of the method), and also provides the radius of convergence. The conditions on r and η do not seem immediate (see, e.g., (2.2)). We can employ some stronger conditions that imply the solvability of (2.2). As an example, suppose that

$$0 < g(\eta) < 1. \tag{2.21}$$

Then there exist $r \geq \eta$ and $\mu \in (0, 1)$ such that

$$g(r) \leq \mu < 1. \tag{2.22}$$

Therefore, by (2.2) and (2.22), we must have

$$\left(1 + \frac{\mu}{1 - \mu}\right)\eta \leq r,$$

or

$$\eta \leq (1 - \mu)r \tag{2.23}$$

(since $\varphi_0(s, t) \leq \varphi(s, t)$). In practice, we choose $\mu \in (0, 1)$ and solve

$$g(t) = \mu. \tag{2.24}$$

By the intermediate value theorem applied to the function $\bar{g}(t) := g(t) - \mu$ and (2.22), equation (2.24) has positive solutions. Denote by r the smallest such solution. Then the sufficient convergence criteria replacing $(\mathcal{A}_3)$ are given by (2.21) and (2.23).

3. Local convergence analysis. The local convergence analysis of method (1.2) is based on the following *conditions* $(\mathcal{C})$:

$(\mathcal{C}_1)$ There exist $x_* \in D$ with $F(x_*) = 0$, $\alpha > 0$ and $\tilde{z} \in D$ with $\|x_* - \tilde{z}\| = \alpha$, such that $[x_*, \tilde{z}; F]^{-1} \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$.

$(\mathcal{C}_2)$ $\|[x_*, \tilde{z}; F]^{-1}([x, y; F] - [x_*, \tilde{z}; F])\| \le w_0(\|x - x_*\|, \|y - \tilde{z}\|)$ for all distinct $x, y \in D$, where $w_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is nondecreasing in each variable. Suppose that there exists $r \in (0, +\infty)$ such that $w_0(r, \alpha + r) < 1$. Let $\rho_0 = \sup\{r \in (0, +\infty) : w_0(r, \alpha + r) < 1\}$. Set $D_2 = D \cap U(x_*, \rho_0)$.

$(\mathcal{C}_2')$ $\|[x_*, \tilde{z}; F]^{-1}([x, y; F] - [u, v; F])\| \le w(\|x - u\|, \|y - v\|)$ for all x, y, u, v in D with $x \neq y$ and $u \neq v$, where $w : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is nondecreasing in each variable.

$(\mathcal{C}_3)$ The equation

$$w(2t, t) + w_0(t, \alpha + t) - 1 = 0 \tag{3.1}$$

has at least one positive zero. Denote by R the smallest such zero.

$(\mathcal{C}_4)$ $U(x_*, R) \subseteq D$ and $w_0(R, \alpha + R) < 1$.

Next, we present an auxiliary Banach perturbation result on the inverse of the divided difference of order one for the operator F.

LEMMA 3.1. *Suppose that conditions $(\mathcal{C})$ hold. If $x, y \in U(x_*, R)$ with $x \neq y$, then $[x, y; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ and*

$$\|[x, y; F]^{-1}[x_*, \tilde{z}; F]\| \le \frac{1}{1 - w_0(\|x - x_*\|, \|y - \tilde{z}\|)} \le \frac{1}{1 - w_0(R, \alpha + R)}. \tag{3.2}$$

Proof. Using $(\mathcal{C}_2)$, $(\mathcal{C}_2')$ and $(\mathcal{C}_4)$, we obtain

$$\begin{aligned}\|[x_*, \tilde{z}; F]^{-1}([x_*, \tilde{z}; F] - [x, y; F])\| &\le w_0(\|x - x_*\|, \|x_* - \tilde{z}\|) \\ &< w_0(R, \alpha + R) < 1.\end{aligned}$$

Then, again by the Banach Lemma on invertible operators [3, 6], we have $[x, y; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$ so that (3.2) is satisfied.

Next, we show the main local convergence result for method (1.2) using conditions $(\mathcal{C})$ and the preceding notation.

THEOREM 3.2. *Suppose that conditions $(\mathcal{C})$ hold and $x_{-1}^{(i)} \neq x_0^{(i)}$ for all $1, \dots, k$. Then sequence $\{x_n\}$ generated for $x_{-1}, x_0 \in U(x_*, R)$ by method (1.2) is well defined, remains in $U(x_*, R)$ for all $n = 0, 1, \dots$ and converges to x_*.*

Proof. By hypothesis $x_{-1}^{(i)} \neq x_0^{(i)}$ for all $1, \dots, k$, we have $x_{-1} \neq x_0$. Then, by Lemma 3.1, $[x_{-1}, x_0; F]^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Hence, x_1 is well defined by method (1.2) for $n = 0$. Using (i) and $F(x_*) = 0$, we get

$$
\begin{aligned}
x_1 - x_* &= x_0 - x_* - [x_{-1}, x_0; F]^{-1}(F(x_0) - F(x_*)) \\
&= x_0 - x_* - [x_{-1}, x_0; F]^{-1}[x_0, x_*; F](x_0 - x_*).
\end{aligned} \tag{3.3}
$$

Then, by (3.2), $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ and (3.3), we have

$$
\begin{aligned}
\|x_1 - x_*\| &\leq \|[x_{-1}, x_0; F]^{-1}[x_*, \tilde{z}; F]\| \\
&\quad \times \|[x_*, \tilde{z}; F]^{-1}([x_{-1}, x_0; F] - [x_0, x_*; F])\| \, \|x_0 - x_*\| \\
&\leq \frac{w(\|x_{-1} - x_0\|, \|x_0 - x_*\|)}{1 - w_0(R, \alpha + R)} \|x_0 - x_*\| \\
&< \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_0 - x_*\| = \|x_0 - x_*\| < R,
\end{aligned} \tag{3.4}
$$

so

$$
\|x_1 - x_*\| < \|x_0 - x_*\| \quad \text{and} \quad x_1 \in U(x_*, R). \tag{3.5}
$$

Case 1: If $x_1^{(i)} = x_0^{(i)}$ for some $i = 1, \dots, k$, we can define $x_2 = x_1 - [x_{-1}, x_0; F]^{-1}F(x_1)$. Then as in (3.3) we have

$$
\begin{aligned}
x_2 - x_* &= x_1 - x_* - [x_{-1}, x_0; F]^{-1}(F(x_1) - F(x_*)) \\
&= x_1 - x_* - [x_{-1}, x_0; F]^{-1}[x_1, x_*; F](x_1 - x_*),
\end{aligned} \tag{3.6}
$$

so

$$
\begin{aligned}
\|x_2 - x_*\| &\leq \|[x_{-1}, x_0; F]^{-1}[x_*, \tilde{z}; F]\| \\
&\quad \times \|[x_*, \tilde{z}; F]^{-1}([x_{-1}, x_0; F] - [x_1, x_*; F])\| \, \|x_1 - x_*\| \\
&\leq \frac{w(\|x_{-1} - x_1\|, \|x_0 - x_*\|)}{1 - w_0(R, \alpha + R)} \|x_1 - x_*\| \\
&< \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_1 - x_*\| = \|x_1 - x_*\| \\
&< \|x_0 - x_*\| < R,
\end{aligned} \tag{3.7}
$$

so

$$
\|x_2 - x_*\| < \|x_1 - x_*\| \quad \text{and} \quad x_2 \in U(x_*, R). \tag{3.8}
$$

Case 2: If $x_1^{(i)} \neq x_0^{(i)}$ for all $i = 1, \dots, k$, define $x_2 = x_1 - [x_0, x_1; F]^{-1}F(x_1)$. Then $x_1 \neq x_0$. Since by Lemma 3.1, $[x_0, x_1; F]^{-1}$ exists, we can write

$$
x_2 - x_* = x_1 - x_* - [x_0, x_1; F]^{-1}(F(x_1) - F(x_*)),
$$

so again

$$(3.9)\quad \begin{aligned} \|x_2 - x_*\| &\le \|[x_0, x_1; F]^{-1}[x_*, \tilde{z}; F]\| \\ &\quad \times \|[x_*, \tilde{z}; F]^{-1}([x_0, x_1; F] - [x_1, x_*; F])\| \, \|x_1 - x_*\| \\ &\le \frac{w(\|x_0 - x_1\|, \|x_1 - x_*\|)}{1 - w_0(R, \alpha + R)} \|x_1 - x_*\| \\ &< \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_1 - x_*\| = \|x_1 - x_*\| \\ &< \|x_0 - x_*\| < R. \end{aligned}$$

As in the proof of Theorem 2.2, $x_0 = x_*$ for $j \ge 2$.

CASE 3: We can write

$$(3.10)\quad \begin{aligned} x_{j+1} - x_* &= x_j - x_* - [u_{j-1}, u_j; F]^{-1} F(x_j) \\ &= [u_{j-1}, u_j; F]^{-1}([u_{j-1}, u_j; F] - [x_j, x_*; F])(x_j - x_*). \end{aligned}$$

Using Lemma 3.1, $(\mathcal{C}_2')$ and (3.10), we get

$$(3.11)\quad \begin{aligned} \|x_{j+1} - x_*\| &\le \|[u_{j-1}, u_j; F]^{-1}[x_*, \tilde{z}; F]\| \\ &\quad \times \|[x_*, \tilde{z}; F]^{-1}([u_{j-1}, u_j; F] - [x_j, x_*; F])\| \, \|x_j - x_*\| \\ &\le \frac{w(\|u_{j-1} - x_j\|, \|u_j - x_*\|)}{1 - w_0(\|u_{j-1} - x_*\|, \|u_j - \tilde{z}\|)} \|x_j - x_*\| \\ &< \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_j - x_*\|. \end{aligned}$$

CASE 4: We have

$$x_{j+1} - x_* = x_j - x_* - [x_{j-1}, x_j; F] F(x_j),$$

and as in (3.10) and (3.11) for $x_{j-1} = u_{j-1}$, $x_j = u_j$, we again arrive at

$$(3.12)\quad \|x_{j+1} - x_*\| \le \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_j - x_*\|.$$

Hence, we have show

$$(3.13)\quad \|x_{n+1} - x_*\| \le \frac{w(2R, R)}{1 - w_0(R, \alpha + R)} \|x_n - x_*\| \le c\|x_n - x_*\| < R,$$

where $c = \frac{w(2R,R)}{1 - w_0(\alpha, \alpha + R)} \in [0, 1)$, which shows that $\lim_{n\to\infty} x_n = x_*$ and $x_{n+1} \in U(x_*, R)$. ■

As in Theorem 2.3 for the uniqueness of the solution x_*, we have the following result.

THEOREM 3.3. *Under conditions* $(\mathcal{C})$, *suppose further that there exists* $R_1 \ge R$ *such that*

$$(3.14)\quad w_0(0, \alpha + R_1) < 1.$$

Then the limit point x_* is the only solution of equation (1.1) *in* $D_3 = U(x_*, R_1) \cap D$.

Proof. Let $y_* \in D_3$ be such that $F(y_*) = 0$. Define again $M = [x_*, y_*; F]$. Then, by using $(\mathcal{C}_2)$ and (2.14), we get

$$\|[x_*, \tilde{z}; F]^{-1}(M - [x_*, \tilde{z}; F])\| \le w_0(\|x_* - x_*\|, \|y_* - \tilde{z}\|) \le w_0(0, \alpha + R_1) < 1.$$

Hence, $M^{-1} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Then, from the identity $0 = F(x_*) - F(y_*) = M(x_* - y_*)$, we conclude that $x_* = y_*$. ∎

REMARK 3.4. Comments similar to the ones given for the semi-local case in Remark 2.4 can also be made for the local case.

4. Numerical examples. We present two numerical examples, involving systems on $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. In the first example, we show how to compute the majorant function appearing in Theorems 2.2 and 3.2. Moreover, we show that the new majorant functions φ_0, φ are tighter than φ, and w_0, w are tighter than ω_0, ω used in [4]. Notice that the results in [4] improved those of [9, 10]. The emphasis in the second example is to show convergence of method (1.2) and to present some error bounds and the solution x_* without necessarily verifying the convergence conditions introduced in the previous sections.

EXAMPLE 4.1 ([4]). Let $k = 3$, $D = U(0, 1)$ and for $h = (h_1, h_2, h_3)^T$ define a mapping F on D by

$$F(h) = (h_1 + 0.0125|h_1|, h_2^2 + h_2 + 0.0125|h_2|, e^{h_3} - 1)^T. \tag{4.1}$$

Clearly, a solution of $F(h) = 0$ is given by $x_* = (0, 0, 0)^T$. Using the definition of the divided differences provided in the introduction, we obtain:

Local case:

- $\|[x_*, \tilde{z}; F]^{-1}([x, y; F] - [u, v; F])\| \le d\big[\frac{1}{2}e(\|x - u\| + \|y - v\|) + 0.025\big]$ for all $x, y, u, v \in D$,
- $\|[x_*, \tilde{z}; F]^{-1}([x, y; F] - [x_*, \tilde{z}; F])\| \le d[\|x - x_*\| + \|y - \tilde{z}\| + 0.025]$ for all $x, y \in D$,
- $\|[x_*, \tilde{z}; F]^{-1}([x, y; F] - [u, v; F])\| \le d\big[\frac{1}{2}e^{\rho_0}(\|x - u\| + \|y - v\|) + 0.025\big]$ for all $x, y, u, v \in D_2$,

where $d = \|[x_*, \tilde{z}; F]^{-1}\|$. Therefore, we can define

$$\begin{aligned} \omega(s, t) &= d\big(\tfrac{1}{2}e(s + t) + 0.025\big), \\ \omega_0(s, t) = w_0(s, t) &= d((s + t) + 0.025), \\ w(s, t) &= d\big(\tfrac{1}{2}e^{\bar{\rho}_0}(s + t) + 0.025\big), \end{aligned}$$

where $\bar{\rho} = \min\{1, \rho_0\}$. Then

$$w_0(s, t) < \omega(s, t)$$

and

$$w(s,t) \le \omega(s,t) \tag{4.2}$$

for all $s,t>0$. Then, in [4] using the secant method (1.3) for $\tilde{z}=(0.2,0.2,0.1)^T$, we find for $\alpha = 0.2$, $d = 0.987654$ by solving the equation $\omega(2t,t)+\omega_0(t,\alpha+t) - 1 = 0$ that $\bar{R} = 0.125464$, $\omega_0(R,\alpha+R) = 0.470053 < 1$, and $\bar{R}_1 = 0.78749$. Therefore, the hypotheses of [4, Theorem 2] are satisfied. Hence, $\lim_{n\to\infty} z_n = x_* \in U(x_*,R) \subseteq D$ and x_* is the only solution of the equation $F(x) = 0$ in $D \cap \bar{U}(x_*,\bar{R}_1)$.

In the case of Theorem 3.2 (i.e., using method (1.2)), we have $\rho_0 = 0.39375016$, so $\bar{\rho}_0 = \rho_0$ and (4.2) holds with a strict inequality. Moreover, by solving equation (3.1) we obtain

$$R = 0.0512, \qquad R_1 = 0.78749. \tag{4.3}$$

However, if we solve (3.1) with w_0, w (following [4]) replaced by ω_0, ω, respectively, we get

$$\bar{\bar{R}} = 0.1255, \qquad \bar{\bar{R}}_1 = 0.78749. \tag{4.4}$$

If we use the secant method, so we can directly compare the present results to the corresponding ones in [4], we must solve the equation (see in [4, (8)])

$$w(2t,t) + w_0(t,\alpha+t) - 1 = 0. \tag{4.5}$$

This time, we obtain

$$R^* = 0.1525, \qquad R_1^* = 0.78749. \tag{4.6}$$

It follows from the above that for the secant method, the new results improve the ones in [4, 9, 10], since

$$\bar{\bar{R}} < R^*, \qquad \bar{\bar{R}}_1 \le R_1^*. \tag{4.7}$$

In view of (4.7), we obtain more initial points, tighter error bounds and better information on the location of the solution.

Semi-local case [4]: (a) For $\tilde{z} = (0.02, 0.02, 0)^T$ we also obtain

$$\begin{aligned} \varphi_0(s,t) &= \bar{\varphi}_0(s,t) = d_0(s+t+0.025), \\ \bar{\varphi}(s,t) &= d_0\big(\tfrac{1}{2}e(s+t) + 0.025\big), \\ \varphi(s,t) &= d_0\big(\tfrac{1}{2}e^{\bar{r}_0}(s+t) + 0.025\big), \end{aligned} \tag{4.8}$$

where $d_0 = \|[\tilde{z}, x_0; F]^{-1}\|, \bar{r}_0 = \min\{1, r_0\}$ and $\bar{\varphi}_0, \bar{\varphi}$ are φ_0, φ, respectively but defined on D (see also [4] and (2.18)). Then again we have

$$\varphi_0(s,t) < \bar{\varphi}(s,t), \tag{4.9}$$

$$\varphi_0(s,t) = \bar{\varphi}_0(s,t) \le \bar{\varphi}(s,t). \tag{4.10}$$

Choose $x_0 = (0.1, 0.1, 0.01)^T$, and $x_{-1} = (0.11, 0.11, 0.02)^T$. It follows that $\delta = 0.01$, $\gamma = 0.08$, $\eta = \|x_1 - x_0\| \le 0.1497$, $d_0 = 0.995008$. Then, by solving the equation [4]

$$\left(1 + \frac{\bar{g}_0(t)}{1 - \bar{g}(t)}\right)\eta - t = 0,$$

where $\bar{g}_0(t) = \frac{\bar{\varphi}(\eta+\delta,0)}{1-\bar{\varphi}_0(\gamma+t,t)}$, $\bar{g}(t) = \frac{\bar{\varphi}(2\eta,0)}{1-\bar{\varphi}_0(\gamma+t,t)}$, we get $\bar{r} = 0.163106, \bar{r}_1 = 0.736785$, $\bar{g}_0(\bar{r}) + \bar{g}(\bar{r}) = 0.822036 < 1$. Therefore, the conclusions of [4, Theorem 5] are satisfied, so $\lim_{n\to\infty} z_n = x_* \in U(x_0, \bar{r})$ and x_* is the only solution of $F(x) = 0$ in $D \cap U(x_0, \bar{r}_1)$.

If we use the secant method, we must solve the equation [4]

$$\left(1 + \frac{g_0(t)}{1 - g(t)}\right)\eta - t = 0, \tag{4.11}$$

where

$$g_0(t) = \frac{\varphi(\eta + \delta, 0)}{1 - \varphi_0(\gamma + t, t)}, \qquad g(t) = \frac{\varphi(2\eta, 0)}{1 - \varphi_0(\gamma + t, t)}.$$

We obtain

$$r^* = 0.1497, \qquad r_1^* = 0.736785. \tag{4.12}$$

Hence, we conclude again as in the local case that the present results improve the corresponding ones in [4, 9, 10].

(b) In order to use Remark 2.4(b), let $\tilde{z} = (0.01, 0.01, 0)^T$, $x_0 = (0.001, 0.001, 0)^T$, and $x_{-1} = (0.002, 0.002, 0.001)^T$. Then we get $\delta = 0.001$, $\gamma = 0.009$, $\eta = 0.0014$ and $d_0 = 0.9877$. In the present case for method (1.2), we have $r_0 = 0.4832$, so $\bar{r}_0 = r_0$ and the right hand inequality of (4.10) strict. Moreover, we have

$$r = 0.0015, \qquad r_1 = 0.762738534131. \tag{4.13}$$

That is, (2.21) and (2.23) hold for $\mu = 0.0666667$.

Therefore, we again obtain improved results over the corresponding ones in [4].

Example 4.2. Let $k = 2$. For $h = (h_1, h_2)^T$ consider the system

$$f_1(h) = 3h_1^2 h_2 + h_2^2 - 1 + |h_1 - 1| = 0, \tag{4.14}$$

$$f_2(h) = h_1^4 + h_1 h_2^3 - 1 + |h_2| = 0, \tag{4.15}$$

which can be written as $F(h) = 0$, where $F = (f_1, f_2)^T$. Using the standard divided difference $([a, b; F]_{ij})_{i,j=1}^2 \in L(\mathbb{R}^k, \mathbb{R}^k)$ (see also the introduction), for $x_{-1} = (1, 0)^T, x_0 = (5, 5)^T$, we obtain by (1.2)

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	5
1	1	0	5
2	0.909090909090909	0.363636363636364	3.0636E−01
3	0.894886945874111	0.329098638203090	3.453E−02
4	0.894655531991499	0.327827544745569	1.271E−03
5	0.894655373334793	0.327826521746906	1.022E−06
6	0.8946655373334687	0.327826521746298	6.089E−13
7	0.8946655373334687	0.327826421746298	2.710E−20

Hence, the solution x_* is given by

$$x_* = (0.894655373334687, 0.3278626421746298)^T.$$

Notice that the mapping F is not differentiable, so the earlier results mentioned in the introduction of this study cannot be used.

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