ACTA ARITHMETICA
191.4 (2019)

On an almost-prime sieve

by

Hieu T. Ngo (Ho Chi Minh City)

1. Introduction. Understanding the distribution of primes is an important topic in analytic number theory. No less interesting are the wider questions concerning the distribution of numbers with more general multiplicative properties, such as positive integers whose number of prime factors satisfies certain conditions. These questions have a history of more than a hundred years, dating back at least to Landau's classic [8] and studied extensively by Hardy and Ramanujan [4], Sathe [13], Selberg [14], and contemporary researchers (see for instance [12, 6, 17]).

The distribution of positive integers with a prescribed number of prime factors has been studied by a wide variety of analytic methods, most of which employed a variant of the Selberg–Delange method [14, 1, 2]. To our best knowledge, the only works which used sieve methods to effectuate good estimates for this distribution were by Hensley [5] and Graham [3]. In Hensley's work [5], an iterative method, a combinatorial sieve, and a variant of Selberg's sieve were devised to study integers with a prescribed number of prime factors. However, the scope of Hensley's results established by these methods was rather limited, since he did not try to push these ideas to their limits. For instance, the variant of Selberg's sieve used in [5] only showed an upper bound for the number of integers with at most one 'small' prime factor. Hensley [5, p. 258] indicated that his computation "appears to be quite complex" and perhaps for this reason his method was written down only for the case of numbers with at most one 'small' prime factor.

In this paper, we propose to use another variant of Selberg's sieve to establish a more general result for the number of positive integers with a prescribed number of prime factors. Two main technical innovations are present

2010 *Mathematics Subject Classification*: Primary 11N36; Secondary 11M06.
Key words and phrases: Fourier analysis, number of prime factors, Riemann zeta function, Selberg sieve, Selberg–Delange method.
Received 18 December 2017; revised 12 October 2018.
Published online 10 October 2019.

DOI: 10.4064/aa171218-21-10

in this paper. Firstly, we incorporate Fourier-analytic techniques in order to study Λ^2-sieve more effectively. The idea of applying Fourier analysis to sieve questions was used in the Polymath project [11] to strengthen the breakthrough results of Zhang [18] and Maynard [9] on bounded gaps between primes. An abstract formulation of this method can be found in [16]. Here we provide a new context to which a variant of Selberg's sieve with Fourier-analytic approach can be applied. Secondly, we employ a genuine variation of the Selberg–Delange method to study functions which analytically resemble complex powers of the Riemann zeta function. The crux of our method is estimating a certain Dirichlet series and its derivatives. The analysis of this Dirichlet series, which can be found in Sections 3.2 and 3.3, is based on the fact that it is analytically close to products of complex powers of the Riemann zeta function; this resemblance can be considered as the starting point and a main 'workhorse' in this paper and in other works which use the Selberg–Delange method. These ideas allow us to generalize the main result of [5].

We now describe the main result of this paper. For every integer n, let $\omega(n) = \sum_{p|n} 1$ be the number of prime factors of n. Let $\mathcal{N}$ denote the set of integers in the interval $[N, 2N]$. Throughout this paper, let $0 < \epsilon_0 < 1/10$ be a fixed constant and let $N^{\epsilon_0} < R < N$ be a parameter to be chosen. Put

$$\omega_R(n) = \max\{\omega(\gcd(n, d)) : d \le R\}.$$

We seek a good estimate for the quantity

$$\pi_{R,k}(\mathcal{N}) = |\{n \in \mathcal{N} : \omega_R(n) \le k\}|.$$

A trivial lower bound for $\pi_{R,k}(\mathcal{N})$ is the number of integers in $\mathcal{N}$ with at most k prime factors, which by a theorem of Landau [8] equals

$$\frac{N(\log\log N)^{k-1}}{(k-1)!\log N}(1 + o(1)).$$

One might guess that $\pi_{R,k}(\mathcal{N})$ is asymptotically close to the number of integers in $\mathcal{N}$ with at most $k+1$ prime factors. In fact, Hensley [5, Theorem 5.2] proved that

$$\pi_{R,1}(\mathcal{N}) \le \frac{N\log\log N}{\log R} + O\biggl(R^2(\log R)^4 + \frac{N}{\log N}\biggr).$$

We are not aware of any other work in which the quantity $\pi_{R,k}(\mathcal{N})$ is studied.

Our main result provides an upper bound for $\pi_{R,k}(\mathcal{N})$.

Theorem 1.1. *Let $0 < \epsilon_0 < 1/10$ be a fixed constant and k be a nonnegative integer. For every $\epsilon > 0$ and every $N^{\epsilon_0} < R < N$,*

$$\pi_{R,k}(\mathcal{N}) \le (1+\epsilon)\frac{N(\log\log N)^k}{(k!)^2\log R} + O\biggl(R^2(\log N)^{2k} + \frac{N(\log\log N)^{k-1}}{\log N}\biggr).$$

REMARK 1.2. (i) For $k=1$, we recover Hensley's theorem with a slightly larger constant in the main term, namely $1+o(1)$ instead of 1, and with a slightly better error term.

(ii) In Theorem 1.1, the error term is smaller than the main term when $R=o\big(\frac{N^{1/2}(\log\log N)^{k/2}}{(\log N)^k}\big)$.

An outline of the proof of Theorem 1.1 is as follows. In Section 2, a certain generalized Möbius function is defined to set up the Selberg sieve. We then analyze the main term of the sieve in Section 3, using Fourier analysis to transform the problem into the study of a certain Dirichlet series, which we then analyze, in particular near its singular points. In Section 4, we solve an optimization problem arising from the sieve analysis. Finally, in Section 5 we gather the estimates obtained to prove Theorem 1.1.

2. Setting up the sieve. We define the arithmetic function $\mu_k(n)$ for $k\geq 0$ by the Dirichlet series identity

$$\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\frac{1}{\zeta(s)}\sum_{\omega(n)\leq k}\frac{1}{n^s}.$$

We have (see [5, p. 250])

$$\mu_k(n)=\mu(n)(-1)^k\binom{\omega(n)-1}{k}. \tag{2.1}$$

In particular, $\mu_k(1)=1$, and $\mu_k(n)\neq 0$ if and only if $n=1$ or $\omega(n)\geq k+1$.

Let F be a smooth, compactly supported function with $\operatorname{supp} F\subset(-\infty,1]$ and $F(0)=1$. Define the sieve weight

$$\lambda_k(d)=\mu_k(d)F\left(\frac{\log d}{\log R}\right). \tag{2.2}$$

PROPOSITION 2.1. *If $k\geq 0$ and $N^{\epsilon_0}<R<N$, then*

$$\pi_{R,k}(\mathcal{N})\leq N\sum_{d_1,d_2=1}^{\infty}\frac{\lambda_k(d_1)\lambda_k(d_2)}{[d_1,d_2]}+O(R^2(\log N)^{2k}).$$

Proof. It is apparent from (2.2) that $\sum_{d\mid n}\lambda_k(d)=1$ whenever $\omega_R(n)\leq k$. Hence

$$\pi_{R,k}(\mathcal{N})\leq\sum_{n\in\mathcal{N}}\Big(\sum_{d\mid n}\lambda_k(d)\Big)^2.$$

On expanding the right hand side, we find that

$$\pi_{R,k}(\mathcal{N})\leq N\sum_{d_1,d_2=1}^{\infty}\frac{\lambda_k(d_1)\lambda_k(d_2)}{[d_1,d_2]}+O\Big(\Big(\sum_{d}|\lambda_k(d)|\Big)^2\Big).$$

It is clear that $|\lambda_k(d)| \ll |\mu_k(d)| \le \omega(d)^k$, whence the error term is

$$O(R^2 \max\{\omega(d)^{2k} : 1 \le d \le R\}).$$

Recall the classical Hardy–Ramanujan inequality [4] which asserts that for every $\epsilon > 0$, we have $\omega(n) < (1+\epsilon)\frac{\log n}{\log\log n}$ for all $n \gg_\epsilon 1$. Therefore $\omega(d)^{2k} \ll (\log N)^{2k}$ for every $1 \le d \le R$. Hence the error term is $O(R^2(\log N)^{2k})$. ∎

3. Sieve analysis

3.1. Fourier transform. Let $f(x_1)$ be the Fourier transform of $F(x)e^x$, namely

$$F(x)e^x = \int_{-\infty}^{\infty} f(x_1)e^{-ixx_1}\,\mathrm{d}x_1. \tag{3.1}$$

Since $F(\cdot)$ is smooth and compactly supported, $f(\cdot)$ is smooth and rapidly decaying, that is, for any $A > 0$ one has $|f(x_1)| \ll 1/(1+|x_1|)^A$ as $x_1 \to \infty$. It follows from (2.2) and (3.1) that

$$\begin{aligned}\mathcal{L} &:= \sum_{d_1,d_2=1}^{\infty} \frac{\lambda_k(d_1)\lambda_k(d_2)}{[d_1,d_2]} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)Z_0\Big(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned} \tag{3.2}$$

where for $\mathrm{Re}(w_1), \mathrm{Re}(w_2) > 0$ we define

$$Z_0(w_1,w_2) = \sum_{n_1,n_2=1}^{\infty} \frac{\mu_k(n_1)\mu_k(n_2)}{[n_1,n_2]\cdot n_1^{w_1}n_2^{w_2}}. \tag{3.3}$$

3.2. The Dirichlet series U and its derivatives. For $w_1, w_2, z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}(w_1), \mathrm{Re}(w_2) > 0$ define

$$\begin{aligned}U(w_1,w_2;z_1,z_2) &= \prod_p \Big(1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_1+z_2}}{p^{1+w_1+w_2}}\Big) \\ &= \sum_{n_1,n_2=1}^{\infty} \frac{\mu(n_1)e^{z_1\omega(n_1)}\mu(n_2)e^{z_2\omega(n_2)}}{[n_1,n_2]\cdot n_1^{w_1}n_2^{w_2}},\end{aligned} \tag{3.4}$$

$$V(w_1,w_2;z_1,z_2) = \zeta(1+w_1)^{-e^{z_1}}\,\zeta(1+w_2)^{-e^{z_2}}\,\zeta(1+w_1+w_2)^{e^{z_1+z_2}}. \tag{3.5}$$

Set

$$\widetilde{U}(w_1,w_2;z_1,z_2) = U(w_1,w_2;z_1,z_2)\cdot V^{-1}(w_1,w_2;z_1,z_2). \tag{3.6}$$

We claim that $\widetilde{U}(w_1,w_2;z_1,z_2)$ extends to a holomorphic function for $w_1, w_2, z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}(w_1), \mathrm{Re}(w_2), \mathrm{Re}(w_1+w_2) > -1/2$. In fact, it follows from (3.4) that $U = \prod_p U_p$ where

$$U_p = U_p(w_1, w_2; z_1, z_2) = 1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_1+z_2}}{p^{1+w_1+w_2}};$$

whereas from (3.5) we deduce that $V^{-1} = \prod_p V_p^{-1}$ where

$$V_p^{-1} = V_p^{-1}(w_1, w_2; z_1, z_2) = 1 + \frac{e^{z_1}}{p^{1+w_1}} + \frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_1+z_2}}{p^{1+w_1+w_2}} + O\biggl(\frac{1}{p^{2+2\operatorname{Re}(w_1)}} + \frac{1}{p^{2+2\operatorname{Re}(w_2)}} + \frac{1}{p^{2+2\operatorname{Re}(w_1+w_2)}}\biggr).$$

Therefore $\widetilde{U} = \prod_p \widetilde{U}_p$ where

$$\widetilde{U}_p = \widetilde{U}_p(w_1, w_2; z_1, z_2) = 1 + O\biggl(\frac{1}{p^{2+2\operatorname{Re}(w_1)}} + \frac{1}{p^{2+2\operatorname{Re}(w_2)}} + \frac{1}{p^{2+2\operatorname{Re}(w_1+w_2)}}\biggr).$$

When $w_1, w_2, z_1, z_2 \in \mathbb{C}$ satisfy $\operatorname{Re}(w_1), \operatorname{Re}(w_2), \operatorname{Re}(w_1 + w_2) > -1/2$, we have

$$\sum_p \biggl(\frac{1}{p^{2+2\operatorname{Re}(w_1)}} + \frac{1}{p^{2+2\operatorname{Re}(w_2)}} + \frac{1}{p^{2+2\operatorname{Re}(w_1+w_2)}}\biggr) < \infty;$$

hence $\widetilde{U}(w_1, w_2; z_1, z_2)$ extends to a holomorphic function there.

The function $Z_0(w_1, w_2)$ given by (3.3) is a linear combination of functions of the type

$$Z_{l_1,l_2}(w_1, w_2) = \frac{d^{l_1}}{dz_1^{l_1}} \frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}} U(w_1, w_2; z_1, z_2) = \sum_{n_1,n_2=1}^{\infty} \frac{\mu(n_1)\omega(n_1)^{l_1}\mu(n_2)\omega(n_2)^{l_2}}{[n_1, n_2] \cdot n_1^{w_1} n_2^{w_2}}. \tag{3.7}$$

More precisely, by (2.1), (3.3), and (3.7),

$$Z_0(w_1, w_2) = \sum_{l_1,l_2=0}^{k} c_{l_1,l_2} Z_{l_1,l_2}(w_1, w_2) \tag{3.8}$$

where c_{l_1,l_2} are constants with $c_{k,k} = 1/(k!)^2$. The goal of this section is to give a good estimate for $Z_{l_1,l_2}(w_1, w_2)$ when $\operatorname{Re}(w_1)$ and $\operatorname{Re}(w_2)$ are positive and small.

LEMMA 3.1.

(a) *Let* $w, z \in \mathbb{C}$ *with* $\operatorname{Re}(w) > 0$, *and set* $V(w, z) = \zeta(1 + w)^{-e^z}$. *Put* $L = \log \zeta(1 + w)$. *For every positive integer* l, *there is a polynomial* $P(X)$ *of degree* l *such that*

$$\frac{d^l}{dz^l} V(w, z) = V(w, z) \cdot P(e^z L).$$

Moreover, the leading coefficient of $P(X)$ *is* $(-1)^l$.

(b) *Let* $w_1, w_2, z_1, z_2 \in \mathbb{C}$ *with* $\mathrm{Re}(w_1), \mathrm{Re}(w_2) > 0$. *Put* $L_1 = \log\zeta(1+w_1)$, $L_2 = \log\zeta(1+w_2)$, *and* $L_3 = \log\zeta(1+w_1+w_2)$. *For every pair* (l_1, l_2) *of nonnegative integers, there is a polynomial* $Q(X,Y,Z)$ *of total weighted degree* l_1+l_2, *in which* $\deg X = \deg Y = 1$ *and* $\deg Z = 2$, *such that*

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}V(w_1,w_2;z_1,z_2)$$
$$= V(w_1,w_2;z_1,z_2)\cdot Q(e^{z_1+z_2}L_3 - e^{z_1}L_1, e^{z_1+z_2}L_3 - e^{z_2}L_2, e^{z_1+z_2}L_3).$$

Moreover, the coefficient of $Z^{\min(l_1,l_2)}$ *in* $Q(X,Y,Z)$ *is* 1.

REMARK 3.2. In (b), we can write

$$Q(X,Y,Z) = \sum_{r=0}^{\min(l_1,l_2)} Z^r P_{l_1-r}(X) R_{l_2-r}(Y)$$

where $P_i(X)$ and $R_j(Y)$ are polynomials of degrees i and j respectively. Furthermore, $P_{\max(0,l_1-l_2)}(X) = 1$ and $R_{\max(0,l_2-l_1)}(Y) = 1$.

Proof of Lemma 3.1. To show (a), we write $V(w,z) = \exp(-e^z L)$, and so

$$\frac{d}{dz}V(w,z) = V(w,z)\cdot(-e^z L).$$

Part (a) then follows by induction.

For (b), we rewrite (3.5) as

$$V(w_1,w_2;z_1,z_2) = \exp(-e^{z_1}L_1 - e^{z_2}L_2 + e^{z_1+z_2}L_3).$$

Hence

$$\frac{dV}{dz_1} = V\cdot(e^{z_1+z_2}L_3 - e^{z_1}L_1), \qquad \frac{dV}{dz_2} = V\cdot(e^{z_1+z_2}L_3 - e^{z_2}L_2),$$
$$\frac{d}{dz_1}\frac{d}{dz_2}V = V\cdot\big((e^{z_1+z_2}L_3 - e^{z_1}L_1)(e^{z_1+z_2}L_3 - e^{z_2}L_2) + e^{z_1+z_2}L_3\big).$$

Part (b) then follows by induction. ∎

LEMMA 3.3.

(a) *Let* $w, z \in \mathbb{C}$ *be such that* $w = w' + iw''$ *with* $0 < w' < 1$, *and set* $V(w,z) = \zeta(1+w)^{-e^z}$. *Then*

$$\frac{d^l}{dz^l}\bigg|_{z=0} V(w,z) \ll_l w'^{-1}(-\log w')^l.$$

(b) *Let* $w_1, w_2, z_1, z_2 \in \mathbb{C}$ *be such that* $w_1 = w_1' + iw_1''$ *and* $w_2 = w_2' + iw_2''$ *with* $w_1', w_2', w_1' + w_2' \in (0,1)$. *Then*

$$\frac{d^{l_1}}{dz^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}} V(w_1,w_2;z_1,z_2) \ll_{l_1,l_2} \frac{(\max\{-\log w_1', -\log w_2'\})^{l_1+l_2}}{w_1'w_2'(w_1'+w_2')}.$$

Proof. We first show (a). Put $L=\log\zeta(1+w)$. By Lemma 3.1(a),

$$\frac{d^l V}{dz^l}\bigg|_{z=0}=\zeta(1+w)^{-1}P(L)$$

where $P(X)$ is a polynomial of degree l. The hypothesis $0<w'<1$ implies that $|\zeta(1+w)|^{\pm1}\ll w'^{-1}$ and $L\ll-\log w'$. This shows (a).

For (b), we proceed similarly. Let L_1, L_2 and L_3 be as in Lemma 3.1(b). By that lemma there is a polynomial $Q(X,Y,Z)$ of total weighted degree l_1+l_2 (with $\deg X=\deg Y=1$ and $\deg Z=2$) such that

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}V=\zeta(1+w_1)^{-1}\zeta(1+w_2)^{-1}\zeta(1+w_1+w_2)\,Q(L_3-L_1,L_3-L_2,L_3).$$

The assumptions $w_1',w_2',w_1'+w_2'\in(0,1)$ imply that

$$|\zeta(1+w_1)|^{\pm1}\ll w_1'^{-1},|\zeta(1+w_2)|^{\pm1}\ll w_2'^{-1},|\zeta(1+w_1+w_2)|^{\pm1}\ll(w_1'+w_2')^{-1},$$

and

$$L_1\ll-\log w_1',\qquad L_2\ll-\log w_2',\qquad L_3\ll-\log(w_1'+w_2').$$

Part (b) follows. ∎

LEMMA 3.4. *Let $w_1,w_2,z_1,z_2\in\mathbb{C}$ be such that $w_1=w_1'+iw_1''$ and $w_2=w_2'+iw_2''$ with $w_1',w_2',w_1'+w_2'>-1/2+\delta$ for some $\delta>0$. For every pair (l_1,l_2) of nonnegative integers, we have*

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}\widetilde{U}(w_1,w_2;z_1,z_2)=O_{l_1,l_2,\delta}(1).$$

Proof. Put $w_3=w_1+w_2$ and $z_3=z_1+z_2$. For $j\in\{1,2,3\}$ define

$$I_{p,j}=I_p(w_j,z_j)=e^{z_j}\big(\log(1-p^{-1-w_j})^{-1}-p^{-1-w_j}\big).$$

Let

$$J_p=\log\bigg(1-\frac{e^{z_1}}{p^{1+w_1}}-\frac{e^{z_2}}{p^{1+w_2}}+\frac{e^{z_3}}{p^{1+w_3}}\bigg)^{-1}-\bigg(\frac{e^{z_1}}{p^{1+w_1}}+\frac{e^{z_2}}{p^{1+w_2}}-\frac{e^{z_3}}{p^{1+w_3}}\bigg).$$

By (3.6), we have $\widetilde{U}=\exp(\sum_p\widetilde{L}_p)$ where $\widetilde{L}_p=I_{p,1}+I_{p,2}-I_{p,3}-J_p$.

To prove the lemma, it suffices to show that

$$\sum_p\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}\widetilde{L}_p(w_1,w_2;z_1,z_2)=O_{l_1,l_2,\delta}(1). \tag{3.9}$$

On the one hand, it is clear that

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}I_{p,1} = \begin{cases} I_{p,1}, & l_2 = 0,\\ 0, & l_2 > 0,\end{cases}$$

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}I_{p,2} = \begin{cases} I_{p,2}, & l_1 = 0,\\ 0, & l_1 > 0,\end{cases}$$

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}I_{p,3} = I_{p,3},$$

and when $z_1 = z_2 = 0$, we have $I_{p,j} = O_\delta(p^{-1-2\delta})$. Therefore

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}(I_{p,1}+I_{p,2}-I_{p,3}) = O_\delta(p^{-1-2\delta}). \tag{3.10}$$

On the other hand,

$$\frac{d}{dz_1}J_p = \frac{\left(\frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_3}}{p^{1+w_3}}\right)\left(\frac{e^{z_1}}{p^{1+w_1}} + \frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_3}}{p^{1+w_3}}\right)}{1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_3}}{p^{1+w_3}}},$$

$$\frac{d}{dz_2}J_p = \frac{\left(\frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_3}}{p^{1+w_3}}\right)\left(\frac{e^{z_1}}{p^{1+w_1}} + \frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_3}}{p^{1+w_3}}\right)}{1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_3}}{p^{1+w_3}}},$$

$$\frac{d}{dz_1}\frac{d}{dz_2}J_p = \frac{\left(\frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_3}}{p^{1+w_3}}\right)\left(\frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_3}}{p^{1+w_3}}\right)}{\left(1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_3}}{p^{1+w_3}}\right)^2} - \frac{\frac{e^{z_3}}{p^{1+w_3}}\left(\frac{e^{z_1}}{p^{1+w_1}} + \frac{e^{z_2}}{p^{1+w_2}} - \frac{e^{z_3}}{p^{1+w_3}}\right)}{1 - \frac{e^{z_1}}{p^{1+w_1}} - \frac{e^{z_2}}{p^{1+w_2}} + \frac{e^{z_3}}{p^{1+w_3}}}.$$

It then follows by induction that

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}J_p = O_{l_1,l_2,\delta}(p^{-1-2\delta}). \tag{3.11}$$

Combining (3.10) and (3.11) yields (3.9). ∎

We are in a position to prove the main estimate of this section.

PROPOSITION 3.5. *Let $w_1, w_2, z_1, z_2 \in \mathbb{C}$ be such that $w_1 = w_1' + iw_1''$ and $w_2 = w_2' + iw_2''$ with $w_1', w_2', w_1' + w_2' \in (0,1)$. Then*

$$Z_{l_1,l_2}(w_1,w_2) \ll_{l_1,l_2} \frac{\left(\max\{-\log w_1', -\log w_2'\}\right)^{l_1+l_2}}{w_1'w_2'(w_1'+w_2')}.$$

Proof. By (3.6) and (3.7), we have

$$Z_{l_1,l_2}(w_1,w_2) = \frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}\left(\widetilde{U}(w_1,w_2;z_1,z_2)\cdot V(w_1,w_2;z_1,z_2)\right).$$

Applying Lemmas 3.3(b) and 3.4, we deduce the proposition. ∎

3.3. The integral $\mathcal{L}$. In this section we analyze the integral $\mathcal{L}$ given by (3.2). Define

$$\mathcal{L}_{l_1,l_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)Z_{l_1,l_2}\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\right) \mathrm{d}x_1\,\mathrm{d}x_2, \tag{3.12}$$

$$\mathcal{L}_{l_1,l_2;\epsilon} = \iint\limits_{\substack{|x_1|<(\log N)^\epsilon \\ |x_2|<(\log N)^\epsilon}} f(x_1)f(x_2)Z_{l_1,l_2}\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\right) \mathrm{d}x_1\,\mathrm{d}x_2. \tag{3.13}$$

By (3.2) and (3.8),

$$\mathcal{L} = \sum_{l_1,l_2=0}^{k} c_{l_1,l_2}\mathcal{L}_{l_1,l_2}.$$

Lemma 3.6. *For every $\epsilon > 0$ and every $W > 1$,*

$$\mathcal{L}_{l_1,l_2} = \mathcal{L}_{l_1,l_2;\epsilon} + O_{\epsilon,W}((\log N)^{-W}).$$

As a consequence,

$$\mathcal{L} = \sum_{l_1,l_2=0}^{k} c_{l_1,l_2}\mathcal{L}_{l_1,l_2;\epsilon} + O_{\epsilon,W}((\log N)^{-W}). \tag{3.14}$$

Proof. Consider the integral (3.12). By Proposition 3.5, for every $\epsilon' > 0$,

$$Z_{l_1,l_2}\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\right) \ll (\log N)^{3+\epsilon'}.$$

Since $f(\cdot)$ is a smooth and rapidly decaying, the integral (3.12) can be restricted to the region $|x_1|, |x_2| < (\log N)^\epsilon$ up to an error term $O_{\epsilon,W}((\log N)^{-W})$. ∎

Lemma 3.7. *If $|x| < (\log R)/4$, then*

$$\log\zeta\left(1+\frac{1+ix}{\log R}\right) = \log\log N - \log(1+ix) + O(1).$$

Proof. Recall a classical estimate: if $|w| < 1/2$, then $\zeta'/\zeta(1+w) = -1/w + O(1)$ (see for instance [10, Theorem 6.7]). Therefore

$$\log\zeta\left(1+\frac{1+ix}{\log R}\right) - \log\zeta\left(1+\frac{1}{\log R}\right) = \int_{\frac{1}{\log R}}^{\frac{1+ix}{\log R}} \frac{\zeta'}{\zeta}(1+w)\,\mathrm{d}w$$

$$= -\frac{i}{\log R}\int_0^x \frac{\log R}{1+ix'}\,\mathrm{d}x' + O(1) = -\log(1+ix) + O(1).$$

Furthermore, it is plain that $\log\zeta\big(1+\frac{1}{\log R}\big) = \log\log N + O(1)$. ∎

The following lemma plays an important role in extracting the main term from the right hand side of (3.14).

Lemma 3.8. *Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth and rapidly decaying function.*

(a) *For any $V, W > 0$,*

$$\int_{-\infty}^{\infty} |f(x)| \cdot |1+ix|^{V} \cdot |\log(1+ix)|^{W}\,\mathrm{d}x < \infty.$$

(b) *For any $V_1, V_2, V_3, W > 0$,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1)f(x_2)| \cdot \frac{|1+ix_1|^{V_1}|1+ix_2|^{V_2}}{|2+ix_1+ix_2|^{V_3}} \cdot (\log(4+x_1^2+x_2^2))^{W}\,\mathrm{d}x_1\,\mathrm{d}x_2 < \infty.$$

Proof. Part (a) is clear on noting that $f(x)$ decays faster than any polynomial of x. For (b), we note that $\log(4+x_1^2+x_2^2) \ll |\log(1+ix_1)| \cdot |\log(1+ix_2)|$ and $|2+ix_1+ix_2|^{-V_3} \ll 1$. The conclusion is now an immediate consequence of Fubini's theorem and part (a). ■

Lemma 3.9. *Let $w_1, w_2, z_1, z_2 \in \mathbb{C}$ be such that $w_1 = w_1' + iw_1''$ and $w_2 = w_2' + iw_2''$ with $w_1', w_2', w_1'+w_2' > -1/2+\delta$ for some $\delta > 0$, and $|w_1|, |w_2| < 1/2$. Then*

$$\frac{d}{dw_1}\widetilde{U}(w_1,w_2;0,0) = O_\delta(1), \qquad \frac{d}{dw_2}\widetilde{U}(w_1,w_2;0,0) = O_\delta(1).$$

Proof. The proof is similar to that of Lemma 3.4. Put $\widetilde{U}_0(w_1,w_2) = \widetilde{U}(w_1,w_2;0,0)$. By symmetry, we just need to prove that

$$\frac{d}{dw_1}\widetilde{U}_0(w_1,w_2) = O_\delta(1).$$

Put $w_3 = w_1 + w_2$, for $j \in \{1,2,3\}$ define

$$I_{p,j} = I_p(w_j) = \log(1-p^{-1-w_j})^{-1} - p^{-1-w_j},$$

and let

$$J_p = \log\left(1 - \frac{1}{p^{1+w_1}} - \frac{1}{p^{1+w_2}} + \frac{1}{p^{1+w_3}}\right)^{-1} - \left(\frac{1}{p^{1+w_1}} + \frac{1}{p^{1+w_2}} - \frac{1}{p^{1+w_3}}\right).$$

By (3.6), we have $\widetilde{U}_0 = \exp(\sum_p \widetilde{L}_p)$ where $\widetilde{L}_p = I_{p,1} + I_{p,2} - I_{p,3} - J_p$.

It suffices to show that

$$\sum_p \frac{d}{dw_1}\widetilde{L}_p(w_1,w_2) = O_\delta(1). \tag{3.15}$$

On the one hand, it is clear that

$$\frac{d}{dw_1} I_{p,1} = \left(1 - \frac{1}{1-p^{-1-w_1}}\right) p^{-1-w_1} \log p = O_\delta(p^{-1-\delta}),$$
$$\frac{d}{dw_1} I_{p,2} = 0,$$
$$\frac{d}{dw_1} I_{p,3} = \left(1 - \frac{1}{1-p^{-1-w_3}}\right) p^{-1-w_3} \log p = O_\delta(p^{-1-\delta}).$$

Therefore

$$\frac{d}{dw_1}(I_{p,1} + I_{p,2} - I_{p,3}) = O_\delta(p^{-1-\delta}). \tag{3.16}$$

On the other hand,

$$\frac{d}{dw_1} J_p = \left(1 - \frac{1}{1 - \frac{1}{p^{1+w_1}} - \frac{1}{p^{1+w_2}} + \frac{1}{p^{1+w_3}}}\right)(p^{-1-w_1} - p^{-1-w_3}) \log p.$$

It follows that

$$\frac{d}{dw_1} J_p = O_\delta(p^{-1-\delta}). \tag{3.17}$$

Combining (3.16) and (3.17) gives (3.15). ∎

The following lemma provides an asymptotic for $\widetilde{U}(w_1, w_2; 0, 0)$ when w_1 and w_2 are near 1.

LEMMA 3.10. *If* $|x_1|, |x_2| < (\log N)^\epsilon$, *then*

$$\widetilde{U}\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; 0, 0\right) = 1 + O((\log N)^{\epsilon-1}).$$

Proof. Put $\widetilde{U}_0(w_1, w_2) = \widetilde{U}(w_1, w_2; 0, 0)$. We first note that $\widetilde{U}_0(0,0) = 1$, whence

$$\begin{aligned}
&\widetilde{U}_0\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\right) - 1 \\
&\quad = \widetilde{U}_0\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}\right) - \widetilde{U}_0\left(\frac{1+ix_1}{\log R}, 0\right) + \widetilde{U}_0\left(\frac{1+ix_1}{\log R}, 0\right) - \widetilde{U}_0(0,0) \\
&\quad = \int_0^{\frac{1+ix_2}{\log R}} \frac{d}{dw_2}\bigg|_{w_2 = w_2'} \widetilde{U}_0\left(\frac{1+ix_1}{\log R}, w_2\right) dw_2' + \int_0^{\frac{1+ix_1}{\log R}} \frac{d}{dw_1}\bigg|_{w_1 = w_1'} \widetilde{U}_0(w_1, 0)\, dw_1'.
\end{aligned}$$

On applying Lemma 3.9, we find that

$$\frac{d}{dw_2}\bigg|_{w_2 = w_2'} \widetilde{U}_0\left(\frac{1+ix_1}{\log R}, w_2\right) \ll 1, \qquad \frac{d}{dw_1}\bigg|_{w_1 = w_1'} \widetilde{U}_0(w_1, 0) \ll 1.$$

Hence the lemma follows. ∎

We now estimate the derivatives of $V(w_1, w_2; z_1, z_2)$ when w_1 and w_2 are near 1.

LEMMA 3.11. *Suppose that* $|x_1|, |x_2| < (\log N)^\epsilon$.

(a) *If* (l_1, l_2) *is a pair of nonnegative integers, then*

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\z_2=0}} V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; z_1, z_2\right)$$
$$\ll_{l_1,l_2} \frac{(\log\log N)^{\min(l_1,l_2)}}{\log N} \cdot \frac{|(1+ix_1)(1+ix_2)|}{|(2+ix_1+ix_2)|} \cdot |\log(4+x_1^2+x_2^2)|^{l_1+l_2}.$$

(b) *If* l *is a nonnegative integer, then*

$$\frac{d^{l}}{dz_1^{l}}\frac{d^{l}}{dz_2^{l}}\bigg|_{\substack{z_1=0\\z_2=0}} V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; z_1, z_2\right) - \frac{(\log\log N)^l}{\log R}\frac{(1+ix_1)(1+ix_2)}{2+ix_1+ix_2}$$
$$\ll_l \frac{(\log\log N)^{l-1}}{\log N} \cdot \frac{|(1+ix_1)(1+ix_2)|}{|(2+ix_1+ix_2)|} \cdot |\log(4+x_1^2+x_2^2)|^{3l-1}.$$

Proof. We first prove (a). From Lemma 3.1(b), we deduce that

$$\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\z_2=0}} V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; z_1, z_2\right)$$
$$= V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; 0, 0\right)$$
$$\cdot Q\left(\log\frac{\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)}{\zeta\left(\frac{1+ix_1}{\log R}\right)}, \log\frac{\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)}{\zeta\left(\frac{1+ix_2}{\log R}\right)}, \log\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)\right).$$

By definition (3.5), the V term equals

$$\frac{\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)}{\zeta\left(\frac{1+ix_1}{\log R}\right)\zeta\left(\frac{1+ix_2}{\log R}\right)} \ll \frac{1}{\log N} \cdot \frac{|(1+ix_1)(1+ix_2)|}{|2+ix_1+ix_2|},$$

whereas Remark 3.2 and Lemma 3.7 imply that the Q term is

$$\ll (\log\log N)^{\min(l_1,l_2)} \cdot (\log(4+x_1^2+x_2^2))^{l_1+l_2}.$$

Part (a) follows.

For (b), we proceed similarly. Lemma 3.1(b) yields

$$\frac{d^{l}}{dz_1^{l}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\z_2=0}} V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; z_1, z_2\right)$$
$$= V\left(\frac{1+ix_1}{\log R}, \frac{1+ix_2}{\log R}; 0, 0\right)$$
$$\cdot Q\left(\log\frac{\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)}{\zeta\left(\frac{1+ix_1}{\log R}\right)}, \log\frac{\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)}{\zeta\left(\frac{1+ix_2}{\log R}\right)}, \log\zeta\left(\frac{2+ix_1+ix_2}{\log R}\right)\right).$$

On the one hand, by definition (3.5) the V term equals

$$\frac{\zeta\big(\frac{2+ix_1+ix_2}{\log R}\big)}{\zeta\big(\frac{1+ix_1}{\log R}\big)\zeta\big(\frac{1+ix_2}{\log R}\big)} = \frac{(1+ix_1)(1+ix_2)}{\log R\cdot(2+ix_1+ix_2)}\big(1+O((\log N)^{\epsilon-1})\big)$$

On the other hand, Remark 3.2 and Lemma 3.7 imply that the Q term equals

$$\begin{aligned}&\biggl(\log\zeta\biggl(\frac{2+ix_1+ix_2}{\log R}\biggr)\biggr)^l\\ &\quad+O\biggl(\biggl|\log\frac{\zeta\big(\frac{2+ix_1+ix_2}{\log R}\big)}{\zeta\big(\frac{1+ix_1}{\log R}\big)}\cdot\log\frac{\zeta\big(\frac{2+ix_1+ix_2}{\log R}\big)}{\zeta\big(\frac{1+ix_2}{\log R}\big)}\biggr|^l\biggl|\log\zeta\biggl(\frac{2+ix_1+ix_2}{\log R}\biggr)\biggr|^{l-1}\biggr)\\ &=(\log\log N)^l+O\big((\log\log N)^{l-1}(\log(4+x_1^2+x_2^2))^{3l-1}\big).\end{aligned}$$

By multiplying the V term and the Q term, we deduce (b). ■

We can now estimate $Z_{l_1,l_2}(w_1,w_2)$ for w_1 and w_2 near 1.

PROPOSITION 3.12. *Suppose that* $|x_1|,|x_2|<(\log N)^\epsilon$.

(a) *If* (l_1,l_2) *is a pair of nonnegative integers, then*

$$\begin{aligned}&Z_{l_1,l_2}\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R}\biggr)\\ &\quad\ll_{l_1,l_2}\frac{(\log\log N)^{\min(l_1,l_2)}}{\log N}\cdot\frac{|(1+ix_1)(1+ix_2)|}{|(2+ix_1+ix_2)|}\cdot(\log(4+x_1^2+x_2^2))^{l_1+l_2}.\end{aligned}$$

(b) *If* l *is a nonnegative integer, then*

$$\begin{aligned}&Z_{l,l}\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R}\biggr)-\frac{(\log\log N)^l}{\log R}\,\frac{(1+ix_1)(1+ix_2)}{2+ix_1+ix_2}\\ &\qquad\ll_l\frac{(\log\log N)^{l-1}}{\log N}\cdot\frac{|(1+ix_1)(1+ix_2)|}{|(2+ix_1+ix_2)|}\cdot|\log(4+x_1^2+x_2^2)|^{3l-1}.\end{aligned}$$

Proof. It follows from (3.6) and (3.7) that

$$\begin{aligned}&Z_{l_1,l_2}\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R}\biggr)\\ &\quad=\frac{d^{l_1}}{dz_1^{l_1}}\frac{d^{l_2}}{dz_2^{l_2}}\bigg|_{\substack{z_1=0\\ z_2=0}}\widetilde{U}\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R};z_1,z_2\biggr)\cdot V\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R};z_1,z_2\biggr).\end{aligned}$$

Part (a) follows from Lemmas 3.4 and 3.11(a). Part (b) follows from Lemmas 3.4, 3.10 and 3.11(b). ■

COROLLARY 3.13. *For every* $\epsilon>0$,

$$\mathcal{L}_{l_1,l_2;\epsilon}\ll_\epsilon\frac{(\log\log N)^{\min(l_1,l_2)}}{\log N}.$$

Proof. This is an immediate consequence of (3.13), Lemma 3.8, and Proposition 3.12. ■

We now derive an asymptotic for the integral $\mathcal{L}$.

PROPOSITION 3.14. *We have*

$$\mathcal{L} = \frac{(\log\log N)^k}{\log R\cdot(k!)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)\frac{(1+ix_1)(1+ix_2)}{(2+ix_1+ix_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + O\biggl(\frac{(\log\log N)^{k-1}}{\log N}\biggr).$$

Proof. Combining (3.14) and the first statement in the cases $l_1+l_2<2k$, recalling that $c_{k,k}=1/(k!)^2$, we deduce that

$$\mathcal{L} = \frac{1}{(k!)^2}\mathcal{L}_{k,k;\epsilon} + O_\epsilon\biggl(\frac{(\log\log N)^{k-1}}{\log N}\biggr). \tag{3.18}$$

Now consider

$$\mathcal{L}_{k,k;\epsilon} = \iint\limits_{\substack{|x_1|<(\log N)^\epsilon\\ |x_2|<(\log N)^\epsilon}} f(x_1)f(x_2)Z_{k,k}\biggl(\frac{1+ix_1}{\log R},\frac{1+ix_2}{\log R}\biggr)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

It follows from Proposition 3.12 (b) and Lemma 3.8 that

$$\mathcal{L}_{k,k;\epsilon} = \frac{(\log\log N)^k}{\log R}\iint\limits_{\substack{|x_1|<(\log N)^\epsilon\\ |x_2|<(\log N)^\epsilon}} f(x_1)f(x_2)\frac{(1+ix_1)(1+ix_2)}{(2+ix_1+ix_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + O_\epsilon\biggl(\frac{(\log\log N)^{k-1}}{\log N}\biggr).$$

Since $f(x)$ is smooth and rapidly decaying, we have

$$\mathcal{L}_{k,k;\epsilon} = \frac{(\log\log N)^k}{\log R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)\frac{(1+ix_1)(1+ix_2)}{(2+ix_1+ix_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + O_\epsilon\biggl(\frac{(\log\log N)^{k-1}}{\log N}\biggr). \tag{3.19}$$

Combining (3.18) and (3.19), we conclude the proof. ∎

4. Optimization. The goal of this section is to establish the following optimization result.

PROPOSITION 4.1. *For every $\epsilon>0$, there is a smooth, compactly supported function $F:\mathbb{R}\to\mathbb{R}$ which satisfies the following conditions:*

(i) $\operatorname{supp} F\subset[-\epsilon,1]$;
(ii) $F(0)=1$;
(iii) $\int_0^{+\infty}(F'(x))^2\,\mathrm{d}x<1+\epsilon$.

REMARK 4.2. (i) Consider a smooth, compactly supported function $f:\mathbb{R}\to\mathbb{R}$ with $\operatorname{supp} f\subset(-\infty,1]$ and $f(0)=1$. Then $\int_0^{+\infty} f'(x)\,\mathrm{d}x=1$, and so by the Cauchy–Schwarz inequality we have $\int_0^{+\infty} f'(x)^2\,\mathrm{d}x \ge 1$. Thus Proposition 4.1 provides an essentially optimal solution for the problem of minimizing $\int_0^{+\infty} f'(x)^2\,\mathrm{d}x$ subject to the conditions that $f : \mathbb{R} \to \mathbb{R}$ is smooth, with compact support in $(-\infty, 1]$, and $f(0) = 1$.

(ii) This optimization proposition was stated in [15] without proof. In this section we will give both the constructions and the necessary estimates.

We introduce some notations. For an interval I, write χ_I for the characteristic function of I, namely $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ if $x \notin I$. If $\delta > 0$, put $v_{-\delta} = \chi_{[-\delta,0]}/\delta$. For $q \ge 0$, let C^q denote the space of functions $f : \mathbb{R} \to \mathbb{C}$ which are q times differentiable with $f^{(q)}$ continuous. Let C^∞ denote the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$.

We need some preliminary lemmas.

Lemma 4.3. *Let $p_0 : \mathbb{R} \to \mathbb{C}$ be a continuous function. If $\delta > 0$ and $p_1 = p_0 * v_{-\delta}$, then $p_1 \in C^1$ and*

$$p_1'(x) = \frac{1}{\delta}(p_0(x+\delta) - p_0(x)).$$

Proof. This lemma is standard; see for instance [7, Section 1.3, p. 19]. ∎

Lemma 4.4. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, compactly supported function, let $\delta > 0$ and suppose that ϕ_δ is a smooth function with* $\operatorname{supp} \phi_\delta \subset (-\delta, \delta)$ *and $\phi_\delta \ge 0$, $\int_{\mathbb{R}} \phi_\delta = 1$. Then for every $x \in \mathbb{R}$,*

$$|f(x) - f * \phi_\delta(x)| \le \sup_{|r| \le \delta} |f(x) - f(x-r)|.$$

Proof. By definition,

$$\begin{aligned}|f(x) - f * \phi_\delta(x)| &= \Big| \int_{-\delta}^{\delta} (f(x) - f(x-r))\phi_\delta(r)\,\mathrm{d}r \Big| \\ &\le \sup_{|r|\le\delta} |f(x) - f(x-r)| \int_{-\delta}^{\delta} |\phi_\delta(r)|\,\mathrm{d}r.\end{aligned}$$

The lemma follows on noting that $\int_{-\delta}^{\delta} |\phi_\delta(r)|\,\mathrm{d}r = 1$. ∎

Proof of Proposition 4.1. Let $\epsilon > 0$. Let $1/2 > \delta_1 \ge \delta_2 \ge \delta_3 \ge \delta > 0$ be (small) positive parameters to be determined. We proceed in four steps. First, we construct a compactly supported function $f_0 \in C^0$ with $f_0(0) = 1$ and $\int_0^\infty f_0'(x)^2\,\mathrm{d}x \approx 1$. Second, we construct a compactly supported function $f_1 \in C^1$ with $f_1(0) \approx 1$ and $\int_0^\infty f_1'(x)^2\,\mathrm{d}x \approx 1$. Next, we mollify f_1 to obtain a smooth, compactly supported function F_δ with $F_\delta(0) \approx 1$ and $\int_0^\infty F_\delta'(x)^2\,\mathrm{d}x \approx 1$. Finally, we rescale F_δ to obtain a smooth, compactly supported function F with $F(0) = 1$ and $\int_0^\infty F'(x)^2\,\mathrm{d}x \approx 1$.

STEP 1. We construct a function $f_0 \in C^0$ satisfying $f_0(0) = 1$ and $\int_0^\infty f_0'(x)^2\,\mathrm{d}x \approx 1$ as follows. Define

$$f_0(x) = \left(1 - \frac{x}{1-\delta_1}\right)\chi_{[0,1-\delta_1]}(x) + \left(1 + \frac{x}{\delta_2}\right)\chi_{[-\delta_2,0]}(x).$$

Heuristically, $f_0(x) \approx \chi_{[0,1]}(x)(1-x)$. By construction, we have

$$f_0 \in C^0,\ \operatorname{supp} f_0 \subset [-\delta_2, 1-\delta_1],\ f_0(0) = 1,\ \int_0^{+\infty} f_0'(x)^2\,\mathrm{d}x = \frac{1}{1-\delta_1}. \tag{4.1}$$

STEP 2. We transform (via convolution) f_0 to an $f_1 \in C^1$ with $f_1(0) \approx 1$ and $\int_0^{+\infty} f_1'(x)^2\,\mathrm{d}x \approx 1$. More precisely, we will construct a compactly supported function $f_1 \in C^1$ which satisfies the following conditions:

$$f_1 \in C^1, \quad \operatorname{supp} f_1 \subset [-\delta_2 - \delta_3, 1-\delta_1], \tag{4.2}$$

$$|f_1(0) - 1| \le \delta_3/\delta_2, \tag{4.3}$$

$$\left|\int_0^{+\infty} f_1'(x)^2\,\mathrm{d}x - 1\right| \ll \delta_1 + \delta_3. \tag{4.4}$$

Define

$$f_1 = f_0 * \upsilon_{-\delta_3}.$$

By (4.1), it is plain that $\operatorname{supp} f_1 \subset [-\delta_2 - \delta_3, 1-\delta_1]$.

Applying Lemma 4.3, we deduce that $f_1 \in C^1$ and we can compute its derivative. We have

$$f_1'(x) = \begin{cases} -\frac{1}{\delta_3}\left(1 - \frac{x}{1-\delta_1}\right) & \text{if } 1-\delta_1-\delta_3 \le x \le 1-\delta_1, \\ -\frac{1}{1-\delta_1} & \text{if } 0 \le x \le 1-\delta_1-\delta_3, \\ -\frac{1}{\delta_3}\left(\frac{x}{\delta_2} + \frac{x+\delta_3}{1-\delta_1}\right) & \text{if } -\delta_3 \le x \le 0. \end{cases} \tag{4.5}$$

In particular,

$$|f_1'(x)| \ll 1/\delta_3 \quad (|x| \le \delta_3). \tag{4.6}$$

We now prove (4.3). By Lemma 4.4 and the definition of f_0, we have

$$|f_1(0) - 1| = |f_1(0) - f_0(0)| \le \sup_{|r|\le\delta_3} |f_0(0) - f_0(r)| \le \delta_3/\delta_2.$$

This shows (4.3).

We next prove (4.4). It follows from (4.5) that

$$\int_0^{+\infty} f_1'(x)^2\,\mathrm{d}x = \frac{1-\delta_1-\delta_3}{(1-\delta_1)^2} + \frac{\delta_3}{3(1-\delta_1)^2}.$$

Hence (4.4) follows.

STEP 3. In this smoothing step, we make use of a smooth, compactly supported function ϕ_δ with $\operatorname{supp} \phi_\delta \subset (-\delta, \delta)$ and $\phi_\delta \ge 0$, $\int_{\mathbb{R}} \phi_\delta = 1$. We mollify f_1 to a smooth, compactly supported function F_δ with $F_\delta(0) \approx 1$ and

$\int_0^{+\infty} F'_\delta(x)^2\,\mathrm{d}x \approx 1$. More precisely, we will construct a smooth, compactly supported function F_δ which satisfies the following conditions:

$$F_\delta \in C^\infty, \quad \operatorname{supp} F_\delta \subset [-\delta - \delta_2 - \delta_3, 1 - (\delta_1 - \delta)], \tag{4.7}$$

$$|F_\delta(0) - 1| \ll \delta_3/\delta_2 + \delta/\delta_3, \tag{4.8}$$

$$\Big| \int_0^{+\infty} F'_\delta(x)^2\,\mathrm{d}x - 1 \Big| \ll \delta_1 + \delta_3 + \delta/\delta_3^2. \tag{4.9}$$

Define

$$F_\delta = f_1 * \phi_\delta.$$

It is clear that $\operatorname{supp} F_\delta \subset [-\delta - \delta_2 - \delta_3, 1 - (\delta_1 - \delta)]$ and $F_\delta \in C^\infty$.

We now show (4.8). We apply Lemma 4.4 and (4.6), noting $0 < \delta < \delta_3$, to infer that

$$|f_1(0) - F_\delta(0)| \le \sup_{|r|\le\delta} |f_1(0) - f_1(r)| \le \delta \sup_{|r|\le\delta,\, r\ne 0} |f'_1(r)| \ll \delta/\delta_3.$$

Combining this estimate and (4.3), we deduce (4.8).

We next show (4.9) by estimating

$$E = \int_0^{+\infty} F'_\delta(x)^2\,\mathrm{d}x - \int_0^{+\infty} f'_1(x)^2\,\mathrm{d}x$$
$$= \int_0^{1-\delta_1+\delta} F'_\delta(x)^2\,\mathrm{d}x - \int_0^{1-\delta_1+\delta} f'_1(x)^2\,\mathrm{d}x.$$

It is evident that

$$|F'_\delta(x)^2 - f'_1(x)^2| \le 2|f'_1(x)(F'_\delta(x) - f'_1(x))| + (F'_\delta(x) - f'_1(x))^2. \tag{4.10}$$

Since $F_\delta = f_1 * \phi_\delta$ and $f_1 \in C^1$, we have $F'_\delta = f'_1 * \phi_\delta$. By applying Lemma 4.4 we infer that

$$|F'_\delta(x) - f'_1(x)| \le \sup_{|r|\le\delta} |f'_1(x) - f'_1(x-r)|. \tag{4.11}$$

We partition the integral range of E as follows:

$$[0, 1-\delta_1+\delta] = [0,\delta] \cup [\delta, 1-\delta_1-\delta_3-\delta] \cup [1-\delta_1-\delta_3-\delta, 1-\delta_1+\delta].$$

On $I_1 = \{0 \le x \le \delta\}$, we have, by (4.11) and (4.6),

$$|F'_\delta(x) - f'_1(x)| \ll 1/\delta_3.$$

Hence, by (4.10),

$$\Big| \int_{I_1} (F'_\delta)^2 - (f'_1)^2 \Big| \ll \int_{I_1} |f'_1(F'_\delta - f'_1)| + \int_{I_1} (F'_\delta - f'_1)^2 \ll \delta/\delta_3^2. \tag{4.12}$$

On $I_2 = \{\delta \le x \le 1-\delta_1-\delta_3-\delta\}$, by (4.11) and (4.5) we have $F'_\delta(x) = f'_1(x)$. Hence

$$\int_{I_2} (F'_\delta)^2 = \int_{I_2} (f'_1)^2. \tag{4.13}$$

On $I_3 = \{1 - \delta_1 - \delta_3 - \delta \le x \le 1 - \delta_1 + \delta\}$ we have, by (4.11) and (4.5),

$$|F'_\delta(x) - f'_1(x)| \ll 1/\delta_3.$$

Hence, by (4.10),

$$\Big|\int_{I_3} (F'_\delta)^2 - (f'_1)^2\Big| \ll \int_{I_3} |f'_1(F'_\delta - f'_1)| + \int_{I_3} (F'_\delta - f'_1)^2 \ll \delta/\delta_3^2. \tag{4.14}$$

We gather the estimates (4.12)–(4.14) to deduce that

$$\Big|\int_0^{+\infty} (F'_\delta)^2 - \int_0^{+\infty} (f'_1)^2\Big| \ll \delta/\delta_3^2.$$

This bound, together with (4.4), yields (4.9).

Step 4. In this rescaling step, we first recall that $\epsilon > 0$ is given and $0 < \delta \le \delta_3 \le \delta_2 \le \delta_1 < 1/2$ are to be determined. We will make these parameters sufficiently small compared to each other, for instance set $\delta_3 = \delta_2^2, \delta = \delta_3^3 = \delta_2^6$ and then make δ_1, δ_2 sufficiently small compared to $\epsilon > 0$. An easy rescaling of F_δ, using (4.7)–(4.9), yields a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} F \subset [-\epsilon, 1]$, $F(0) = 1$, and $|\int_0^{+\infty} F'(x)^2\,dx - 1| \le \epsilon$. This completes the proof of the proposition. ∎

5. Proof of the main theorem. We now deduce an estimate for $\pi_{R,k}(\mathcal{N})$.

Corollary 5.1. *If $k \ge 0$ and $N^{\epsilon_0} < R < N$, then*

$$\pi_{R,k}(\mathcal{N}) \le \frac{N(\log\log N)^k}{(k!)^2 \log R} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)\frac{(1+ix_1)(1+ix_2)}{(2+ix_1+ix_2)}\,dx_1\,dx_2 + O\Big(R^2(\log N)^{2k} + \frac{N(\log\log N)^{k-1}}{\log N}\Big).$$

Proof. This is a consequence of Propositions 2.1 and 3.14. ∎

Proof of Theorem 1.1. By [16, Lemma 3.5], we deduce that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)f(x_2)\frac{(1+ix_1)(1+ix_2)}{(2+ix_1+ix_2)}\,dx_1\,dx_2 = \int_0^{\infty} (F'(x))^2\,dx.$$

Therefore, by Corollary 5.1,

$$\pi_{R,k}(\mathcal{N}) \le \frac{N(\log\log N)^k}{(k!)^2 \log R}\int_0^{\infty} F'(x)^2\,dx + O\Big(R^2(\log N)^{2k} + \frac{N(\log\log N)^{k-1}}{\log N}\Big).$$

By Proposition 4.1, for every $\epsilon > 0$ we have

$$\pi_{R,k}(\mathcal{N}) \le (1+\epsilon)\frac{N(\log\log N)^k}{(k!)^2 \log R} + O\Big(R^2(\log N)^{2k} + \frac{N(\log\log N)^{k-1}}{\log N}\Big). \ ∎$$

Acknowledgements. We would like to thank Jeffrey C. Lagarias for his literature help. We are grateful to Djordje Milićević for his helpful comments and to the anonymous referee for the suggestions that helped improve the presentation of the paper.

References

[1] H. Delange, *Sur des formules dues à Atle Selberg*, Bull. Sci. Math. (2) 83 (1959), 101–111.

[2] H. Delange, *Sur des formules de Atle Selberg*, Acta Arith. 19 (1971), 105–146.

[3] S. W. Graham, *An upper bound weighted sieve*, in: Recent Progress in Analytic Number Theory, Vol. 1 (Durham, 1979), Academic Press, London, 1981, 47–60.

[4] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quart. J. Math. 48 (1917), 76–92.

[5] D. Hensley, *An almost-prime sieve*, J. Number Theory 10 (1978), 250–262; Corrigendum, ibid. 12 (1980), 437.

[6] A. Hildebrand and G. Tenenbaum, *On the number of prime factors of an integer*, Duke Math. J. 56 (1988), 471–501.

[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Grundlehren Math. Wiss. 256, Springer, 1983.

[8] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 1, Teubner, Leipzig, 1909.

[9] J. Maynard, *Small gaps between primes*, Ann. of Math. (2) 181 (2015), 383–413.

[10] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, 2006.

[11] D. H. J. Polymath, *Variants of the Selberg sieve, and bounded intervals containing many primes*, Res. Math. Sci. 1 (2014), art. 12, 83 pp.

[12] C. Pomerance, *On the distribution of round numbers*, in: K. Aladi (ed.), Number Theory, Lecture Notes in Math. 1122, Springer, Berlin, 1984, 173–200.

[13] L. G. Sathe, *On a problem of Hardy and Ramanujan on the distribution of integers having a given number of prime factors*, J. Indian Math. Soc. 17 (1953), 63–141.

[14] A. Selberg, *Note on a paper of L. G. Sathe*, J. Indian Math. Soc. 18 (1954), 83–87.

[15] T. Tao, *254A, Notes 4: Some sieve theory*, https://terrytao.wordpress.com/2015/01/21/254a-notes-4-some-sieve-theory/, 2015.

[16] A. Vatwani, *A higher rank Selberg sieve and applications*, Czechoslovak Math. J. 68 (143) (2018), 169–193.

[17] D. Wolke and T. Zhan, *On the distribution of integers with a fixed number of prime factors*, Math. Z. 213 (1993), 133–147.

[18] Y. Zhang, *Bounded gaps between primes*, Ann. of Math. (2) 179 (2014), 1121–1174.

Hieu T. Ngo
Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City, Vietnam
E-mail: ngotrunghieu@tdt.edu.vn