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POTENTIAL THEORY

Condensers with infinitely many touching Borel plates and minimum energy problems

by

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Dedicated to the memory of Professor Bogdan Bojarski

Summary. Defining a condenser in a locally compact space as a locally finite, countable collection of Borel sets A_i, $i \in I$, with the sign $s_i = \pm 1$ prescribed such that $A_i \cap A_j = \emptyset$ whenever $s_i s_j = -1$, we consider a minimum energy problem with an external field over infinite-dimensional vector measures $(\mu^i)_{i\in I}$, where μ^i is a suitably normalized positive Radon measure carried by A_i and such that $\mu^i \leq \xi^i$ for all $i \in I_0$, $I_0 \subset I$ and constraints ξ^i, $i \in I_0$, being given. If $I_0 = \emptyset$, the problem reduces to the (unconstrained) Gauss variational problem, which is in general unsolvable even for a condenser of two closed, oppositely signed plates in $\mathbb{R}^3$ and the Coulomb kernel. Nevertheless, we provide sufficient conditions for the existence of solutions to the stated problem in its full generality, establish the vague compactness of the solutions, analyze their uniqueness, describe their weighted potentials, and single out their characteristic properties. The strong and the vague convergence of minimizing nets to the minimizers is studied. The phenomena of non-existence and non-uniqueness of solutions to the problem are illustrated by examples. The results obtained are new even for the classical kernels on $\mathbb{R}^n$, $n \geq 2$, and closed A_i, $i \in I$, which is important for applications.

1. Introduction. The interest in minimum energy problems with external fields, initially inspired by Gauss [23] and further experiencing a new growth due to work of Frostman [15] and Polish and Japanese mathematicians (Leja et al. and Ohtsuka; see [31, 37] and the references cited therein), has been motivated by their direct relations with the Dirichlet and bal-

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ayage problems. A new impulse to this part of potential theory (which is often referred to as *Gauss variational problem* or *weighted minimum energy problems*) came in the 1980's when Gonchar and Rakhmanov [24, 26] and Mhaskar and Saff [33] applied logarithmic potentials with external fields in the investigation of orthogonal polynomials and rational approximations to analytic functions.

In the present paper we study weighted minimum energy problems in a very general setting, over infinite-dimensional vector measures on a locally compact (Hausdorff) space (l.c.s.) X [3, Chapter I, Section 9, n° 7], associated with a generalized condenser. To be precise, a *generalized condenser* $\mathbf{A}$ in X is a locally finite, countable collection of *Borel* sets $A_i \subset X$, $i \in I$, termed *plates*, with the sign $s_i := \operatorname{sign} A_i = \pm 1$ prescribed such that $A_i \cap A_j = \emptyset$ whenever $s_i s_j = -1$. We emphasize that although any two oppositely charged plates of a generalized condenser are disjoint, their closures in X may have points in common. A generalized condenser $\mathbf{A}$ is said to be *standard* if the A_i, $i \in I$, are closed in X. The concept of a standard condenser with infinitely many (closed) plates has been introduced first in our earlier study [43], while that of a generalized condenser seems to be new. *Unless explicitly stated otherwise, when speaking of a condenser, we shall tacitly assume it is generalized.*

We denote by $\mathfrak{M}(X)$ the linear space of all real-valued scalar Radon measures on X, equipped with the *vague topology*, i.e., the (Hausdorff) topology of pointwise convergence on the class $C_0(X)$ of all continuous functions on X with compact support ([1]). For any set $Q \subset X$, let $\mathfrak{M}^+(Q)$ stand for the cone of all *positive* $\nu \in \mathfrak{M}(X)$ *carried by* Q (for a definition, see Section 2.1 below). These and other notions of the theory of measures and integration on a l.c.s., to be used throughout the paper, can be found in [14, 4]; see also [16] for a short survey.

A vector measure $\boldsymbol{\mu} = (\mu^i)_{i \in I}$ is said to be *associated* with a (generalized) condenser $\mathbf{A} = (A_i)_{i \in I}$ if $\mu^i \in \mathfrak{M}^+(A_i)$ for all $i \in I$. Denoting by $\mathfrak{M}^+(\mathbf{A})$ the class of all those $\boldsymbol{\mu}$, we thus have ([2])

$$\mathfrak{M}^+(\mathbf{A}) := \prod_{i \in I} \mathfrak{M}^+(A_i).$$

The trace of the vague product space topology on $\mathfrak{M}^+(X)^{\operatorname{Card} I}$ on $\mathfrak{M}^+(\mathbf{A})$ is likewise called the *vague topology* on $\mathfrak{M}^+(\mathbf{A})$.

For any topological space Y, let $\Psi(Y)$ consist of all lower semicontinuous (l.s.c.) functions $\psi : Y \to (-\infty, \infty]$, nonnegative unless Y is compact.

([1]) When speaking of a continuous function, we understand that the values are *finite* real numbers.

([2]) If I is a singleton, we keep the normal fonts instead of the bold ones.

A *kernel* on X is defined as a symmetric function $\kappa \in \Psi(X \times X)$. In the present paper we shall be concerned with a *positive definite* kernel κ, which means that the *energy* $\kappa(\nu, \nu) := \int \kappa(x, y)\, d(\nu \otimes \nu)(x, y)$ of any (signed) $\nu \in \mathfrak{M}(X)$ is nonnegative whenever defined. (By definition, $\kappa(\nu, \nu)$ is well defined provided that $\kappa(\nu^+, \nu^+) + \kappa(\nu^-, \nu^-)$ or $\kappa(\nu^+, \nu^-)$ is finite, ν^+ and ν^- being respectively the positive and negative parts in the Hahn–Jordan decomposition of ν.) Then the set $\mathcal{E}_\kappa(X)$ of all $\nu \in \mathfrak{M}(X)$ with $\kappa(\nu, \nu)$ finite is a pre-Hilbert space with the inner product

$$\langle \mu, \nu \rangle_\kappa := \kappa(\mu, \nu) := \int \kappa(x, y)\, d(\mu \otimes \nu)(x, y), \qquad \mu, \nu \in \mathcal{E}_\kappa(X),$$

and the seminorm $\|\nu\|_\kappa := \sqrt{\kappa(\nu, \nu)}$. The topology on $\mathcal{E}_\kappa(X)$ determined by $\|\cdot\|_\kappa$ is termed *strong*. A (positive definite) kernel κ is said to be *strictly positive definite* if the seminorm $\|\cdot\|_\kappa$ is a norm.

In accordance with an electrostatic interpretation of a condenser, assume that the interaction between the components μ^i, $i \in I$, of $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is characterized by the matrix $(s_i s_j)_{i,j \in I}$, so that the *energy* of $\boldsymbol{\mu}$ is given by [3]

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) := \sum_{i,j \in I} s_i s_j \kappa(\mu^i, \mu^j). \tag{1.1}$$

Let $\mathcal{E}^+_\kappa(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ with $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$ finite (see footnote 3).

To define admissible measures in the extremal problem we shall be dealing with, fix a numerical vector $\mathbf{a} = (a_i)_{i \in I}$ with $a_i > 0$, a vector-valued function $\mathbf{g} = (g_i)_{i \in I}$ with continuous $g_i : X \to (0, \infty)$, and a vector-valued *external field* $\mathbf{f} = (f_i)_{i \in I}$ with universally measurable $f_i : X \to [-\infty, \infty]$. Let $\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ such that $\langle g_i, \mu^i \rangle := \int g_i\, d\mu^i = a_i$ for all $i \in I$ and $\langle \mathbf{f}, \boldsymbol{\mu} \rangle := \sum_{i \in I} \langle f_i, \mu^i \rangle$ is finite (see footnote 3); then so is the *weighted energy*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) := \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2\langle \mathbf{f}, \boldsymbol{\mu} \rangle, \qquad \boldsymbol{\mu} \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Fix also $I_0 \subset I$ and $\xi^i \in \mathfrak{M}^+(A_i)$, $i \in I_0$, such that $\langle g_i, \xi^i \rangle > a_i$; these ξ^i, $i \in I_0$, will serve as (upper) *constraints* acting on positive measures carried by A_i, $i \in I_0$. *We shall be concerned with the problem of minimizing the weighted energy $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})$ over all $\boldsymbol{\mu} \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with the additional property that $\mu^i \le \xi^i$ for all $i \in I_0$.*

If $I_0 = \emptyset$, the problem reduces to the (unconstrained) Gauss variational problem, which is in general unsolvable even for a standard condenser of

(3) An expression $\sum_{i \in I} c_i$ involving numerical values c_i is meant to be well defined provided that every summand is so and the sum does not depend on the order of summation. By the Riemann series theorem, the sum is finite if and only if the series converges absolutely. Thus, $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$ is finite if κ is $(\mu^i \otimes \mu^j)$-integrable for all $i, j \in I$ and the series in (1.1) converges absolutely.

two closed, oppositely charged plates in $\mathbb{R}^n$, $n \geq 3$, and the Riesz kernels $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$, $\alpha \in (0,n)$. (Here, $|x-y|$ denotes the Euclidean distance between $x, y \in \mathbb{R}^n$.) See Theorem 1.6 below providing necessary and sufficient conditions for the solvability of this problem for $\alpha \in (0,2]$. The phenomenon of unsolvability is illustrated by Example 1.7.

Nevertheless, we provide sufficient conditions for the existence of solutions to the stated problem in its full generality and establish the vague compactness of the solutions (Theorems 6.1, 6.3, and 6.5), analyze their uniqueness (Section 4.2), describe their weighted potentials, and single out their characteristic properties (Theorem 8.2 and Corollary 8.3). The strong and the vague convergence of minimizing nets to the minimizers is also studied (Eq. (6.5) and Corollary 6.7). We discover the phenomenon of non-uniqueness of solutions to the problem, which is illustrated by Example 4.6.

Remark 1.1. The results obtained are new even for the classical kernels on $\mathbb{R}^n$, $n \geq 2$ (in particular, for $-\log|x-y|$ on $\mathbb{R}^2$), and closed A_i, $i \in I$, which is important for applications. While our investigation is focused on theoretical aspects in a very general context, and possible applications are outside the scope of the present paper, it is worth remarking that minimum energy problems in the constrained and unconstrained settings for the logarithmic kernel and finite-dimensional vector measures have been considered for several decades in relation to Hermite–Padé approximants [25, 1] and random matrix ensembles [29, 2].

The results of the present paper, mentioned above, are obtained for a condenser with *nearly closed* plates, which differ from closed sets in a set of zero inner capacity $c_\kappa(\cdot)$ (Definition 2.8) (4). Nevertheless, we develop an efficient approach to the study of energies and potentials of infinite-dimensional vector measures for an *arbitrary* generalized condenser (Section 3), which we intend to use in future work.

The approach developed is based on the observation that, since $(A_i)_{i\in I}$ is locally finite, the A_i, $i \in I$, are Borel, and $A_i \cap A_j = \emptyset$ whenever $s_i s_j = -1$, the mapping

$$\mathfrak{M}^+(\mathbf{A}) \ni \boldsymbol{\mu} \mapsto R\boldsymbol{\mu} := \sum_{i\in I} s_i \mu^i$$

maps $\mathfrak{M}^+(\mathbf{A})$ onto a certain set of *signed* scalar Radon measures on X. Furthermore, $\mathcal{E}^+_\kappa(\mathbf{A})$ becomes a *semimetric* space with the semimetric

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+_\kappa(\mathbf{A})} := \Big[\sum_{i,j\in I} s_i s_j \kappa(\mu_1^i - \mu_2^i, \mu_1^j - \mu_2^j)\Big]^{1/2}, \tag{1.2}$$

and R maps $\mathcal{E}^+_\kappa(\mathbf{A})$ *isometrically* onto its (scalar) R-image, contained in

(4) These closed sets may not form a condenser.

the pre-Hilbert space $\mathcal{E}_\kappa(X)$ (see Section 3.5). In view of this isometry, the topology on the semimetric space $\mathcal{E}^+_\kappa(\mathbf{A})$ is likewise termed *strong.*

Another fact crucial to our approach is a strong completeness result for a certain subspace of $\mathcal{E}^+_\kappa(\mathbf{A})$, where $\mathbf{A}$ is a standard condenser (see Theorem 1.2 below, established in our earlier paper [43]). Let A^+, resp. A^-, denote the union of the A_i, $i \in I$, with $s_i = +1$, resp. $s_i = -1$. Write

$$\mathcal{E}^+_\kappa(\mathbf{A}, \leq \mathbf{a}, \mathbf{g}) := \{\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A}) : \langle g_i, \mu^i\rangle \leq a_i \text{ for all } i \in I\}.$$

THEOREM 1.2. *Assume the A_i, $i \in I$, are closed, κ is consistent* ([5]), *and*

$$\sum_{i\in I} a_i g_{i,\inf}^{-1} := C < \infty, \quad \textit{where} \quad g_{i,\inf} := \inf_{x\in A_i} g_i(x). \tag{1.3}$$

If moreover $\kappa|_{A^+\times A^-}$ is upper bounded, then the following assertions hold.

- $\mathcal{E}^+_\kappa(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ *is complete in the induced strong topology. In more detail, any strong Cauchy net in $\mathcal{E}^+_\kappa(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ converges strongly to any of its vague cluster points.*
- *If moreover κ is strictly positive definite and the A_i, $i \in I$, are mutually disjoint, then the strong topology on $\mathcal{E}^+_\kappa(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ is finer than the induced vague topology.*

1.1. Minimum α-Riesz energy problem for a standard condenser. We next show that the problem in question is in general unsolvable even in the case where $\mathbf{A} = (A_1, A_2)$ is a standard condenser in $\mathbb{R}^n$, $n \geq 3$, with $s_1 = +1$, $s_2 = -1$, $f_1 \equiv f_2 \equiv 0$, $g_1 \equiv g_2 \equiv 1$, $a_1 = a_2 = 1$, $I_0 = \emptyset$, and $\kappa(x, y) := \kappa_\alpha(x, y) := |x - y|^{\alpha-n}$, $\alpha \in (0, 2]$. Under these requirements, the problem can equivalently be rewritten as follows:

$$w_\alpha(\mathbf{A}) := \inf \kappa_\alpha(\mu^1 - \mu^2, \mu^1 - \mu^2), \tag{1.4}$$

where μ^i, $i = 1, 2$, ranges over the class

$$\mathcal{E}^+_{\kappa_\alpha}(A_i, 1) := \{\nu \in \mathfrak{M}^+(A_i) \cap \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n) : \nu(A_i) = 1\}.$$

To formulate the corresponding result and to briefly explain the phenomenon of unsolvability, we first recall the concept of α-thinness at infinity.

Throughout Section 1.1, F denotes a closed set in $\mathbb{R}^n$, $n \geq 3$, such that $F^c := \mathbb{R}^n \setminus F \neq \emptyset$, and F^* the inverse of F relative to $\{x \in \mathbb{R}^n : |x - x_0| = 1\}$, $x_0 \in F^c$ being fixed. Let ν^F stand for the α-Riesz swept measure of $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$ onto F, determined uniquely by [20, Theorem 3.6].

DEFINITION 1.3. F is said to be *α-thin at infinity* if any of the following four equivalent conditions holds:

(i) F^* is α-thin at x_0.
(ii) Either F is compact, or x_0 is an α-irregular boundary point of F^*.

([5]) We refer to [16, 18] for the concept of *consistency* (see also Section 2.2 below).

(iii) If F_k denotes $F \cap \{x \in \mathbb{R}^n : q^k \leq |x - x_0| < q^{k+1}\}$, where $q \in (1, \infty)$, then

$$\sum_{k \in \mathbb{N}} \frac{c_{\kappa_\alpha}(F_k)}{q^{k(n-\alpha)}} < \infty. \tag{1.5}$$

(iv) There is a connected component D of F^c such that for every nonzero $\nu \in \mathfrak{M}^+(D)$ we have $\nu^F(\mathbb{R}^n) < \nu(\mathbb{R}^n)$.

The equivalence of (i) and (ii) is due to [6, Theorem VII.13] or [30, Theorem 5.10], that of (ii) and (iii) holds by the Wiener criterion, and that of (iii) and (iv) has been established in [45, Theorems 8.6, 8.7] (see also earlier papers [39, Theorem B] and [40, Theorem 4]).

Theorem 1.4. *If F is not α-thin at infinity, then $c_{\kappa_\alpha}(F) = \infty$. This cannot be reversed, i.e., there is F with $c_{\kappa_\alpha}(F) = \infty$ that is α-thin at infinity.*

Proof. According to [30, Lemma 5.5], $c_{\kappa_\alpha}(F) < \infty \Leftrightarrow \sum_{k \in \mathbb{N}} c_{\kappa_\alpha}(F_k) < \infty$, F_k being defined in Definition 1.3(iii). When compared with (1.5), this yields the theorem. ■

Remark 1.5. For $\alpha = 2$, the concept of α-thinness at infinity thus defined is, in fact, equivalent to that of Doob [10, pp. 175–176], while for $\alpha \neq 2$, it seems to appear first in our earlier work [40]. Due to its deep relation to balayage, it plays an important role in the investigation of condenser problems in Riesz potential theory (see e.g. [11, 21, 22]; for an illustration, see Example 1.7 below). Note that for $\alpha = 2$, a different concept of α-thinness at infinity has been introduced by Brelot [5, p. 313], which is actually more restrictive than Doob's (equivalently, our) concept. Indeed, a closed set $F \subset \mathbb{R}^n$ is 2-thin at infinity in the sense of Brelot if and only if $c_2(F) < \infty$ (see [8, p. 277, footnote] or [6, Chapter IX, Section 6]); while according to Theorem 1.4, $c_\alpha(F) < \infty$ is only sufficient, but not necessary for F to be α-thin at infinity in the sense of our Definition 1.3. We emphasize that each of items (i)–(iv) in Definition 1.3 is indeed equivalent to the existence of the α-Riesz equilibrium measure γ_F on F, but treated in an *extended* sense where $\gamma_F(F) = \kappa_\alpha(\gamma_F, \gamma_F) = \infty$ is allowed; see [45, Section 5] for details.

Returning to problem (1.4), we can certainly assume that $c_{\kappa_\alpha}(A_i) > 0$, $i = 1, 2$, for if not, then $w_\alpha(\mathbf{A}) = +\infty$, and hence the problem makes no sense. There is also no loss of generality in assuming $c_{\kappa_\alpha}(A_1) < \infty$, because if $c_{\kappa_\alpha}(A_i) = \infty$ for $i = 1, 2$, then $w_\alpha(\mathbf{A}) = 0$; and hence this infimum cannot be an actual minimum, κ_α being strictly positive definite [30, Theorem 1.15].

Theorem 1.6 (see [40, Theorem 5]). *Assume, for simplicity, A_2^c is connected. If moreover the Euclidean distance between A_1 and A_2 is > 0, then problem* (1.4) *is* (*uniquely*) *solvable if and only if either $c_{\kappa_\alpha}(A_2) < \infty$, or A_2 is not α-thin at infinity.*

It follows that if A_2 is α-thin at infinity, but $c_{\kappa_\alpha}(A_2) = \infty$ (such an A_2 exists by Theorem 1.4), then $w_\alpha(\mathbf{A})$ cannot be attained among the admissible measures. The reason is that, under the quoted assumptions, any minimizing sequence converges strongly and vaguely to a (unique) $\gamma = \gamma^+ - \gamma^-$ such that $\gamma^+ \in \mathcal{E}^+_{\kappa_\alpha}(A_1, 1)$, while $\gamma^- = (\gamma^+)^{A_2}$ [40, Eq. (27)]. Since A_2 is α-thin at infinity, we get $(\gamma^+)^{A_2}(A_2) < 1$ by Definition 1.3(iv), and problem (1.4) therefore has *no* solution.

EXAMPLE 1.7. Let $n = 3$ and $\alpha = 2$. Define A_2 to be a rotation body

$$A_2 := \{x \in \mathbb{R}^3 : 0 \le x_1 < \infty,\ x_2^2 + x_3^2 \le \varrho^2(x_1)\},$$

where ϱ is given by one of the following three formulae:

$$\varrho(x_1) = x_1^{-s} \quad \text{with } s \in [0, \infty), \tag{1.6}$$

$$\varrho(x_1) = \exp(-x_1^s) \quad \text{with } s \in (0, 1], \tag{1.7}$$

$$\varrho(x_1) = \exp(-x_1^s) \quad \text{with } s \in (1, \infty), \tag{1.8}$$

and let A_1 be a closed ball in $\mathbb{R}^3 \setminus A_2$. Then A_2 is not 2-thin at infinity if ϱ is defined by (1.6), A_2 is 2-thin at infinity but has infinite Newtonian capacity if ϱ is given by (1.7), and finally $c_{\kappa_2}(A_2) < \infty$ if (1.8) holds [41, Example 5.3]. By Theorem 1.6, problem (1.4) is therefore solvable for $\mathbf{A} = (A_1, A_2)$ if A_2 is determined either by (1.6), or by (1.8), but problem (1.4) is *unsolvable* if A_2 is given by (1.7) (see Figure 1.1).

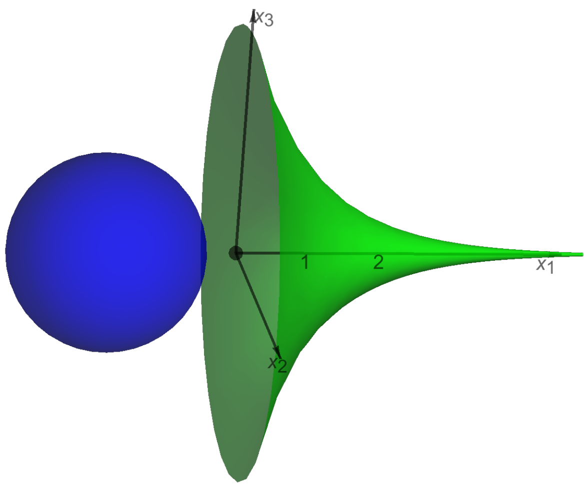

Fig. 1. A standard condenser $\mathbf{A} = (A_1, A_2)$ in $\mathbb{R}^3$, where $A_2 = \{0 \le x_1 < \infty,\ x_2^2 + x_3^2 \le \rho^2(x_1)\}$ with $\rho(x_1) = \exp(-x_1)$ and A_1 is a closed ball in $\mathbb{R}^3 \setminus A_2$

REMARK 1.8. Theorem 1.6 and Example 1.7 have been illustrated in [35, 27] by means of numerical experiments.

2. Preliminaries

2.1. Measures, vague convergence, capacity. We shall tacitly use the notation of Section 1. The vague topology on $\mathfrak{M}(X)$ in general does not possess a countable base, and hence it cannot be described in terms of convergence of sequences. We follow Moore and Smith's theory of convergence, based on the concept of *nets* [34] (see also [28, Chapter 2] and [14, Chapter 0]). However, if X is metrizable and *countable at infinity*, where the latter means that X can be written as a countable union of compact sets [3, Chapter I, Section 9, n° 9], then $\mathfrak{M}(X)$ satisfies the first axiom of countability [19, Remark 2.4], and the use of nets may be avoided.

LEMMA 2.1 (see, e.g., [16, Section 1.1]). *For any $\psi \in \Psi(X)$ the map $\nu \mapsto \langle\psi, \nu\rangle$ is vaguely l.s.c. on $\mathfrak{M}^+(X)$.*

Let a set $Q \subset X$ and a measure $\nu \in \mathfrak{M}^+(X)$ be given. If Q is ν-measurable, then the indicator function 1_Q of Q is locally ν-integrable, and hence one can consider the *trace* (restriction) $\nu|_Q = 1_Q \cdot \nu$ of ν on Q [14, Section 4.14.7]. As in [14, Section 4.7.3], Q is said to be *ν-σ-finite* if Q is contained in a countable union of ν-integrable open sets ([6]). If Q is open or ν-measurable and ν-σ-finite, then $\nu_*(Q) = \nu^*(Q) \in [0, \infty]$, where $\nu_*(Q)$ and $\nu^*(Q)$ denote the *inner* and the *outer* ν-measure of Q, respectively [14, Eqs. (4.7.3), (4.7.4)]; and we write $\nu(Q) := \nu_*(Q) = \nu^*(Q)$.

LEMMA 2.2. *If Q is ν-measurable and ν-σ-finite, then for any nonnegative l.s.c. function ψ on X we have $\langle\psi, \nu|_Q\rangle = \langle\psi|_Q, \nu\rangle$.*

Proof. Applying first [14, Proposition 4.14.1(b)] and [14, Eq. (4.14.8)] to $\psi|_Q$ and ν, and then applying [14, Proposition 4.14.1(a)] to ψ and $\nu|_Q$, we arrive at our claim. ■

THEOREM 2.3. *Let X be metrizable and countable at infinity. If a sequence $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}^+(X)$ converges to ν vaguely, then for any relatively compact Borel set $Q \subset X$ with $\nu(\partial_X Q) = 0$ we have $\nu_k|_Q \to \nu|_Q$ vaguely as $k \to \infty$* ([7]).

Proof. The Portmanteau theorem in the form stated in [32, Theorem 2.1] shows that under the hypotheses of Theorem 2.3,

$$\lim_{k\to\infty} \nu_k(Q) = \nu(Q).$$

Applying now to X, Q, ν_k and ν the same arguments as in [30, proof of Theorem 0.5′], the only difference being in using the preceding display in place of [30, Theorem 0.5], we establish the theorem. ■

([6]) This necessarily holds if X is countable at infinity or ν is *bounded*, i.e., $\nu(X) < \infty$.

([7]) If X is an open subset of $\mathbb{R}^n$, $n \geq 2$, then Theorem 2.3 is, in fact, [30, Theorem 0.5′].

Let $\mathfrak{M}^+(Q)$ consist of all $\nu \in \mathfrak{M}^+(X)$ *carried by* Q, which means that $Q^c := X \setminus Q$ is locally ν-negligible, or equivalently that Q is ν-measurable and $\nu = \nu|_Q$. If Q^c is open or ν-σ-finite, then the concept of local ν-negligibility for Q^c coincides with that of ν-negligibility; and hence $\nu \in \mathfrak{M}^+(Q)$ if and only if $\nu^*(Q^c) = 0$. Therefore, ν is carried by a *closed* Q if and only if it is supported by Q, that is, $S(\nu) \subset Q$, where $S(\nu)$ is the support of ν.

In all that follows, κ is a *positive definite* kernel on X (Section 1). For any $Q \subset X$, write $\mathcal{E}^+_\kappa(Q) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q)$. The (*inner*) *capacity* of Q is given by the formula

$$c_\kappa(Q) := \Big[\inf_{\nu \in \mathcal{E}^+_\kappa(Q):\, \nu(Q)=1} \kappa(\nu, \nu) \Big]^{-1} \tag{2.1}$$

(see, e.g., [16, 37]). Then $0 \le c_\kappa(Q) \le \infty$. (As usual, the infimum over the empty set is taken to be $+\infty$. We also set $1/(+\infty) = 0$ and $1/0 = +\infty$.)

A proposition $\mathcal{P}(x)$ involving a variable point $x \in X$ is said to hold c_κ-*nearly everywhere* (c_κ-*n.e.*) on Q if $c_\kappa(N) = 0$, where N consists of all $x \in Q$ for which $\mathcal{P}(x)$ fails. We write briefly 'n.e.' in place of 'c_κ-n.e.' if this does not cause any misunderstanding, and we omit 'on Q' if $Q = X$.

Lemma 2.4 (see [16, Lemma 2.3.1]). $c_\kappa(Q) = 0 \Leftrightarrow \mathcal{E}^+_\kappa(Q) = \{0\}$.

2.2. Consistent and perfect kernels. In addition to the strong topology on $\mathcal{E}_\kappa(X)$, determined by the seminorm $\|\cdot\|_\kappa$ (see Section 1), it is often useful to consider the so-called *weak* topology on $\mathcal{E}_\kappa(X)$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \mu)|$, where $\mu \in \mathcal{E}_\kappa(X)$ [16]. By the Cauchy–Schwarz (Bunyakovski) inequality

$$|\kappa(\mu, \nu)| \le \|\mu\|_\kappa \cdot \|\nu\|_\kappa, \quad \text{where } \mu, \nu \in \mathcal{E}_\kappa(X),$$

the strong topology on $\mathcal{E}_\kappa(X)$ is *finer* than the weak topology.

Following Fuglede [16, 18], we call a (positive definite) kernel κ *consistent* if it satisfies either of the following two *equivalent* properties:

(C_1) *Every strong Cauchy net in* $\mathcal{E}^+_\kappa(X)$ *converges strongly to any of its vague cluster points* (*whenever these exist*).

(C_2) *Every strongly bounded and vaguely convergent net in* $\mathcal{E}^+_\kappa(X)$ *converges weakly to its vague limit.*

A kernel κ is called *perfect* if it is consistent and strictly positive definite [16, Theorem 3.3], or equivalently if the following two conditions are fulfilled (see [16, p. 166]):

(P_1) $\mathcal{E}^+_\kappa(X)$ *is complete in the induced strong topology.*

(P_2) *The strong topology on* $\mathcal{E}^+_\kappa(X)$ *is finer than the induced vague topology on* $\mathcal{E}^+_\kappa(X)$.

Example 2.5. On $X = \mathbb{R}^n$, $n \geq 3$, the α-Riesz kernel κ_α, $\alpha \in (0, n)$, is strictly positive definite and consistent, and hence perfect [9]; thus so is the Newtonian kernel $\kappa_2(x, y) = |x - y|^{2-n}$ [7]. Recently it has been shown that if $X = D$ where D is an arbitrary open set in $\mathbb{R}^n$, $n \geq 3$, and G^α_D, $\alpha \in (0, 2]$, is the α-Green kernel on D [30, Chapter IV, Section 5], then $\kappa = G^\alpha_D$ is likewise perfect [20, Theorems 4.9, 4.11]. Furthermore, the 2-Green kernel on a planar 2-Greenian set is strictly positive definite by [10, Chapter XIII, Section 7] and it is consistent by [13], and hence perfect. The logarithmic kernel $-\log|x - y|$ on a closed disc in $\mathbb{R}^2$ of radius < 1 is strictly positive definite, as shown in [30, Theorem 1.16]. It is therefore perfect (see [36]), because it satisfies Frostman's maximum principle by [30, Theorem 1.6], and hence is regular by [37, Eq. (1.3)]. For analogous results concerning the logarithmic kernel on closed balls of arbitrary finite dimension, see [17].

Remark 2.6. In contrast to (P_1), for a perfect kernel κ the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly *incomplete*, and this is the case even for the α-Riesz kernel of order $\alpha \in (1, n)$ on $\mathbb{R}^n$, $n \geq 3$ [7].

Remark 2.7. The concept of consistent kernel is an efficient tool in minimum energy problems over classes of *positive scalar* Radon measures with finite energy. Indeed, if Q is closed, $c_\kappa(Q) \in (0, \infty)$, and κ is consistent, then the minimum energy problem in (2.1) has a solution λ [16, Theorem 4.1]; we shall call this λ an (*inner*) *κ-capacitary measure* on Q. (This λ is unique if κ is strictly positive definite.) Later the concept of consistency has been shown to be efficient also in minimum energy problems over classes of vector measures of finite or infinite dimensions associated with a standard condenser [41]–[44]. The approach developed in [41]–[44] substantially used the assumption of the boundedness of the kernel on the Cartesian product of the oppositely charged plates of a condenser, which made it possible to extend Cartan's proof [7] of the strong completeness of the cone $\mathcal{E}^+_{\kappa_2}(\mathbb{R}^n)$ of all positive measures on $\mathbb{R}^n$ with finite Newtonian energy to an arbitrary consistent kernel κ on a l.c.s. X and suitable classes of (*signed*) measures $\mu \in \mathcal{E}_\kappa(X)$ (compare with Theorem 1.2 as well as Remark 2.6 above).

2.3. Nearly closed sets. The following concept seems to be new (a private communication by Bent Fuglede).

Definition 2.8. A set $Q \subset X$ is said to be *nearly closed*, resp. *nearly compact*, if there exists a closed, resp. compact, set $\breve{Q} \subset X$ such that

$$c_\kappa(Q \triangle \breve{Q}) = 0, \quad \text{where} \quad Q \triangle \breve{Q} := (Q \setminus \breve{Q}) \cup (\breve{Q} \setminus Q).$$

Lemma 2.9. *For any nearly closed set Q,*

$$\mathcal{E}^+_\kappa(Q) = \mathcal{E}^+_\kappa(\breve{Q}).$$

Proof. Note that $Q = [\breve{Q} \cup (Q \setminus \breve{Q})] \setminus (\breve{Q} \setminus Q)$. Since no set in X with $c_\kappa(\cdot) = 0$ can carry a nonzero measure from $\mathcal{E}^+_\kappa(X)$ (cf. Lemma 2.4), $\mathcal{E}^+_\kappa(Q) \subset \mathcal{E}^+_\kappa(\breve{Q})$. Interchanging Q and $\breve{Q}$, we obtain the converse inclusion. ∎

LEMMA 2.10. *If a set $Q \subset X$ is nearly closed, then the truncated cone $\{\nu \in \mathcal{E}^+_\kappa(Q) : \|\nu\|_\kappa \le 1\}$ is closed in the induced vague topology.*

Proof. As seen from Lemma 2.9, it is enough to establish the lemma for $\breve{Q}$ in place of Q. Since $\mathfrak{M}^+(\breve{Q})$ is vaguely closed, $\breve{Q}$ being closed in X, and since the energy $\kappa(\nu,\nu)$ is vaguely l.s.c. on $\mathfrak{M}^+(X)$ [16, Lemma 2.2.1(e)], the lemma follows. ∎

3. Vector measures. Their energies and potentials

3.1. Vector measures. Fix a countable set I of indices $i \in \mathbb{N}$, and consider the Cartesian product $\mathfrak{M}^+(X)^{\operatorname{Card} I}$, equipped with the vague product space topology. Elements $\boldsymbol{\mu} = (\mu^i)_{i\in I}$ of $\mathfrak{M}^+(X)^{\operatorname{Card} I}$, where $\mu^i \in \mathfrak{M}^+(X)$ for all $i \in I$, are termed positive $(\operatorname{Card} I)$-dimensional *vector measures* on X.

DEFINITION 3.1. A set $\mathfrak{F} \subset \mathfrak{M}^+(X)^{\operatorname{Card} I}$ is said to be *vaguely bounded* if for every $\varphi \in C_0(X)$,

$$\sup_{\boldsymbol{\mu}\in\mathfrak{F}} |\mu^i(\varphi)| < \infty \quad \text{for all } i \in I.$$

LEMMA 3.2. *A vaguely bounded set $\mathfrak{F} \subset \mathfrak{M}^+(X)^{\operatorname{Card} I}$ is vaguely relatively compact.*

Proof. It is clear from the above definition that for every $i \in I$, the set

$$\mathfrak{F}^i := \{\mu^i \in \mathfrak{M}^+(X) : \boldsymbol{\mu} = (\mu^j)_{j\in I} \in \mathfrak{F}\}$$

is vaguely bounded, and hence vaguely relatively compact in $\mathfrak{M}^+(X)$ [4, Chapter III, Section 2, Proposition 9]. Since $\mathfrak{F} \subset \prod_{i\in I} \mathfrak{F}^i$, the lemma follows from Tikhonov's theorem on the product of compact spaces [3, Chapter I, Section 9, Theorem 3]. ∎

Since $\mathfrak{M}^+(X)$ is Hausdorff in the vague topology, so is $\mathfrak{M}^+(X)^{\operatorname{Card} I}$ [3, Chapter I, Section 8, Proposition 7], and hence a vague limit of any net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^+(X)^{\operatorname{Card} I}$ is *unique* if it exists. (Throughout the paper, S denotes an upper directed set of indices s.)

3.2. Generalized and standard condensers. Assume $I = I^+ \cup I^-$, where $I^+ \cap I^- = \emptyset$ and I^- is allowed to be empty, and to every $i \in I$ there corresponds a nonempty *Borel* set $A_i \subset X$.

DEFINITION 3.3. A collection $\mathbf{A} = (A_i)_{i\in I}$ is termed a *generalized (I^+, I^-)-condenser* (or simply a *generalized condenser*) in X if every compact subset of X intersects at most finitely many A_i, and moreover

$$A_i \cap A_j = \emptyset \quad \text{for all } i \in I^+,\ j \in I^-. \tag{3.1}$$

Writing

$$s_i := \operatorname{sign} A_i := \begin{cases} +1 & \text{if } i \in I^+, \\ -1 & \text{if } i \in I^-, \end{cases} \tag{3.2}$$

we call A_i, $i \in I^+$, and A_j, $j \in I^-$, *positive* and *negative plates* of the generalized condenser $\mathbf{A}$. Note that any two equally signed plates may intersect or even coincide. Also note that although any two oppositely signed plates are disjoint by (3.1), their closures in X may intersect (actually, even in a set of nonzero capacity) ([8]). Furthermore, it follows from the above definition that the sets $A^+ := \bigcup_{i\in I^+} A_i$ and $A^- := \bigcup_{j\in I^-} A_j$ are *Borel* and *disjoint*, which will be used substantially in all that follows.

Lemma 3.4. *If the A_i, $i \in I$, are nearly closed, then so are A^+ and A^-.*

Proof. With $\breve{A}_i$, $i \in I$, determined by Definition 2.8 for $Q = A_i$, write

$$\breve{A}^+ := \bigcup_{i\in I^+} \breve{A}_i \quad \text{and} \quad \breve{A}^- := \bigcup_{j\in I^-} \breve{A}_j. \tag{3.3}$$

Then $\breve{A}^\pm$ is closed, for the collection $(\breve{A}_i)_{i\in I^\pm}$ of (closed) sets $\breve{A}_i$ is locally finite. Since $c_\kappa(A_i \bigtriangleup \breve{A}_i) = 0$ for all $i \in I$, the countable subadditivity of inner capacity on Borel sets [16, Lemma 2.3.5] yields $c_\kappa(A^\pm \bigtriangleup \breve{A}^\pm) = 0$. ■

Definition 3.5. A generalized condenser is *standard* if its plates are closed.

Unless explicitly stated otherwise, in all that follows $\mathbf{A} = (A_i)_{i\in I}$ is a generalized condenser in X. Let $\mathfrak{M}^+(\mathbf{A})$ consist of all $\boldsymbol{\mu} = (\mu^i)_{i\in I} \in \mathfrak{M}^+(X)^{\operatorname{Card} I}$ with $\mu^i \in \mathfrak{M}^+(A_i)$ for all $i \in I$. In other words, $\mathfrak{M}^+(\mathbf{A})$ stands for the Cartesian product $\prod_{i\in I} \mathfrak{M}^+(A_i)$, equipped with the vague topology induced from $\mathfrak{M}^+(X)^{\operatorname{Card} I}$. Elements of $\mathfrak{M}^+(\mathbf{A})$ are said to be (vector) measures *associated with* $\mathbf{A}$.

Lemma 3.6. *If a condenser $\mathbf{A}$ is standard, then $\mathfrak{M}^+(\mathbf{A})$ is vaguely closed in $\mathfrak{M}^+(X)^{\operatorname{Card} I}$.*

Proof. Noting that the $\mathfrak{M}^+(A_i)$, $i \in I$, are vaguely closed in $\mathfrak{M}^+(X)$ (A_i being closed in X), we obtain the lemma from [3, Chapter I, Section 4, Corollary to Proposition 7]. ■

3.3. Mapping $R : \mathfrak{M}^+(\mathbf{A}) \to \mathfrak{M}(X)$. Since each compact subset of X has points in common with at most finitely many A_i, for any given $\boldsymbol{\mu} = (\mu^i)_{i\in I} \in \mathfrak{M}^+(\mathbf{A})$ and $\varphi \in C_0(X)$ only a finite number of $\mu^i(\varphi)$ are nonzero. This

([8]) This remains valid even when the A_i, $i \in I$, are nearly closed.

implies that to every positive vector measure $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ there corresponds a unique (signed) scalar Radon measure $R\boldsymbol{\mu} = R(\boldsymbol{\mu}) \in \mathfrak{M}(X)$ such that

$$R\boldsymbol{\mu}(\varphi) = \sum_{i\in I} s_i\mu^i(\varphi) \quad \text{for all } \varphi \in C_0(X),$$

s_i being determined by (3.2). Since the positive scalar measures $\sum_{i\in I^+}\mu^i$ and $\sum_{i\in I^-}\mu^i$ are carried by the nonintersecting Borel sets A^+ and A^-, respectively, these two measures are, in fact, the positive and negative parts in the Hahn–Jordan decomposition of $R\boldsymbol{\mu}$; i.e., $R\boldsymbol{\mu} = (R\boldsymbol{\mu})^+ - (R\boldsymbol{\mu})^-$, where

$$(R\boldsymbol{\mu})^+ := \sum_{i\in I^+}\mu^i \quad \text{and} \quad (R\boldsymbol{\mu})^- := \sum_{i\in I^-}\mu^i.$$

When the dependence of the mapping R on the (generalized) condenser $\mathbf{A}$ needs to be indicated explicitly, we shall write $R_{\mathbf{A}}$ in place of R.

The mapping $\mathfrak{M}^+(\mathbf{A}) \to \mathfrak{M}(X)$ thus defined is in general non-injective, i.e., there exist $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathfrak{M}^+(\mathbf{A})$ such that $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, but $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$. We say that $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathfrak{M}^+(\mathbf{A})$ are *R-equivalent* if $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$. R-equivalence on $\mathfrak{M}^+(\mathbf{A})$ implies identity (and hence these two relations on $\mathfrak{M}^+(\mathbf{A})$ are equivalent) if and only if $A_i \cap A_j = \emptyset$ for all $i \neq j$ (compare with Lemma 3.15 below).

Lemma 3.7. *If a net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^+(\mathbf{A})$ converges vaguely to $\boldsymbol{\mu}_0 \in \mathfrak{M}^+(\mathbf{A})$, then $R\boldsymbol{\mu}_s \to R\boldsymbol{\mu}_0$ vaguely in $\mathfrak{M}(X)$ as s increases along S.*

Proof. This follows directly from the observation that the (compact) support of any $\varphi \in C_0(X)$ can intersect only finitely many A_i. ∎

Remark 3.8. Lemma 3.7 cannot in general be reversed. However, if the A_i, $i \in I$, are closed and mutually disjoint, then for any $(\boldsymbol{\mu}_s)_{s\in S}$ and $\boldsymbol{\mu}_0$ in $\mathfrak{M}^+(\mathbf{A})$, the vague convergence of $(R\boldsymbol{\mu}_s)_{s\in S}$ to $R\boldsymbol{\mu}_0$ implies the vague convergence of $(\boldsymbol{\mu}_s)_{s\in S}$ to $\boldsymbol{\mu}_0$. This follows from the Tietze–Urysohn extension theorem [14, Theorem 0.2.13].

3.4. Energies and potentials of vector measures and those of their scalar R-images. For a (positive definite) kernel κ and vector measures $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$, define the *mutual energy* ([9])

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\nu}) := \sum_{i,j\in I} s_i s_j \kappa(\mu^i, \nu^j) \tag{3.4}$$

and the *vector potential* $\kappa_{\boldsymbol{\mu}}(\cdot)$ as a vector-valued function on X with the components

$$\kappa^i_{\boldsymbol{\mu}}(\cdot) := \sum_{j\in I} s_i s_j \kappa(\cdot, \mu^j), \qquad i \in I, \tag{3.5}$$

([9]) With regard to (3.4) and (3.5), see footnote 3.

where $\kappa(\cdot,\nu) := \int \kappa(\cdot,y)\,d\nu(y)$ denotes the *potential* of $\nu \in \mathfrak{M}(X)$. For $\boldsymbol{\mu} = \boldsymbol{\nu}$, $\kappa(\boldsymbol{\mu},\boldsymbol{\nu})$ becomes the *energy* $\kappa(\boldsymbol{\mu},\boldsymbol{\mu})$ of $\boldsymbol{\mu}$ (cf. (1.1)).

Let $\mathcal{E}^+_\kappa(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ with finite $\kappa(\boldsymbol{\mu},\boldsymbol{\mu})$, which means that κ is $(\mu^i \otimes \mu^j)$-integrable for all $i,j \in I$ and the series $\sum_{i,j\in I} |\kappa(\mu^i,\mu^j)|$ is convergent (the latter can be omitted if X is compact, for then I is finite).

LEMMA 3.9. *For $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ to have finite energy, it is sufficient that $\mu^i \in \mathcal{E}^+_\kappa(A_i)$ for all $i \in I$ and moreover $\sum_{i\in I} \|\mu^i\|_\kappa < \infty$.*

Proof. In fact, applying the Cauchy–Schwarz inequality in $\mathcal{E}_\kappa(X)$, we get

$$\sum_{i,j\in I} |\kappa(\mu^i,\mu^j)| \le \sum_{i,j\in I} \|\mu^i\|_\kappa \|\mu^j\|_\kappa = \Big(\sum_{i\in I} \|\mu^i\|_\kappa\Big)^2. \ \blacksquare$$

The following lemma is crucial for the establishment of relations between energies and potentials of vector measures $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ and those of their (signed scalar) R-images $R\boldsymbol{\mu} \in \mathfrak{M}(X)$.

LEMMA 3.10. *Given a generalized (L^+,L^-)-condenser $\mathbf{B} = (B_\ell)_{\ell\in L}$ in a l.c.s. Y, consider $\boldsymbol{\omega} = (\omega^\ell)_{\ell\in L} \in \mathfrak{M}^+(\mathbf{B})$ and $\psi \in \Psi(Y)$. For ψ to be $|R_{\mathbf{B}}\boldsymbol{\omega}|$-integrable, it is necessary and sufficient that $\sum_{\ell\in L} |\langle\psi,\omega^\ell\rangle| < \infty$; and in that case,*

$$\langle\psi, R_{\mathbf{B}}\boldsymbol{\omega}\rangle = \sum_{\ell\in L} s_\ell \langle\psi,\omega^\ell\rangle.$$

Proof. We can certainly assume L is infinite, for otherwise the lemma is obvious. Then Y is noncompact, and hence ψ is nonnegative. Therefore

$$\langle\psi,(R_{\mathbf{B}}\boldsymbol{\omega})^+\rangle \ge \sum_{\ell\in L^+,\,\ell\le N} \langle\psi,\omega^\ell\rangle \quad \text{for all } N \in L^+.$$

On the other hand, since $\mathbf{B}$ is locally finite, the sum of ω^ℓ over all $\ell \in L^+$ that do not exceed N approaches $(R_{\mathbf{B}}\boldsymbol{\omega})^+$ vaguely as N increases along L^+. Hence, by Lemma 2.1 ([10]),

$$\langle\psi,(R_{\mathbf{B}}\boldsymbol{\omega})^+\rangle \le \lim_{N\in L^+} \sum_{\ell\in L^+,\,\ell\le N} \langle\psi,\omega^\ell\rangle.$$

Combining these two displays and then letting N along L^+, we get

$$\langle\psi,(R_{\mathbf{B}}\boldsymbol{\omega})^+\rangle = \sum_{\ell\in L^+} \langle\psi,\omega^\ell\rangle.$$

Since the same holds for $(R_{\mathbf{B}}\boldsymbol{\omega})^-$ and L^-, the lemma follows by subtraction. ■

([10]) The symbol $\lim_{s\in S}$ denotes a limit as s increases along an upper directed set S.

Corollary 3.11. *Fix* $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathfrak{M}^+(\mathbf{A})$ *and* $x\in X$. *Then*

$$\kappa(R\boldsymbol{\mu},R\boldsymbol{\nu})=\sum_{i,j\in I}s_is_j\kappa(\mu^i,\nu^j),\tag{3.6}$$

$$\kappa(x,R\boldsymbol{\mu})=\sum_{i\in I}s_i\kappa(x,\mu^i),\tag{3.7}$$

each of the identities being understood in the sense that either of its sides is finite whenever so is the other and then they coincide. By (3.4) *and* (3.6) *with* $\boldsymbol{\mu}=\boldsymbol{\nu}$,

$$\boldsymbol{\mu}\in\mathcal{E}^+_\kappa(\mathbf{A})\iff R\boldsymbol{\mu}\in\mathcal{E}_\kappa(X).\tag{3.8}$$

Proof. Relation (3.7) follows directly from Lemma 3.10 with $Y=X$, $\mathbf{B}=\mathbf{A}$, and $\psi(\cdot)=\kappa(x,\cdot)$. We next apply Lemma 3.10 to the (generalized) condenser $\mathbf{A}\times\mathbf{A}:=(A_i\times A_j)_{(i,j)\in I\times I}$ in $X\times X$ with $s_{(i,j)}:=s_is_j$, the function $\psi:=\kappa\in\Psi(X\times X)$, and the vector measure $\boldsymbol{\mu}\otimes\boldsymbol{\nu}\in\mathfrak{M}^+(\mathbf{A}\times\mathbf{A})$, where $\boldsymbol{\mu}\otimes\boldsymbol{\nu}:=(\mu^i\otimes\nu^j)_{(i,j)\in I\times I}$. Noting that

$$R_{\mathbf{A}\times\mathbf{A}}(\boldsymbol{\mu}\otimes\boldsymbol{\nu})=\sum_{i,j\in I}s_is_j\mu^i\otimes\nu^j=(R_{\mathbf{A}}\boldsymbol{\mu})\otimes(R_{\mathbf{A}}\boldsymbol{\nu}),$$

we arrive at (3.6). ∎

Corollary 3.12. *Given* $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathcal{E}^+_\kappa(\mathbf{A})$, *we have*

$$\kappa(\boldsymbol{\mu},\boldsymbol{\nu})=\kappa(R\boldsymbol{\mu},R\boldsymbol{\nu})=\sum_{i,j\in I}s_is_j\kappa(\mu^i,\nu^j).\tag{3.9}$$

Furthermore, for every $i\in I$, $\kappa^i_{\boldsymbol{\mu}}(x)$ *is finite n.e. and can be written in the form*

$$\kappa^i_{\boldsymbol{\mu}}(x)=s_i\kappa(x,R\boldsymbol{\mu})=\sum_{j\in I}s_is_j\kappa(x,\mu^j).\tag{3.10}$$

The series in (3.9) *as well as in* (3.10) *converges absolutely, the latter being valid n.e.*

Proof. It is seen from (3.8) that $R\boldsymbol{\mu},R\boldsymbol{\nu}\in\mathcal{E}_\kappa(X)$; hence, $\kappa(R\boldsymbol{\mu},R\boldsymbol{\nu})$ is finite (see, e.g., [16, Lemma 3.1.1]), which yields (3.6) with the absolutely convergent series on the right-hand side. Compared with (3.4), this implies (3.9). Being the potential of a (scalar) measure of finite energy relative to the positive definite kernel, $\kappa(\cdot,R\boldsymbol{\mu})$ is finite n.e. [16, p. 164]. Hence, the series on the right-hand side in (3.7) converges absolutely n.e., which together with (3.5) establishes (3.10). ∎

Remark 3.13. Since the kernel is positive definite, (3.9) with $\boldsymbol{\nu}=\boldsymbol{\mu}$ yields the *positivity of the energy* $\kappa(\boldsymbol{\mu},\boldsymbol{\mu})$, which a priori was not obvious:

$$\kappa(\boldsymbol{\mu},\boldsymbol{\mu})\geq 0\quad\text{for all }\boldsymbol{\mu}\in\mathcal{E}^+_\kappa(\mathbf{A}).\tag{3.11}$$

REMARK 3.14. It is clear from the above that $\mathcal{E}^+_\kappa(\mathbf{A})$ is a *convex cone.* Indeed, since $\mathfrak{M}^+(\mathbf{A})$ is so, it is enough to observe that $R(\beta_1\boldsymbol{\mu}_1 + \beta_2\boldsymbol{\mu}_2) \in \mathcal{E}_\kappa(X)$ for any $\beta_1, \beta_2 \in (0,\infty)$ and $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{E}^+_\kappa(\mathbf{A})$. As $R\boldsymbol{\mu}_1, R\boldsymbol{\mu}_2 \in \mathcal{E}_\kappa(X)$ by (3.8), while $R(\beta_1\boldsymbol{\mu}_1 + \beta_2\boldsymbol{\mu}_2) = \beta_1 R\boldsymbol{\mu}_1 + \beta_2 R\boldsymbol{\mu}_2$, the convexity of $\mathcal{E}^+_\kappa(\mathbf{A})$ follows from the linearity of $\mathcal{E}_\kappa(X)$.

3.5. Semimetric space of vector measures with finite energy. We next show that the cone $\mathcal{E}^+_\kappa(\mathbf{A})$ can be thought of as a semimetric space, isometric to its (scalar) R-image.

LEMMA 3.15. *R-equivalence on $\mathcal{E}^+_\kappa(\mathbf{A})$ is equivalent to identity if and only if the A_i, $i \in I$, are mutually essentially disjoint, i.e.,*

$$c_\kappa(A_i \cap A_j) = 0 \quad \textit{for all } i \neq j. \tag{3.12}$$

Proof. The sufficiency part is obvious by Lemma 2.4. Assume now that there are A_k and A_ℓ, $k \neq \ell$, with $c_\kappa(A_k \cap A_\ell) > 0$; then necessarily $s_k s_\ell = +1$. It follows from Lemma 2.4 that there exists a nonzero $\tau \in \mathcal{E}^+_\kappa(A_k \cap A_\ell)$. Choose $\boldsymbol{\mu} = (\mu^i)_{i\in I} \in \mathcal{E}^+_\kappa(\mathbf{A})$ such that $\mu^k|_{A_k\cap A_\ell} - \tau \geq 0$, and define $\boldsymbol{\mu}_m = (\mu^i_m)_{i\in I} \in \mathcal{E}^+_\kappa(\mathbf{A})$, $m = 1, 2$, where $\mu^k_1 := \mu^k - \tau$ and $\mu^i_1 := \mu^i$ for all $i \neq k$, while $\mu^\ell_2 := \mu^\ell + \tau$ and $\mu^i_2 := \mu^i$ for all $i \neq \ell$. Then $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$, and hence $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are R-equivalent, but $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$. ∎

THEOREM 3.16. *The cone $\mathcal{E}^+_\kappa(\mathbf{A})$ is a semimetric space with the semimetric $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+_\kappa(\mathbf{A})}$ defined by* (1.2), *and this space is isometric to its R-image. Assume now κ is strictly positive definite. Then $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+_\kappa(\mathbf{A})}$ becomes a metric if and only if* (3.12) *holds.*

Proof. Fix any $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{E}^+_\kappa(\mathbf{A})$. Applying (3.9) to $\kappa(R\boldsymbol{\mu}_k, R\boldsymbol{\mu}_t)$, $k, t = 1, 2$, and then combining the equalities obtained, we get

$$\|R\boldsymbol{\mu}_1 - R\boldsymbol{\mu}_2\|^2_\kappa = \sum_{i,j\in I} s_i s_j \kappa(\mu^i_1 - \mu^i_2, \mu^j_1 - \mu^j_2),$$

where the series converges absolutely. Hence, the sum on the right-hand side in (1.2) is ≥ 0. When compared with (1.2), the last display yields

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = \|R\boldsymbol{\mu}_1 - R\boldsymbol{\mu}_2\|_\kappa. \tag{3.13}$$

Since $\|\cdot\|_\kappa$ is a seminorm on $\mathcal{E}_\kappa(X)$, the first assertion of the theorem follows.

Assume now κ is strictly positive definite. By (3.13), $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0$ if and only if $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are R-equivalent, while by Lemma 3.15, R-equivalence on $\mathcal{E}^+_\kappa(\mathbf{A})$ is equivalent to identity if and only if (3.12) holds. ∎

In view of the isometry between $\mathcal{E}^+_\kappa(\mathbf{A})$ and its R-image, contained in the pre-Hilbert space $\mathcal{E}_\kappa(X)$, the topology on the semimetric space $\mathcal{E}^+_\kappa(\mathbf{A})$ is likewise termed *strong.* As usual, $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ are said to be *equivalent* in the semimetric space $\mathcal{E}^+_\kappa(\mathbf{A})$ if $\|\boldsymbol{\mu} - \boldsymbol{\nu}\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0$.

COROLLARY 3.17. *$\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ are equivalent in $\mathcal{E}_\kappa^+(\mathbf{A})$ if and only if*

$$\kappa_{\boldsymbol{\mu}}^i(\cdot) = \kappa_{\boldsymbol{\nu}}^i(\cdot) \quad \text{n.e. for all } i \in I.$$

Proof. In view of (3.13), $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are equivalent in $\mathcal{E}_\kappa^+(\mathbf{A})$ if and only if $R\boldsymbol{\mu}$ and $R\boldsymbol{\nu}$ are equivalent in $\mathcal{E}_\kappa(X)$, which in turn holds if and only if $\kappa(\cdot, R\boldsymbol{\mu}) = \kappa(\cdot, R\boldsymbol{\nu})$ n.e. [16, Lemma 3.2.1(a)]. Combining this with (3.10) establishes the corollary. ∎

Being nonlinear, $\mathcal{E}_\kappa^+(\mathbf{A})$ is not normed. Nevertheless, for any of its elements $\boldsymbol{\mu}$ it is convenient to write $\|\boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} := \|\boldsymbol{\mu} - \mathbf{0}\|_{\mathcal{E}_\kappa^+(\mathbf{A})}$. Then

$$\|\boldsymbol{\mu}\|^2_{\mathcal{E}_\kappa^+(\mathbf{A})} = \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) = \kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}) = \|R\boldsymbol{\mu}\|^2_\kappa. \tag{3.14}$$

4. Minimum energy problems for a generalized condenser

4.1. Formulation of the problems. For a (positive definite) kernel κ on X and a (generalized) condenser $\mathbf{A} = (A_i)_{i\in I}$, we shall consider minimum energy problems with external fields over certain subclasses of $\mathcal{E}_\kappa^+(\mathbf{A})$.

Fix a vector-valued *external field* $\mathbf{f} = (f_i)_{i\in I}$, where each $f_i : X \to [-\infty, \infty]$ is μ-measurable for every $\mu \in \mathfrak{M}^+(X)$. The $\mathbf{f}$*-weighted vector potential* and the $\mathbf{f}$*-weighted energy* of $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ are defined by

$$\mathbf{W}^{\boldsymbol{\mu}}_{\kappa,\mathbf{f}} := \kappa_{\boldsymbol{\mu}} + \mathbf{f}, \tag{4.1}$$

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) := \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2\langle \mathbf{f}, \boldsymbol{\mu}\rangle, \tag{4.2}$$

respectively. Let $\mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ with $\langle \mathbf{f}, \boldsymbol{\mu}\rangle$ finite, which means that every f_i, $i \in I$, is μ^i-integrable and the series $\sum_{i\in I} \langle f_i, \mu^i\rangle$ converges absolutely.

Fix a numerical vector $\mathbf{a} = (a_i)_{i\in I}$ with $a_i > 0$, $i \in I$, and a vector-valued function $\mathbf{g} = (g_i)_{i\in I}$, where all the $g_i : X \to (0, \infty)$ are (finitely) continuous, and write

$$\mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \{\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}) : \langle g_i, \mu^i\rangle = a_i \text{ for all } i \in I\}.$$

If $\mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty, or equivalently if

$$G_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \inf_{\boldsymbol{\mu}\in\mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A},\mathbf{a},\mathbf{g})} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) < \infty,$$

then the following (*unconstrained*) $\mathbf{f}$*-weighted minimum energy problem*, also known in the literature as the *Gauss variational problem* (see, e.g., [23, 37, 38, 43, 44, 22]), makes sense.

PROBLEM 4.1. *Does there exist $\boldsymbol{\lambda}_\mathbf{A} \in \mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_\mathbf{A}) = G_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})?$$

Let $\mathfrak{C}(A_i)$, $i \in I$, consist of all $\xi^i \in \mathfrak{M}^+(A_i)$ with $\langle g_i, \xi^i\rangle > a_i$; those ξ^i are said to be (upper) *constraints* for elements of $\mathfrak{M}^+(A_i, a_i, g_i)$. Given $\xi^i \in \mathfrak{C}(A_i)$, write

$$\mathfrak{M}^{\xi^i}(A_i, a_i, g_i) := \{\mu^i \in \mathfrak{M}^+(A_i, a_i, g_i) : \mu^i \le \xi^i\},$$
$$\mathcal{E}_\kappa^{\xi^i}(A_i, a_i, g_i) := \mathcal{E}_\kappa^+(A_i) \cap \mathfrak{M}^{\xi^i}(A_i, a_i, g_i),$$

where $\mu^i \le \xi^i$ means that $\xi^i - \mu^i \ge 0$.

Fix $I_0 \subset I$, which might be empty. We generalize Problem 4.1 by assuming that for every $i \in I_0$, the i-components μ^i of the (new) admissible measures $\boldsymbol{\mu}$ are now additionally required not to exceed a fixed constraint $\xi^i \in \mathfrak{C}(A_i)$; that is, $\mu^i \in \mathfrak{M}^{\xi^i}(A_i, a_i, g_i)$ for all $i \in I_0$. To be precise, write $\boldsymbol{\sigma} := (\sigma^i)_{i\in I}$, where

$$\sigma^i := \begin{cases} \xi^i & \text{if } i \in I_0, \\ \infty & \text{if } i \in I \setminus I_0, \end{cases}$$

and define

$$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \prod_{i\in I} \mathfrak{M}^{\sigma^i}(A_i, a_i, g_i),$$
$$\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathcal{E}_\kappa^+(\mathbf{A}) \cap \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}),$$
$$\mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A}) \cap \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Here the formal notation $\mathfrak{M}^\infty(A_i, a_i, g_i)$ means that *no* active upper constraint is imposed on $\mu^i \in \mathfrak{M}^+(A_i, a_i, g_i)$, i.e.,

$$\mathfrak{M}^{\sigma^i}(A_i, a_i, g_i) = \mathfrak{M}^+(A_i, a_i, g_i) \quad \text{for all } i \in I \setminus I_0.$$

If $\mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty (*which will always be tacitly required*), or equivalently if ([11])

$$G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \inf_{\boldsymbol{\mu}\in\mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) < \infty, \tag{4.3}$$

then the following generalization of Problem 4.1 makes sense.

Problem 4.2. *Does there exist* $\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}} \in \mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *with*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}}) = G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})?$$

Observe that under the (standing) assumption (4.3), Problem 4.1 also makes sense. In fact, $\mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathcal{E}_{\kappa,\mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$, and hence

$$G_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \le G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty. \tag{4.4}$$

Problem 4.2 reduces to Problem 4.1 if $I_0 = \emptyset$, while for $I_0 = I$, Problem 4.2 is known as the *constrained Gauss variational problem* (see, e.g.,

([11]) See Lemma 5.5 below providing sufficient conditions for (4.3) to hold. Also note that the (nonempty) class $\mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is *convex* (cf. Remark 3.14).

[12, 42, 19, 22, 11]). However, the Gauss variational problem in either constrained or unconstrained setting has not been studied yet under the present requirements, where $\mathbf{A}$ is a collection of infinitely many touching Borel plates (cf. Remark 4.3 below). Finally, in the case where I_0 is a nonempty proper subset of I, Problem 4.2 seems to be new (even for a standard condenser), though such a problem with mixed upper boundary conditions looks quite natural and also promising in relation to its possible applications (cf. Remark 1.1).

REMARK 4.3. The most general study of Problem 4.1 for a standard condenser of infinitely many (closed) plates seems to have been provided in [43, 44]. It includes, e.g., a complete description of the set of all $\mathbf{a} = (a_i)_{i \in I}$ for which minimizers $\boldsymbol{\lambda}_{\mathbf{A}}$ exist as well as an analysis of their uniqueness, vague compactness, and strong and vague continuity of $\boldsymbol{\lambda}_{\mathbf{A}}$ when $\mathbf{A}$ varies. The weighted potentials of minimizers are described, and their characteristic properties are singled out.

4.2. Uniqueness of solutions. We next show that the set of solutions to Problem 4.2 is contained in a certain equivalence class in $\mathcal{E}^+_\kappa(\mathbf{A})$.

LEMMA 4.4. *Any two solutions* $\boldsymbol{\lambda}$ *and* $\widehat{\boldsymbol{\lambda}}$ *to Problem* 4.2 (*whenever these exist*) *are equivalent in* $\mathcal{E}^+_\kappa(\mathbf{A})$, *i.e.*, $\|\boldsymbol{\lambda} - \widehat{\boldsymbol{\lambda}}\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0$.

Proof. This can be shown in a way similar to that in [43, proof of Lemma 5.1], based on the convexity of $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, isometry between $\mathcal{E}^+_\kappa(\mathbf{A})$ and its (scalar) R-image, and the pre-Hilbert structure on $\mathcal{E}_\kappa(X)$. Indeed, from (4.3), (4.2), and (3.14) we get

$$4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \le 4G_{\kappa,\mathbf{f}}\left(\frac{\boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}}{2}\right) = \|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2_\kappa + 4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle.$$

On the other hand, applying the parallelogram identity in $\mathcal{E}_\kappa(X)$ to $R\boldsymbol{\lambda}$ and $R\widehat{\boldsymbol{\lambda}}$ and then adding and subtracting $4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle$, we obtain

$$\|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2_\kappa = -\|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2_\kappa - 4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\kappa,\mathbf{f}}(\widehat{\boldsymbol{\lambda}}).$$

When combined with the preceding relation, this yields

$$0 \le \|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2_\kappa \le -4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\kappa,\mathbf{f}}(\widehat{\boldsymbol{\lambda}}) = 0,$$

which in view of (3.13) establishes the lemma. ∎

COROLLARY 4.5. *If* κ *is strictly positive definite and the* A_i, $i \in I$, *are mutually essentially disjoint, then a solution to Problem* 4.2 *is unique.*

Proof. This follows from Lemma 4.4 combined with Theorem 3.16. ∎

The following example shows that Corollary 4.5 fails in general if the assumption of mutual essential disjointness of the A_i, $i \in I$, is omitted from its hypotheses.

Example 4.6. Let $X=\mathbb{R}^n$, $n\geq 3$, $\kappa=\kappa_2$, $I=I^+=\{1,2\}$, $I_0=\{1\}$, $a_1=a_2=1$, $g_1\equiv g_2\equiv 1$, $f_1\equiv f_2\equiv 0$, and let $A_1=A_2=K_0$, K_0 being an $(n-1)$-dimensional unit sphere. Let λ denote the κ_2-capacitary measure on K_0, which exists (cf. Remark 2.7 and Example 2.5). For symmetry reasons, λ coincides up to a normalizing factor with the $(n-1)$-dimensional surface measure m_{n-1} on K_0. Define $\xi^1:=3\lambda$, and consider Problem 4.2 with these data. It is obvious that $\boldsymbol{\lambda}=(\lambda,\lambda)$ is one of its solutions. Choose now compact disjoint sets $K_k\subset K_0$, $k=1,2$, so that $m_{n-1}(K_1)=m_{n-1}(K_2)>0$, and define $\nu=\lambda|_{K_1}-\lambda|_{K_2}$. Then $\widehat{\boldsymbol{\lambda}}=(\lambda-\nu,\lambda+\nu)$ is an admissible measure for Problem 4.2 such that $R\widehat{\boldsymbol{\lambda}}=R\boldsymbol{\lambda}$, and hence $\kappa(\widehat{\boldsymbol{\lambda}},\widehat{\boldsymbol{\lambda}})=\kappa(\boldsymbol{\lambda},\boldsymbol{\lambda})$. Thus $\widehat{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda}$ both solve Problem 4.2, though $\widehat{\boldsymbol{\lambda}}\neq\boldsymbol{\lambda}$.

5. Standing assumptions. Supplementary results. In all that follows we require that either X is countable at infinity, or

$$g_{i,\inf}:=\inf_{x\in A_i} g_i(x)>0 \quad \text{for all } i\in I. \tag{5.1}$$

Lemma 5.1. *Let $\mu^i\in\mathcal{E}^+_\kappa(A_i)$ be such that $\langle g_i,\mu^i\rangle=c<\infty$. Then a proposition $\mathcal{P}(x)$ holds μ^i-almost everywhere (μ^i-a.e.) if it holds n.e. on A_i.*

Proof. The set N of all $x\in A_i$ for which $\mathcal{P}(x)$ fails has inner capacity zero, and hence it is locally μ^i-negligible [16, Lemma 2.3.1(iii)]. Furthermore, N is μ^i-σ-finite. This is obvious if X is countable at infinity, while otherwise (5.1) holds, and therefore

$$\mu^i(X)\leq c g_{i,\inf}^{-1}<\infty. \tag{5.2}$$

Being locally μ^i-negligible and μ^i-σ-finite, N is μ^i-negligible as claimed. ∎

When speaking of an external field $\mathbf{f}=(f_i)_{i\in I}$, we shall henceforth tacitly assume that Case I or Case II holds, where:

I. *For every $i\in I$, $f_i\in\Psi(X)$.*
II. *For every $i\in I$, $f_i=s_i\kappa(\cdot,\zeta)$, where a (signed) $\zeta\in\mathcal{E}_\kappa(X)$ is given.*

Lemma 5.2. *If Case* II *takes place, then $\mathcal{E}^+_\kappa(\mathbf{A})=\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$, and moreover*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})=\|R\boldsymbol{\mu}+\zeta\|^2_\kappa-\|\zeta\|^2_\kappa \quad \textit{for all } \boldsymbol{\mu}\in\mathcal{E}^+_\kappa(\mathbf{A}). \tag{5.3}$$

Proof. Applying Lemma 3.10 to $\boldsymbol{\mu}\in\mathcal{E}^+_\kappa(\mathbf{A})$ and each of $\kappa(\cdot,\zeta^+),\kappa(\cdot,\zeta^-)$ $\in\Psi(X)$, we get by subtraction

$$\langle\mathbf{f},\boldsymbol{\mu}\rangle=\sum_{i\in I}s_i\int\kappa(x,\zeta)\,d\mu^i(x)=\kappa(\zeta,R\boldsymbol{\mu}).$$

Substituting this together with (3.14) into (4.2), we arrive at (5.3). ∎

LEMMA 5.3. *In either Case* I *or Case* II ([12]),

$$G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) \ge G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) > -\infty. \tag{5.4}$$

Proof. Since in Case II relation (5.4) follows directly from (5.3), it remains to consider Case I. Assume X is compact, for if not, then $f_i \ge 0$ for all $i \in I$, and (5.4) holds by (3.11). But then I has to be finite, while every f_i, being l.s.c., is bounded from below on the (compact) space X by $-c_i$, where $0 < c_i < \infty$. In addition, (5.1) and hence (5.2) with $c = a_i$ both hold for every $i \in I$ and every $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$, g_i being a strictly positive continuous function on X. Combining all this gives

$$-\infty < -c_i a_i g^{-1}_{i,\inf} \le -c_i \sup_{\boldsymbol{\mu}\in\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})} \mu^i(X) \le \langle f_i, \mu^i\rangle,$$

which in view of the finiteness of I again leads to (5.4). ∎

LEMMA 5.4. *Under the* (*standing*) *requirement* (4.3), *for all* $i \in I$,

$$c_\kappa(A_i^\circ) > 0, \quad \textit{where} \quad A_i^\circ := \{x \in A_i : |f_i(x)| < \infty\}. \tag{5.5}$$

Proof. Suppose that, on the contrary, $c_\kappa(A_j^\circ) = 0$ for some $j \in I$. Then, by Lemma 5.1, for every $\boldsymbol{\mu} \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ (which exists by (4.4)) we have $|f_j| = \infty$ μ^j-a.e. But this is impossible because $\mu^j \ne 0$ while f_j is μ^j-integrable. ∎

For any $M \in (0,\infty)$ and $i \in I$, write $A_i^M := \{x \in A_i^\circ : |f_i(x)| \le M\}$.

LEMMA 5.5. *Assume there exist* $M, M_1 \in (0,\infty)$ *that are independent of* $i \in I$ *and satisfy*

$$\sum_{i\in I_0} \|\xi^i|_{A_i^M}\|_\kappa < \infty, \tag{5.6}$$

$$\langle g_i, \xi^i|_{A_i^M}\rangle \in (a_i,\infty) \quad \textit{for all } i \in I_0, \tag{5.7}$$

$$\inf_{i\in I\setminus I_0} c_\kappa(A_i^{M_1}) =: M_3 \in (0,\infty]. \tag{5.8}$$

If moreover (1.3) *is fulfilled, then* (4.3) *holds.*

Proof. Fix $\varepsilon \in (0,\infty)$, and for every $i \in I \setminus I_0$ choose $\tau_i \in \mathcal{E}^+_\kappa(A_i^{M_1})$ of compact support so that $\tau_i(A_i^{M_1}) = 1$ and $\|\tau_i\|^2_\kappa \le c_\kappa(A_i^{M_1})^{-1} + \varepsilon$. (Such a τ_i exists since $c_\kappa(A_i^{M_1})$ would be the same if the measures ν in its definition were required to be of compact support $S(\nu) \subset A_i^{M_1}$, cf. [16, p. 153].) In view of (5.8), we thus get

$$\|\tau_i\|^2_\kappa \le \varepsilon + M_3^{-1} =: M_4^2 \in (0,\infty).$$

Write

$$\widetilde{\nu}_i := \frac{a_i\nu_i}{\langle g_i,\nu_i\rangle} \quad \text{for all } i \in I,$$

([12]) As seen from (4.3), (4.4), and (5.4), $G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ are both finite.

where

$$\nu^i := \begin{cases} \tau_i & \text{if } i \in I \setminus I_0, \\ \xi^i|_{A_i^M} & \text{if } i \in I_0. \end{cases}$$

Note that $0 < \langle g_i, \nu_i \rangle < \infty$ for all $i \in I$. In fact, for $i \in I \setminus I_0$ this holds because

$$0 < \min_{x \in S(\tau^i)} g_i(x) \le \langle g_i, \nu_i \rangle \le \max_{x \in S(\tau^i)} g_i(x) < \infty,$$

while for $i \in I_0$ it is valid by (5.7). Also observing that, again by (5.7), $\widetilde{\nu}_i \le \xi^i$ for all $i \in I_0$, we thus get $\widetilde{\nu}_i \in \mathfrak{M}^{\sigma^i}(A_i, a_i, g_i)$ for all $i \in I$. Furthermore,

$$\begin{aligned} \sum_{i \in I} \|\widetilde{\nu}_i\|_\kappa &\le \sum_{i \in I_0} \frac{a_i}{\langle g_i, \xi^i|_{A_i^M} \rangle} \|\xi^i|_{A_i^M}\|_\kappa + M_4 \sum_{i \in I \setminus I_0} a_i g_{i,\inf}^{-1} \\ &\le \sum_{i \in I_0} \|\xi^i|_{A_i^M}\|_\kappa + M_4 \sum_{i \in I \setminus I_0} a_i g_{i,\inf}^{-1} < \infty, \end{aligned}$$

where the second inequality follows from (5.7) and the third from (5.6) and (1.3). Therefore, by Lemma 3.9, $\widetilde{\boldsymbol{\nu}} := (\widetilde{\nu}_i)_{i \in I} \in \mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Finally,

$$\sum_{i \in I} |\langle f_i, \widetilde{\nu}_i \rangle| \le (M + M_1) \sum_{i \in I} \frac{a_i \nu^i(X)}{\langle g_i, \nu_i \rangle} \le (M + M_1) \sum_{i \in I} a_i g_{i,\inf}^{-1} < \infty,$$

the last inequality coming from (1.3). Altogether, $\widetilde{\boldsymbol{\nu}} \in \mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. ■

If I is finite, Lemma 5.5 takes the following much simpler form.

COROLLARY 5.6. *Let I be finite, and let $c_\kappa(A_i^\circ) > 0$ for all $i \in I \setminus I_0$, with A_i° defined in* (5.5). *Then* (4.3) *holds if for every $i \in I_0$,*

$$\langle g_i, \xi^i|_{A_i^\circ} \rangle > a_i \quad \text{and} \quad \xi^i|_{K_i} \in \mathcal{E}_\kappa^+(K_i) \quad \text{for every compact } K_i \subset A_i^\circ.$$

We omit the proof, since it is similar to that of Lemma 5.5. Combining this corollary with Lemma 5.4 yields the following assertion.

COROLLARY 5.7. *If I is finite and $I_0 = \emptyset$, then* (4.3) *and* (5.5) *are equivalent.*

DEFINITION 5.8. A net $(\boldsymbol{\mu}_s)_{s \in S} \subset \mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is *minimizing* in Problem 4.2 if

$$\lim_{s \in S} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_s) = G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{5.9}$$

Let $\mathbb{M}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ consist of all those $(\boldsymbol{\mu}_s)_{s \in S}$; it is nonempty because of (4.3).

LEMMA 5.9. *For any $(\boldsymbol{\mu}_s)_{s \in S}$ and $(\boldsymbol{\nu}_t)_{t \in T}$ in $\mathbb{M}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$,*

$$\lim_{(s,t) \in S \times T} \|\boldsymbol{\mu}_s - \boldsymbol{\nu}_t\|_{\mathcal{E}_\kappa^+(\mathbf{A})} = 0, \tag{5.10}$$

$S \times T$ being the upper directed product ([13]) *of the upper directed sets S and T.*

([13]) See, e.g., [28, Chapter 2, Section 3].

Proof. In the same manner as in the proof of Lemma 4.4 we get

$$\begin{aligned}0 &\le \|R\boldsymbol{\mu}_s - R\boldsymbol{\nu}_t\|^2_\kappa \\ &\le -4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_s) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\nu}_t),\end{aligned}$$

which yields (5.10) when combined with (3.13), (4.3), (5.4), and (5.9). ∎

COROLLARY 5.10. *Every* $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is strong Cauchy in* $\mathcal{E}^+_\kappa(\mathbf{A})$.

6. Sufficient conditions for solvability of Problem 4.2. Throughout Section 6 we require the standing assumptions stated in Sections 4.1 and 5. Furthermore, the A_i, $i \in I$, are assumed to be nearly closed. According to Lemma 3.4, then so are both A^+ and A^-. Let $\breve{A}^+$ and $\breve{A}^-$ be the (closed) sets defined by (3.3). We denote by $(\boldsymbol{\mu}_s)'_{s\in S}$ the cluster set of any $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^+(\mathbf{A})$ in the vague product space topology on $\mathfrak{M}^+(X)^{\operatorname{Card} I}$, and $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ the class of solutions to Problem 4.2.

THEOREM 6.1. *Suppose that the kernel* κ *is consistent, and the assumptions*

$$\sup_{(x,y)\in \breve{A}^+\times\breve{A}^-} \kappa(x,y) < \infty \tag{6.1}$$

and (1.3) *are both fulfilled. Also assume that*

$$\langle g_i, \xi^i\rangle < \infty \quad \text{for all } i \in I_0, \tag{6.2}$$

while for every $i \in I \setminus I_0$, *the following two conditions hold:*

- *Either* A_i *is nearly compact, or* $c_\kappa(A_i) < \infty$ ([14]).
- *Either* g_i *is upper bounded, or there are* $r_i \in (1,\infty)$ *and* $\nu_i \in \mathcal{E}_\kappa(X)$ *such that*

$$g_i^{r_i}(x) \le \kappa(x,\nu_i) \quad \text{n.e. on } A_i. \tag{6.3}$$

Then in either Case I *or Case* II, $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is nonempty, vaguely compact, and given by*

$$\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \bigcup_{(\boldsymbol{\nu}_t)_{t\in T}\in\mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})} (\boldsymbol{\nu}_t)'_{t\in T}. \tag{6.4}$$

Furthermore, for every $(\boldsymbol{\nu}_t)_{t\in T} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *and every* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$,

$$\lim_{t\in T} \|\boldsymbol{\nu}_t - \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0. \tag{6.5}$$

([14]) A compact set $K \subset X$ may be of infinite capacity; $c_\kappa(K)$ is necessarily finite if κ is strictly positive definite [16]. On the other hand, even for the Newtonian kernel, closed sets of finite capacity may be noncompact (see, e.g., Example 1.7 above).

Definition 6.2. Denoting by ∞_X the Alexandroff point of X [3, Chapter I, Section 9, n° 8], we say that a kernel κ *has the property* (∞_X) if $\kappa(\cdot, y) \to 0$ as $y \to \infty_X$ uniformly on compact sets in X.

The Riesz kernel κ_α, $\alpha \in (0, n)$, on $\mathbb{R}^n$, $n \geq 3$, has the property (∞_X). So does the 2-Green kernel G^2_D on an open bounded set $D \subset \mathbb{R}^n$, $n \geq 2$, provided that D is regular in the sense of the solvability of the classical Dirichlet problem.

Theorem 6.3. *Assume a l.c.s. X is metrizable and countable at infinity* ([15]), *while a kernel $\kappa(x, y)$ is continuous for $x \neq y$ and has the property (∞_X). Let I^+, resp. I^-, be finite, the A_i, $i \in I$, be nearly compact, and suppose*

$$\breve{A}^+ \cap \breve{A}^- = \emptyset. \tag{6.6}$$

If, moreover, Case I *takes place and* (1.3) *holds, then for any I_0 and $\boldsymbol{\sigma}$ the class $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty, vaguely compact, and given by* (6.4).

Remark 6.4. In contrast to Theorem 6.1, in Theorem 6.3 the kernel κ is not required to be consistent. However, if it is, then Theorem 6.3 becomes valid in both Cases I and II; and moreover, then (6.5) also holds.

Recall that a kernel κ is said to satisfy the *continuity principle* (or to be *regular*) if for any $\mu \in \mathfrak{M}^+(X)$ with compact support $S(\mu)$, $\kappa(\cdot, \mu)$ is continuous on X whenever its restriction to $S(\mu)$ is continuous. The Riesz kernel κ_α, $\alpha \in (0, n)$, on $\mathbb{R}^n$, $n \geq 3$, is regular [30, Theorem 1.7]. So is the α-Green kernel G^α_D, $\alpha \in (0, 2]$, on an open set $D \subset \mathbb{R}^n$, $n \geq 3$ [20, Corollary 4.8], as well as the logarithmic kernel on $\mathbb{R}^2$ (combine [30, Theorem 1.6] and [37, Eq. (1.3)]).

Theorem 6.5. *Assume I is finite and the A_i, $i \in I$, are nearly compact. Suppose the kernel κ is regular, and the $\kappa(\cdot, \xi^i)|_{\breve{A}_i}$, $i \in I_0$, as well as the $\kappa|_{\breve{A}_i \times \breve{A}_i}$, $i \in I \setminus I_0$, are continuous. Then in either Case* I *or Case* II *and for any* $\mathbf{a}$ *and* $\mathbf{g}$, *the conclusion of Theorem* 6.1 *remains valid* ([16]).

Remark 6.6. In contrast to Theorem 6.3, in Theorem 6.5 the sets $\breve{A}^+$ and $\breve{A}^-$ may have points in common. But then necessarily $c_\kappa(\breve{A}^+ \cap \breve{A}^-) = 0$, and hence $\breve{A}^+ \cap \breve{A}^-$ cannot carry any nonzero $\nu \in \mathcal{E}_\kappa(X)$ (see Lemma 2.4).

Corollary 6.7. *Under the hypotheses of any of Theorems* 6.1, 6.3, *or* 6.5, *if moreover κ is strictly positive definite, while the A_i, $i \in I$, are mutually essentially disjoint, then $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ reduces to a unique element $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$, and every $(\boldsymbol{\nu}_t)_{t \in T} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ converges to this $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ vaguely.*

([15]) Theorem 6.3 remains valid for an *arbitrary* l.c.s. X if we assume instead that only finitely many $\breve{A}_i$, $i \in I^-$, resp. $\breve{A}_i$, $i \in I^+$, can intersect one another (see Remark 7.3).

([16]) Theorem 6.5 is applicable to the classical kernels only provided that $I_0 = I$.

7. Proofs of Theorems 6.1, 6.3, and 6.5 and Corollary 6.7

7.1. Auxiliary results. Throughout Section 7.1, the A_i, $i \in I$, are assumed to be nearly closed. Write

$$\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \le\mathbf{a}, \mathbf{g}) := \{\boldsymbol{\nu} \in \mathcal{E}^+_\kappa(\mathbf{A}) : \nu^i \le \sigma^i,\ \langle g_i, \nu^i\rangle \le a_i \text{ for all } i \in I\}.$$

LEMMA 7.1. *If* (1.3) *and* (6.1) *both hold, then the vague cluster set* $(\boldsymbol{\mu}_s)'_{s\in S}$ *of any* $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is nonempty, and moreover*

$$(\boldsymbol{\mu}_s)'_{s\in S} \subset \mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \le\mathbf{a}, \mathbf{g}). \tag{7.1}$$

Proof. Fix a net $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. It is strong Cauchy in the semimetric space $\mathcal{E}^+_\kappa(\mathbf{A})$ by Corollary 5.10, and hence strongly bounded, i.e.,

$$\sup_{s\in S} \|\boldsymbol{\mu}_s\|^2_{\mathcal{E}^+_\kappa(\mathbf{A})} = \sup_{s\in S} \|R\boldsymbol{\mu}_s\|^2_\kappa < \infty, \tag{7.2}$$

the equality being valid by (3.14). Furthermore, it follows from (1.3) that (5.1) and hence (5.2) (with a_i and μ^i_s in place of c and μ^i) both hold. Thus,

$$\sup_{s\in S} |R\boldsymbol{\mu}_s|(X) = \sup_{s\in S} \sum_{i\in I} \mu^i_s(A_i) \le \sum_{i\in I} a_i g^{-1}_{i,\mathrm{inf}} = C < \infty. \tag{7.3}$$

By Lemma 2.9 with $Q = A_i$, the μ^i_s, $s \in S$, are supported by $\breve{A}_i$, A_i being nearly closed. Hence, $R\boldsymbol{\mu}^\pm_s$ is supported by $\breve{A}^\pm$ (cf. (3.3)), and therefore

$$\sup_{(x,y)\in S(R\boldsymbol{\mu}^+_s)\times S(R\boldsymbol{\mu}^-_s)} \kappa(x,y) \le \sup_{(x,y)\in \breve{A}^+\times\breve{A}^-} \kappa(x,y) < \infty \quad \text{for all } s \in S,$$

where the latter inequality holds by (6.1). Combining this with (7.3) establishes the inequality

$$\kappa(R\boldsymbol{\mu}^+_s, R\boldsymbol{\mu}^-_s) \le M < \infty \quad \text{for all } s \in S,$$

which together with (7.2) yields

$$\sup_{s\in S} \|R\boldsymbol{\mu}^\pm_s\|_\kappa < \infty. \tag{7.4}$$

We next observe that for every $i \in I$,

$$\sup_{s\in S} \|\mu^i_s\|_\kappa < \infty. \tag{7.5}$$

In view of (7.4), this will follow once we have established the inequality

$$\inf_{s\in S} \sum_{k,m\in I^\pm,\, k\ne m} \kappa(\mu^k_s, \mu^m_s) > -\infty. \tag{7.6}$$

Assume X is compact, for if not, then $\kappa \ge 0$ and the left-hand side in (7.6) is ≥ 0. But then the l.s.c. function κ on $X \times X$ is $\ge -c$, where $c \in (0, \infty)$, while I is finite; and hence (7.6) follows from (7.3) in a way similar to that in the proof of Lemma 5.3.

As seen from (7.3), the net $(\boldsymbol{\mu}_s)_{s\in S}$ is vaguely bounded, and hence, by Lemma 3.2, it is relatively compact in the vague topology of $\mathfrak{M}^+(X)^{\operatorname{Card} I}$. Thus, there is a subnet $(\boldsymbol{\mu}_t)_{t\in T}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ such that for every $i \in I$,

$$\mu_t^i \to \mu^i \quad \text{vaguely as } t \text{ increases along } T, \tag{7.7}$$

where $\mu^i \in \mathfrak{M}^+(X)$. It follows from (7.5) and (7.7) by Lemmas 2.9 and 2.10 that $\mu^i \in \mathcal{E}_\kappa^+(\breve{A}_i) = \mathcal{E}_\kappa^+(A_i)$, and hence $\boldsymbol{\mu} := (\mu^i)_{i\in I} \in \mathfrak{M}^+(\mathbf{A})$.

Moreover, $R\boldsymbol{\mu}^\pm$ is the vague limit of $R\boldsymbol{\mu}_t^\pm$ as t increases along T, which is obtained from (7.7) according to Lemma 3.7. Applying [16, Lemma 2.2.1(e)], we therefore see from (7.4) that the energy of $R\boldsymbol{\mu}^\pm$ is finite. Since κ is positive definite, so is $\kappa(R\boldsymbol{\mu}^+, R\boldsymbol{\mu}^-)$ (see, e.g., [16, Lemma 3.1.1]), and altogether $R\boldsymbol{\mu} \in \mathcal{E}_\kappa(X)$. In view of (3.8), we thus have

$$\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A}).$$

Applying Lemma 2.1 to the continuous function $g_i > 0$, we also obtain from (7.7)

$$\langle g_i, \mu^i\rangle \le \lim_{t\in T}\, \langle g_i, \mu_t^i\rangle = a_i. \tag{7.8}$$

Noting that $\xi^i - \mu_t^i \ge 0$ for all $i \in I_0$ and $t \in T$ as well as that the vague limit of a net of positive (scalar) measures is positive, we finally see from (7.7) that $\mu^i \le \sigma^i$ for all $i \in I$. This together with the preceding two displays shows that, actually, $\boldsymbol{\mu} \in \mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A}, \le\mathbf{a}, \mathbf{g})$, which establishes (7.1). ■

Lemma 7.2. *Let* (1.3) *and* (6.1) *both hold, and let κ be consistent. For every $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and every $\boldsymbol{\mu} \in (\boldsymbol{\mu}_s)'_{s\in S}$,*

$$\lim_{s\in S}\, \|\boldsymbol{\mu}_s - \boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} = 0, \tag{7.9}$$

$$-\infty < G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \le \lim_{s\in S} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_s) = G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty. \tag{7.10}$$

Proof. We tacitly use the notation and assertions from the proof of the preceding lemma. Being consistent, the kernel κ has the property (C_2) (see Section 2.2). The strongly bounded net $(R\boldsymbol{\mu}_t^\pm)_{t\in T} \subset \mathcal{E}_\kappa^+(X)$ therefore converges weakly to its vague limit $R\boldsymbol{\mu}^\pm \in \mathcal{E}_\kappa^+(X)$, which by the definition of weak convergence implies that $R\boldsymbol{\mu}_t \to R\boldsymbol{\mu}$ weakly as t increases along T. By (3.13), this gives

$$\|\boldsymbol{\mu}_t - \boldsymbol{\mu}\|^2_{\mathcal{E}_\kappa^+(\mathbf{A})} = \|R\boldsymbol{\mu}_t - R\boldsymbol{\mu}\|^2_\kappa = \lim_{t'\in T} \kappa(R\boldsymbol{\mu}_t - R\boldsymbol{\mu}, R\boldsymbol{\mu}_t - R\boldsymbol{\mu}_{t'}),$$

and hence, by the Cauchy–Schwarz inequality in $\mathcal{E}_\kappa(X)$,

$$\|\boldsymbol{\mu}_t - \boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} \le \liminf_{t'\in T}\, \|\boldsymbol{\mu}_t - \boldsymbol{\mu}_{t'}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} \quad \text{for all } t \in T,$$

which establishes the relation

$$\lim_{t\in T}\, \|\boldsymbol{\mu}_t - \boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} = 0,$$

because $\|\boldsymbol{\mu}_t - \boldsymbol{\mu}_{t'}\|_{\mathcal{E}^+_\kappa(\mathbf{A})}$ becomes arbitrarily small when $t, t' \in T$ are sufficiently large. Since a strong Cauchy net converges strongly to any of its strong cluster points, we obtain (7.9) from the last display.

As for (7.10), we first note that the equality and the third inequality here are valid by (5.9) and the standing assumption (4.3), respectively. If Case II takes place, then the first inequality is obvious by (5.3), while the second inequality holds (with equality) again by (5.3), applied respectively to $\boldsymbol{\mu}_s$, $s \in S$, and $\boldsymbol{\mu}$, and the subsequent use of (7.9). Assume now Case I holds. Applying Lemma 2.1 to $f_i \in \Psi(X)$, we see from (7.7) after summation over $i \in I$ that ([17])

$$-\infty < \langle \mathbf{f}, \boldsymbol{\mu} \rangle \le \liminf_{t \in T} \langle \mathbf{f}, \boldsymbol{\mu}_t \rangle. \tag{7.11}$$

The former inequality here is obvious if X is noncompact, while otherwise it can be obtained from (1.3) and (7.8) in the same manner as in the proof of Lemma 5.3. Combining (7.11) and (7.9) completes the proof of (7.10). ∎

7.2. Proof of Theorem 6.1. Fix $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\boldsymbol{\mu} \in (\boldsymbol{\mu}_s)'_{s\in S}$ (such a $\boldsymbol{\mu}$ exists by Lemma 7.1). For these $(\boldsymbol{\mu}_s)_{s\in S}$ and $\boldsymbol{\mu}$, Lemmas 7.1 and 7.2 as well as the assertions in their proofs all hold.

We assert that $\boldsymbol{\mu}$ solves Problem 4.2. We first show that, actually,

$$\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{7.12}$$

As seen from (7.1) and (7.10), it is enough to prove that for any given $i \in I$,

$$\langle g_i, \mu^i \rangle = a_i. \tag{7.13}$$

It follows from Lemma 2.9 with $Q = A_i$ that the μ^i_s, $s \in S$, and μ^i are carried by $A_i \cap \breve{A}_i$. There is therefore no loss of generality in replacing each ξ^i, $i \in I_0$, by the extension of $\xi^i|_{A_i \cap \breve{A}_i}$ by 0 to all of X, denoted again by ξ^i ([18]). Observe that for this (new) ξ^i, assumption (6.2) remains valid.

Consider the exhaustion of the (closed) set $\breve{A}_i$ by the upper directed family (K) of all compact subsets of $\breve{A}_i$ ([19]). Let $(\boldsymbol{\mu}_t)_{t\in T}$ be a subnet of $(\boldsymbol{\mu}_s)_{s\in S}$ converging vaguely to $\boldsymbol{\mu}$ (see the proof of Lemma 7.1). Since the indicator function 1_K of K is upper semicontinuous, Lemma 2.1 with

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([17]) Note that, while proving (7.11) in Case I, we have not used the consistency of the kernel.

([18]) As $A_i \cap \breve{A}_i$ is ξ^i-measurable, $\xi^i|_{A_i \cap \breve{A}_i}$ exists. Given $Q \subset X$, the *extension* of $\nu \in \mathfrak{M}^+(Q)$ by 0 to all of X is $\tilde{\nu} \in \mathfrak{M}^+(X)$ determined uniquely by the relation $\tilde{\nu}(\varphi) := \langle \varphi|_Q, \nu \rangle$ for all $\varphi \in C_0(X)$.

([19]) A family $\mathfrak{Q}$ of sets $Q \subset X$ is said to be *upper directed* if for any $Q_1, Q_2 \in \mathfrak{Q}$ there exists $Q_3 \in \mathfrak{Q}$ such that $Q_1 \cup Q_2 \subset Q_3$.

$\psi = -g_i 1_K = -g_i|_K$ and [16, Lemma 1.2.2] yield

$$a_i \geq \langle g_i, \mu^i \rangle = \lim_{K \in (K)} \langle g_i, \mu^i|_K \rangle = \lim_{K \in (K)} \langle g_i|_K, \mu^i \rangle \geq \limsup_{(t,K) \in T \times (K)} \langle g_i|_K, \mu^i_t \rangle$$
$$= \limsup_{(t,K) \in T \times (K)} \langle g_i, \mu^i_t|_K \rangle = a_i - \liminf_{(t,K) \in T \times (K)} \langle g_i, \mu^i_t|_{\breve{A}_i \setminus K} \rangle,$$

$T \times (K)$ being the upper directed product of the upper directed sets T and (K) [28, Chapter 2, Section 3]. The first inequality here holds by (7.8), while the second and third equalities follow from Lemma 2.2, the μ^i_t, $t \in T$, and μ^i being bounded. Hence, (7.13) will be established once we have proven

$$\liminf_{(t,K) \in T \times (K)} \langle g_i, \mu^i_t|_{\breve{A}_i \setminus K} \rangle = \liminf_{(t,K) \in T \times (K)} \langle g_i|_{\breve{A}_i \setminus K}, \mu^i_t \rangle = 0, \tag{7.14}$$

the former equality here being valid again according to Lemma 2.2.

Assume first that $i \in I_0$. Since by (6.2) and [16, Lemma 1.2.2],

$$\infty > \langle g_i, \xi^i \rangle = \lim_{K \in (K)} \langle g_i, \xi^i|_K \rangle,$$

we have

$$\lim_{K \in (K)} \langle g_i, \xi^i|_{\breve{A}_i \setminus K} \rangle = 0.$$

When combined with

$$\langle g_i, \mu^i_t|_{\breve{A}_i \setminus K} \rangle \leq \langle g_i, \xi^i|_{\breve{A}_i \setminus K} \rangle \quad \text{for all } t \in T,$$

this implies the latter equality in (7.14) and hence (7.13).

Let now $i \in I \setminus I_0$. Assuming first that $\breve{A}_i$ is compact, we obtain (7.13) from (7.7) in view of the continuity of g_i. In fact, there is $\varphi_i \in C_0(X)$ such that $\varphi_i|_{\breve{A}_i} = g_i|_{\breve{A}_i}$. Since $\breve{A}_i^c$ is ν-negligible for any $\nu \in \mathfrak{M}^+(\breve{A}_i)$, we thus get

$$a_i = \lim_{t \in T} \langle g_i, \mu^i_t \rangle = \lim_{t \in T} \langle \varphi_i, \mu^i_t \rangle = \langle \varphi_i, \mu^i \rangle = \langle g_i, \mu^i \rangle. \tag{7.15}$$

Assume next $\breve{A}_i$ is noncompact. Then, by the stated hypotheses,

$$c_\kappa(\breve{A}_i) < \infty. \tag{7.16}$$

Since the kernel κ is consistent, for every $Q \subset \breve{A}_i$ there exists an interior equilibrium measure γ_Q [16, Theorem 4.1]. Recall that if $\mathit{\Gamma}_Q$ denotes the convex cone of all $\nu \in \mathcal{E}_\kappa(X)$ with $\kappa(x, \nu) \geq 1$ n.e. on Q, then $\gamma_Q \in \mathit{\Gamma}_Q$, i.e.,

$$\kappa(x, \gamma_Q) \geq 1 \quad \text{n.e. on } Q, \tag{7.17}$$

and moreover

$$c_\kappa(Q) = \|\gamma_Q\|^2_\kappa = \min_{\nu \in \mathit{\Gamma}_Q} \|\nu\|^2_\kappa. \tag{7.18}$$

Observe that there is no loss of generality in assuming g_i to satisfy (6.3) with some $r_i \in (1, \infty)$ and $\nu_i \in \mathcal{E}_\kappa(X)$. Indeed, otherwise g_i must be bounded from above by $M \in (0,\infty)$, which combined with (7.17) for $Q = \breve{A}_i$ results in (6.3) with $\nu_i := M^{r_i}\gamma_{\breve{A}_i}$, $r_i \in (1,\infty)$ being arbitrary.

Consider interior equilibrium measures $\gamma_{\breve{A}_i\setminus K}$ and $\gamma_{\breve{A}_i\setminus K'}$, where $K, K' \in (K)$. Because of (7.17) and (7.18), from [16, Lemma 4.1.1] we obtain

$$\|\gamma_{\breve{A}_i\setminus K} - \gamma_{\breve{A}_i\setminus K'}\|^2_\kappa \le \|\gamma_{\breve{A}_i\setminus K}\|^2_\kappa - \|\gamma_{\breve{A}_i\setminus K'}\|^2_\kappa \quad \text{whenever} \quad K \subset K'.$$

As seen from (7.16) and (7.18), the net $(\|\gamma_{\breve{A}_i\setminus K}\|_\kappa)_{K\in(K)}$ is bounded and decreasing, and hence it is Cauchy in $\mathbb{R}$. The preceding inequality thus shows that the net $(\gamma_{\breve{A}_i\setminus K})_{K\in(K)}$ is strong Cauchy in $\mathcal{E}^+_\kappa(X)$. Since, clearly, this net converges vaguely to zero, the property (C_1) implies that zero is also one of its strong limits. Hence,

$$\lim_{K\in(K)} \|\gamma_{\breve{A}_i\setminus K}\|_\kappa = 0. \tag{7.19}$$

Write $q_i := r_i(r_i-1)^{-1}$, where $r_i \in (1,\infty)$ is the number involved in condition (6.3). Combining (6.3) with (7.17) shows that the inequality

$$g_i(x)1_{\breve{A}_i\setminus K}(x) \le \kappa(x,\nu_i)^{1/r_i}\kappa(x,\gamma_{\breve{A}_i\setminus K})^{1/q_i}$$

holds n.e. on $\breve{A}_i$, and hence μ^i_t-a.e. on X (see Lemma 5.1). We integrate this relation with respect to μ^i_t, then apply the Hölder and the Cauchy–Schwarz inequalities to get

$$\begin{aligned}\langle g_i 1_{\breve{A}_i\setminus K}, \mu^i_t\rangle &\le \Big[\int \kappa(x,\nu_i)\,d\mu^i_t(x)\Big]^{1/r_i}\Big[\int \kappa(x,\gamma_{\breve{A}_i\setminus K})\,d\mu^i_t(x)\Big]^{1/q_i}\\ &\le \|\nu_i\|^{1/r_i}_\kappa \|\gamma_{\breve{A}_i\setminus K}\|^{1/q_i}_\kappa \|\mu^i_t\|_\kappa.\end{aligned}$$

Taking limits here as (t,K) increases along $T\times(K)$ and using (7.5) and (7.19), we again obtain the latter equality in (7.14), and hence (7.13).

Having thus proven (7.12), we get $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \ge G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Since the converse inequality holds by (7.10), $\boldsymbol{\mu}$ is a solution to Problem 4.2, i.e., $\boldsymbol{\mu} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. As $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $\boldsymbol{\mu} \in (\boldsymbol{\mu}_s)'_{s\in S}$ have been chosen arbitrarily, we obtain

$$\bigcup_{(\boldsymbol{\nu}_t)_{t\in T}\in\mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})} (\boldsymbol{\nu}_t)'_{t\in T} \subset \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}).$$

The converse inclusion is obvious because the trivial net $(\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}})$, where $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ is any element of $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, is minimizing and converges vaguely to $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$. Thus, (6.4) indeed holds.

Any net in $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is obviously minimizing, and hence, according to (6.4), any of its vague cluster points again belongs to $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. This establishes the vague compactness of $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Choosing finally any $(\boldsymbol{\nu}_t)_{t\in T} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, we arrive at (6.5) by combining (7.9) with Lemmas 4.4 and 5.9. The proof is complete.

7.3. Proof of Theorem 6.3. Under the stated hypotheses, there is no loss of generality in assuming I^+ to be finite. Then $\breve{A}^+$ is compact, because $\breve{A}_i$, $i \in I$, are. Fix $\varepsilon > 0$. As seen from the property (∞_X), there exists a compact set $K \subset X$ such that

$$\kappa(x,y) < \frac{\varepsilon}{3C^2} \quad \text{for all } x \in \breve{A}^+,\ y \in K^c, \tag{7.20}$$

where $C \in (0,\infty)$ is given by (1.3). Since $\breve{A}^- \cap K$ is compact, it follows from (6.6) and the (finite) continuity of $\kappa(x,y)$ for $x \neq y$ that $\kappa|_{\breve{A}^+ \times (\breve{A}^- \cap K)}$ is upper bounded. This together with (7.20) yields (6.1). Since (1.3) holds by assumption, we are thus able to use Lemma 7.1 as well as the assertions established in the course of its proof.

According to Lemma 2.9 with $Q = A_i$, for every $\boldsymbol{\nu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ we have $\nu^i \in \mathcal{E}^+_\kappa(A_i \cap \breve{A}_i)$ for all $i \in I$. There is therefore no loss of generality in replacing each ξ^i, $i \in I_0$, by the extension of $\xi^i|_{A_i \cap \breve{A}_i}$ by 0 to all of X (cf. footnote 18), denoted again by ξ^i. We next replace (again with the notation preserved) the A_i, $i \in I$, by the (compact) sets $\breve{A}_i$, which again involves no loss of generality.

By (7.1), any vague cluster point $\boldsymbol{\mu}$ of any $(\boldsymbol{\mu}_s)_{s \in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ belongs to $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, {\leq}\,\mathbf{a}, \mathbf{g})$. Choose a subsequence $\{\boldsymbol{\mu}_k\}_{k \in \mathbb{N}}$ of $(\boldsymbol{\mu}_s)_{s \in S}$ that converges vaguely to $\boldsymbol{\mu}$. Since g_i is continuous and A_i is compact, equality holds in (7.8) (cf. (7.15)), and hence

$$\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{7.21}$$

Thus, $|R\boldsymbol{\mu}|(X) \leq C$ (cf. (7.3)) and the above K can be chosen so that

$$|R\boldsymbol{\mu}^-|(\partial_X K) = 0. \tag{7.22}$$

We next use the fact that the map $(\nu_1, \nu_2) \mapsto \nu_1 \otimes \nu_2$ from $\mathfrak{M}^+(X) \times \mathfrak{M}^+(X)$ into $\mathfrak{M}^+(X \times X)$ is vaguely continuous [4, Chapter 3, Section 5, Exercise 5]. Applying Lemma 2.1 to $\kappa \in \Psi(X \times X)$, we therefore obtain

$$\kappa(R\boldsymbol{\mu}^\pm, R\boldsymbol{\mu}^\pm) \leq \liminf_{k \to \infty} \kappa(R\boldsymbol{\mu}^\pm_k, R\boldsymbol{\mu}^\pm_k). \tag{7.23}$$

Furthermore,

$$\begin{aligned} |\kappa(R\boldsymbol{\mu}^+, R\boldsymbol{\mu}^-) - \kappa(R\boldsymbol{\mu}^+_k, R\boldsymbol{\mu}^-_k)| &\leq |\kappa(R\boldsymbol{\mu}^+, R\boldsymbol{\mu}^-|_{K^c})| \\ &+ |\kappa(R\boldsymbol{\mu}^+_k, R\boldsymbol{\mu}^-_k|_{K^c})| + |\kappa(R\boldsymbol{\mu}^+, R\boldsymbol{\mu}^-|_K) - \kappa(R\boldsymbol{\mu}^+_k, R\boldsymbol{\mu}^-_k|_K)|. \end{aligned} \tag{7.24}$$

As seen from (7.20) and (7.3), each of the first two summands on the right-hand side in (7.24) is $< \varepsilon/3$. Since $\kappa|_{A^+ \times (A^- \cap K)}$ is continuous on the (compact) space $A^+ \times (A^- \cap K)$ and $R\boldsymbol{\mu}^+_k \otimes (R\boldsymbol{\mu}^-_k|_K) \to R\boldsymbol{\mu}^+ \otimes (R\boldsymbol{\mu}^-|_K)$ vaguely, the latter being clear from Theorem 2.3 in view of (7.22), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the last summand in (7.24) is $< \varepsilon/3$. Altogether,

$$\kappa(R\boldsymbol{\mu}^+, R\boldsymbol{\mu}^-) = \lim_{k \to \infty} \kappa(R\boldsymbol{\mu}^+_k, R\boldsymbol{\mu}^-_k),$$

for ε has been chosen arbitrarily. Combining this with (7.23) and then substituting (3.9) and (3.11) into the inequality obtained yields

$$0 \le \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) \le \liminf_{k\to\infty} \kappa(\boldsymbol{\mu}_k, \boldsymbol{\mu}_k).$$

Since Case I takes place, in view of footnote 17 we obtain (7.11), which together with the last display establishes the relation

$$-\infty < G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \le \liminf_{k\to\infty} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_k) = G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty.$$

The equality and the third inequality here are valid by (5.9) and (4.3), respectively. In view of (7.21), we thus actually have $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, and therefore $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \ge G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. This together with the preceding display shows that $\boldsymbol{\mu}$ is in fact a solution to Problem 4.2.

It has thus been proven that any vague cluster point (which exists) of any minimizing net (sequence) belongs to $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. In the same way as at the end of the proof of Theorem 6.1, this implies (6.4) as well as the vague compactness of $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. The proof is complete.

Remark 7.3. Assume the conditions of footnote 15 hold. Then the corresponding version of Theorem 6.3 can be proven as above (of course, with a subnet $(\boldsymbol{\mu}_t)_{t\in T}$ in place of a subsequence), the only difference being in the fact that Theorem 2.3 may fail. Instead, choose a compact set K so that $\breve{A}^+ \cap \partial_X K = \emptyset$. Since this K has points in common with only finitely many $\breve{A}_i$, $i \in I^+$, $(R\boldsymbol{\mu}^+_t|_K)_{t\in T}$ again converges vaguely to $R\boldsymbol{\mu}^+|_K$. Reversing the roles of '+' and '−', we arrive at our claim.

7.4. Proof of Theorem 6.5. In the same manner as in the proof of Theorem 6.3, there is no loss of generality in replacing each ξ^i, $i \in I_0$, by the extension of $\xi^i|_{A_i\cap\breve{A}_i}$ by 0 to all of X (denoted again by ξ^i).

We begin by showing that under the stated hypotheses the potential $\kappa(\cdot, \nu^i)$ of any $\nu^i \in \mathfrak{M}^+(\breve{A}_i)$, $i \in I$, such that $\nu^i \le \sigma^i$ is continuous on X. Let first $i \in I_0$. Being relatively continuous on $\breve{A}_i \supset S(\xi^i)$ by assumption, $\kappa(\cdot, \xi^i)$ is continuous on X by the regularity of the kernel. Since $\kappa(\cdot, \nu^i)$ is l.s.c. and since $\kappa(\cdot, \nu^i) = \kappa(\cdot, \xi^i) - \kappa(\cdot, \xi^i - \nu^i)$ with $\kappa(\cdot, \xi^i)$ continuous and $\kappa(\cdot, \xi^i - \nu^i)$ l.s.c., $\kappa(\cdot, \nu^i)$ is also upper semicontinuous, hence continuous. Let now $i \in I \setminus I_0$. Since $-\kappa|_{\breve{A}_i\times\breve{A}_i}$ is continuous by assumption, $-\kappa(x, y) \ge -c$ for all $(x, y) \in \breve{A}_i \times \breve{A}_i$, where $c \in (0, \infty)$. Integrating this inequality with respect to the (bounded) $\nu^i \in \mathfrak{M}^+(\breve{A}_i)$, we observe that $\kappa(\cdot, \nu^i)$ is relatively upper semicontinuous on $\breve{A}_i$. Being also l.s.c. on X, it is relatively continuous on $\breve{A}_i \supset S(\nu^i)$, and hence on all of X, again by the regularity of the kernel κ.

Choose any $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, which exists by (4.3). By Lemma 2.9 with $Q = A_i$, $i \in I$, we have $(\mu^i_s)_{s\in S} \subset \mathcal{E}^+_\kappa(\breve{A}_i)$. Since g_i is continuous and strictly positive, while $\breve{A}_i$ is compact,

$$\mu_s^i(X) \le a_i \Bigl[\min_{x\in\breve{A}_i} g_i(x)\Bigr]^{-1} < \infty \quad \text{for all } s \in S.$$

Therefore, $(\boldsymbol{\mu}_s)_{s\in S}$ is bounded and hence relatively compact in the vague topology on $\mathfrak{M}^+(X)^{\operatorname{Card} I}$ (Lemma 3.2). Fix any of its vague cluster points

$$\boldsymbol{\mu} = (\mu^i)_{i\in I} \in \mathfrak{M}^+(X)^{\operatorname{Card} I},$$

and choose a subnet $(\boldsymbol{\mu}_t)_{t\in T}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ converging vaguely to $\boldsymbol{\mu}$. Since $\mathfrak{M}^{\sigma^i}(\breve{A}_i, a_i, g_i)$ is vaguely closed (cf. (7.15)),

$$\mu^i \in \mathfrak{M}^{\sigma^i}(\breve{A}_i, a_i, g_i) \quad \text{for all } i \in I.$$

As shown in the second paragraph of the present proof, $\kappa(\cdot, \mu^i)$ is continuous on X, and hence bounded on the (compact) $\breve{A}_i$. Combined with $\mu^i(\breve{A}_i) < \infty$, this gives $\mu^i \in \mathcal{E}_\kappa^+(\breve{A}_i)$. By Lemma 2.9 for $Q = A_i$ and the preceding display, we thus get $\boldsymbol{\mu} \in \mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ (I being finite).

Furthermore, since every $\kappa(\cdot, \mu_t^i)$, $t \in T$, is likewise continuous on X,

$$\begin{aligned}\lim_{t\in T}\lim_{t'\in T} \kappa(\mu_t^i, \mu_{t'}^j) &= \lim_{t\in T}\lim_{t'\in T}\int \kappa(\cdot, \mu_t^i)\, d\mu_{t'}^j = \lim_{t\in T}\int \kappa(\cdot, \mu_t^i)\, d\mu^j\\ &= \lim_{t\in T}\int \kappa(\cdot, \mu^j)\, d\mu_t^i = \kappa(\mu^j, \mu^i) \quad \text{for all } i, j \in I.\end{aligned}$$

Summing these equalities, multiplied by $s_i s_j$, over all $i, j \in I$ shows that $R\boldsymbol{\mu}_t \to R\boldsymbol{\mu}$ strongly in $\mathcal{E}_\kappa(X)$; and hence, by (3.13),

$$\lim_{t\in T} \|\boldsymbol{\mu}_t - \boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})} = 0.$$

Since a strong Cauchy net converges strongly to any of its strong cluster points, we see that $(\boldsymbol{\mu}_s)_{s\in S}$ converges to $\boldsymbol{\mu}$ strongly in $\mathcal{E}_\kappa^+(\mathbf{A})$, which is (7.9).

Applying now to $(\boldsymbol{\mu}_s)_{s\in S}$ and $\boldsymbol{\mu}$ the same arguments as in the last paragraph of the proof of Lemma 7.2, we arrive at (7.10). Hence, $\boldsymbol{\mu} \in \mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. The rest of the proof repeats word for word the last two paragraphs in the proof of Theorem 6.1.

7.5. Proof of Corollary 6.7. Let the assumptions of any of Theorems 6.1, 6.3, or 6.5 be fulfilled. As seen from these theorems, the class $\mathfrak{S}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ of solutions to Problem 4.2 is then nonempty and given by (6.4).

Assume moreover that the kernel κ is strictly positive definite, while the A_i, $i \in I$, are mutually essentially disjoint. By Corollary 4.5, a solution to Problem 4.2 is then unique, which implies in view of (6.4) that the vague cluster sets of the minimizing nets are identical to one another, and all these reduce to a unique $\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}} \in \mathfrak{S}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Since the vague topology on $\mathfrak{M}^+(X)^{\operatorname{Card} I}$ is Hausdorff, $\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}}$ must be the vague limit of every $(\boldsymbol{\nu}_t)_{t\in T} \in \mathbb{M}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ [3, Chapter I, Section 9, n° 1], as was to be proven.

8. Weighted potentials of solutions to Problem 4.2. For any $\boldsymbol{\nu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ we denote by $W^{\boldsymbol{\nu},i}_{\kappa,\mathbf{f}}$, $i \in I$, the i-component of the $\mathbf{f}$-weighted vector potential $W^{\boldsymbol{\nu}}_{\kappa,\mathbf{f}}$ (cf. (4.1)).

LEMMA 8.1. $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *solves Problem* 4.2 *if and only if*

$$\sum_{i\in I}\langle W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}}, \nu^i - \lambda^i\rangle \ge 0 \quad \text{for all } \boldsymbol{\nu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{8.1}$$

Proof. For any $\boldsymbol{\mu},\boldsymbol{\nu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $h \in (0,1]$, we obtain by straightforward verification

$$G_{\kappa,\mathbf{f}}(h\boldsymbol{\nu} + (1-h)\boldsymbol{\mu}) - G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) = 2h\sum_{i\in I}\langle W^{\boldsymbol{\mu},i}_{\kappa,\mathbf{f}}, \nu^i - \mu^i\rangle + h^2\|\boldsymbol{\nu}-\boldsymbol{\mu}\|^2_{\mathcal{E}^+_\kappa(\mathbf{A})}.$$

If $\boldsymbol{\mu} = \boldsymbol{\lambda}$ solves Problem 4.2, then the left-hand side of this display is ≥ 0, for the class $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is convex, which leads to (8.1) by letting $h \to 0$. Conversely, if (8.1) holds, then the preceding formula with $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $h = 1$ yields $G_{\kappa,\mathbf{f}}(\boldsymbol{\nu}) \ge G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda})$ for all $\boldsymbol{\nu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, and hence $\boldsymbol{\lambda} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. ∎

We next provide a description of the weighted potentials of the solutions to Problem 4.2 and single out their characteristic properties. The standing assumptions stated in Sections 4.1 and 5 are required.

THEOREM 8.2. *Let the A_i, $i \in I$, be nearly closed, and assume that* (1.3), (6.1), *and*

$$\sup_{x\in\breve{A}_i} g_i(x) < \infty \quad \text{for all } i \in I \tag{8.2}$$

hold. Assume also that for every $i \in I_0$,

$$\xi^i|_K \in \mathcal{E}^+_\kappa(X) \quad \text{for every compact } K \subset A^\circ_i, \tag{8.3}$$

$$\xi^i(A_i \setminus A^\circ_i) = 0, \tag{8.4}$$

A°_i being defined in (5.5). *If moreover the $f_i|_{\breve{A}_i}$, $i\in I$, are lower bounded* ([20]), *then for any $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ the following two assertions are equivalent:*

(i) $\boldsymbol{\lambda} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$.

(ii) *There exists $(w^i_{\boldsymbol{\lambda}})_{i\in I} \in \mathbb{R}^{\operatorname{Card} I}$ such that for all $i \in I_0$,*

$$W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \ge w^i_{\boldsymbol{\lambda}} g_i \quad (\xi^i - \lambda^i)\text{-a.e.}, \tag{8.5}$$

$$W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \le w^i_{\boldsymbol{\lambda}} g_i \quad \lambda^i\text{-a.e.}, \tag{8.6}$$

while for all $i \in I \setminus I_0$,

$$W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \ge w^i_{\boldsymbol{\lambda}} g_i \quad \text{n.e. on } A_i, \tag{8.7}$$

$$W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} = w^i_{\boldsymbol{\lambda}} g_i \quad \lambda^i\text{-a.e.} \tag{8.8}$$

([20]) If Case I holds, then the f_i, $i \in I$, are necessarily lower bounded on X.

Proof. As seen from (8.3), the ξ^i, $i \in I_0$, are c_κ-absolutely continuous, which will be frequently used in the proof ([21]). There is therefore no loss of generality in replacing each ξ^i, $i \in I_0$, by the extension of $\xi^i|_{A_i \cap \breve{A}_i}$ by 0 to all of X (denoted again by ξ^i). We next replace (again with the notation preserved) the A_i, $i \in I$, by the (closed) sets $\breve{A}_i$, which also involves no loss of generality. Note that then (8.3) and (8.4) remain valid.

For every $\boldsymbol{\nu} = (\nu^\ell)_{\ell \in I} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and every $i \in I$, write $\boldsymbol{\nu}_i := (\nu^\ell_i)_{\ell \in I}$, where $\nu^\ell_i := \nu^\ell$ for all $\ell \neq i$ and $\nu^i_i = 0$; then $\boldsymbol{\nu}_i \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$. According to (3.5) and (3.10), $\kappa^i_{\boldsymbol{\nu}_i}$ is given by

$$\kappa^i_{\boldsymbol{\nu}_i}(x) = s_i \sum_{\ell \in I,\ \ell \neq i} s_\ell \kappa(x, \nu^\ell) = s_i \kappa(x, R\boldsymbol{\nu}_i),$$

and it is well defined and finite n.e. (Corollary 3.12).

Furthermore, under the stated assumptions, $\kappa^i_{\boldsymbol{\nu}_i}$ *is lower bounded on* A_i. In fact, in the same manner as in the proof of Lemma 7.1 we see from (1.3) that $|R\boldsymbol{\nu}_i|(X) \leq C$, where $C \in (0, \infty)$. This together with (6.1) implies that $\kappa(\cdot, R\boldsymbol{\nu}_i^-)$, resp. $\kappa(\cdot, R\boldsymbol{\nu}_i^+)$, is upper bounded on A^+, resp. on A^-, which in view of the preceding display yields our claim.

Fix $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. With $\tilde{f}_i := f_i + \kappa^i_{\boldsymbol{\lambda}_i}$, define the function

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} := \kappa(\cdot, \lambda^i) + \tilde{f}_i = \kappa(\cdot, \lambda^i) + f_i + \kappa^i_{\boldsymbol{\lambda}_i}. \tag{8.9}$$

Comparing this with (3.5) and (4.1), we get

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} = W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \quad \text{for all } i \in I. \tag{8.10}$$

Note that $W^{\lambda^i}_{\kappa,\tilde{f}_i}$ is finite n.e. on A_i° and lower bounded on A_i, because this is the case for each of the summands $\kappa(\cdot, \lambda^i)$, f_i, and $\kappa^i_{\boldsymbol{\lambda}_i}$.

To establish the equivalence of (i) and (ii), suppose first that (i) holds, i.e., $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ solves Problem 4.2. Fix $i \in I$. By (3.4) and (4.2), for any $\boldsymbol{\nu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with $\boldsymbol{\nu}_i = \boldsymbol{\lambda}_i$ (in particular, for $\boldsymbol{\nu} = \boldsymbol{\lambda}$) we get

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\nu}) = G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_i) + G_{\kappa,\tilde{f}_i}(\nu^i).$$

Combined with $G_{\kappa,\mathbf{f}}(\boldsymbol{\nu}) \geq G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda})$, this yields $G_{\kappa,\tilde{f}_i}(\nu^i) \geq G_{\kappa,\tilde{f}_i}(\lambda^i)$, and hence λ^i minimizes $G_{\kappa,\tilde{f}_i}(\nu)$, where ν ranges over the class $\mathcal{E}^{\sigma^i}_{\kappa,\tilde{f}_i}(A_i, a_i, g_i)$. This enables us to show that there exists $w_{\lambda^i} \in \mathbb{R}$ such that

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} \geq w_{\lambda^i} g_i \quad (\xi^i - \lambda^i)\text{-a.e.}, \tag{8.11}$$

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} \leq w_{\lambda^i} g_i \quad \lambda^i\text{-a.e.}, \tag{8.12}$$

([21]) As in [30, p. 134], we call $\mu \in \mathfrak{M}(X)$ c_κ-*absolutely continuous* if $\mu(K) = 0$ for every compact $K \subset X$ with $c_\kappa(K) = 0$. Then $|\mu|_*(Q) = 0$ for any $Q \subset X$ with $c_\kappa(Q) = 0$. Every $\mu \in \mathcal{E}_\kappa(X)$ is c_κ-absolutely continuous (but not conversely [30, pp. 134–135]).

whenever $i \in I_0$, while otherwise (for $i \in I \setminus I_0$)

$$W^{\lambda^i}_{\kappa,\tilde f_i} \geq w_{\lambda^i} g_i \quad \text{n.e. on } A_i, \tag{8.13}$$

$$W^{\lambda^i}_{\kappa,\tilde f_i} = w_{\lambda^i} g_i \quad \lambda^i\text{-a.e.} \tag{8.14}$$

To this end, for any $w \in \mathbb{R}$ write

$$A_i^+(w) := \{x \in A_i : W^{\lambda^i}_{\kappa,\tilde f_i}(x) > w g_i(x)\},$$
$$A_i^-(w) := \{x \in A_i : W^{\lambda^i}_{\kappa,\tilde f_i}(x) < w g_i(x)\},$$

and assume first that $i \in I_0$. Then (8.11) holds with

$$w_{\lambda^i} := L_i := \sup\{t \in \mathbb{R} : W^{\lambda^i}_{\kappa,\tilde f_i} \geq t g_i \ (\xi^i - \lambda^i)\text{-a.e.}\}.$$

In turn, (8.11) with $w_{\lambda^i} = L_i$ yields $L_i < \infty$, because

$$\widetilde W^{\lambda^i}_{\kappa,\tilde f_i} := W^{\lambda^i}_{\kappa,\tilde f_i}/g_i < \infty \quad \text{n.e. on } A_i^\circ,$$

hence $(\xi^i - \lambda^i)$-a.e. on A_i°, for ξ^i and λ^i are both c_κ-absolutely continuous, and finally $(\xi^i - \lambda^i)$-a.e. on A_i by (8.4). Also, $L_i > -\infty$ since, in consequence of (8.2), $\widetilde W^{\lambda^i}_{\kappa,\tilde f_i}$ along with $W^{\lambda^i}_{\kappa,\tilde f_i}$ is lower bounded on A_i.

We next establish (8.12) with $w_{\lambda^i} = L_i$. Assume, on the contrary, that this fails to hold. Since $\widetilde W^{\lambda^i}_{\kappa,\tilde f_i}$ is λ^i-measurable, one can choose $w_i \in (L_i, \infty)$ so that $\lambda^i(A_i^+(w_i)) > 0$. At the same time, as $w_i > L_i$, it follows from the definition of L_i that $(\xi^i - \lambda^i)(A_i^-(w_i)) > 0$. Therefore, there exist compact sets $K_1 \subset A_i^+(w_i)$ and $K_2 \subset A_i^-(w_i)$ such that

$$0 < \langle g_i, \lambda^i|_{K_1}\rangle < \langle g_i, (\xi^i - \lambda^i)|_{K_2}\rangle. \tag{8.15}$$

Write $\tau^i := (\xi^i - \lambda^i)|_{K_2}$; then $\kappa(\tau^i, \tau^i) < \infty$ by (8.3). Since $\langle W^{\lambda^i}_{\kappa,\tilde f_i}, \tau^i\rangle \leq \langle w_i g_i, \tau^i\rangle < \infty$, in view of (8.9) we get $\langle \tilde f_i, \tau^i\rangle < \infty$. Define

$$\theta^i := \lambda^i - \lambda^i|_{K_1} + c_i \tau^i, \quad \text{where} \quad c_i := \langle g_i, \lambda^i|_{K_1}\rangle / \langle g_i, \tau^i\rangle.$$

Observing from (8.15) that $c_i \in (0,1)$, we obtain by straightforward verification $\langle g_i, \theta^i\rangle = a_i$ and also $\theta^i \leq \xi^i$. Hence, $\theta^i \in \mathcal{E}^{\xi^i}_{\kappa,\tilde f_i}(A_i, a_i, g_i)$. But

$$\begin{aligned}\langle W^{\lambda^i}_{\kappa,\tilde f_i}, \theta^i - \lambda^i\rangle &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \theta^i - \lambda^i\rangle \\ &= -\langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \lambda^i|_{K_1}\rangle + c_i \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \tau^i\rangle < 0,\end{aligned}$$

which is impossible in view of the scalar version of Lemma 8.1 (with $I = \{i\}$). The contradiction obtained establishes (8.12).

Let now $i \in I \setminus I_0$. Since λ^i minimizes $G_{\kappa,\tilde f_i}(\nu)$ among $\nu \in \mathcal{E}^+_{\kappa,\tilde f_i}(A_i, a_i, g_i)$, it follows from [43, Theorem 7.1] that (8.13) and (8.14) hold with

$$w_{\lambda^i} := \langle W^{\lambda^i}_{\kappa,\tilde f_i}, \lambda^i\rangle / a_i \in \mathbb{R}.$$

Substituting (8.11)–(8.14) into (8.10) establishes (8.5)–(8.8) with $w^i_{\boldsymbol\lambda} := w_{\lambda^i}$, $i \in I$. Hence, (i)⇒(ii).

To complete the proof, suppose finally that (ii) holds. By (8.10), for every $i \in I_0$, resp. $i \in I \setminus I_0$, (8.11) and (8.12), resp. (8.13) and (8.14), are then fulfilled with $w_{\lambda^i} := w^i_{\boldsymbol\lambda}$ and $\tilde f_i := f_i + \kappa^i_{\boldsymbol\lambda_i}$. We observe from (8.11) and (8.12) that $\lambda^i(A_i^+(w_{\lambda^i})) = 0$ and $(\xi^i - \lambda^i)(A_i^-(w_{\lambda^i})) = 0$ for all $i \in I_0$. If we fix $\boldsymbol\nu \in \mathcal{E}^{\boldsymbol\sigma}_{\kappa,\mathbf f}(\mathbf A, \mathbf a, \mathbf g)$, we get, for all $i \in I_0$,

$$
\begin{aligned}
(8.16)\qquad \langle W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}, \nu^i - \lambda^i\rangle &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i - \lambda^i\rangle \\
&= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i|_{A_i^+(w_{\lambda^i})}\rangle + \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, (\nu^i - \xi^i)|_{A_i^-(w_{\lambda^i})}\rangle \ge 0.
\end{aligned}
$$

Furthermore, it follows from (8.13) and (8.14) that

$$\lambda^i(A_i^+(w_{\lambda^i})) = \lambda^i(A_i^-(w_{\lambda^i})) = \nu^i(A_i^-(w_{\lambda^i})) = 0 \quad \text{for all } i \in I \setminus I_0,$$

ν^i being c_κ-absolutely continuous. Hence, for all $i \in I \setminus I_0$,

$$
\begin{aligned}
(8.17)\qquad \langle W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}, \nu^i - \lambda^i\rangle &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i - \lambda^i\rangle \\
&= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i|_{A_i^+(w_{\lambda^i})}\rangle \ge 0.
\end{aligned}
$$

Summing the inequalities in (8.16) and (8.17) over all $i \in I$, we see from Lemma 8.1 in view of the arbitrary choice of $\boldsymbol\nu \in \mathcal{E}^{\boldsymbol\sigma}_{\kappa,\mathbf f}(\mathbf A, \mathbf a, \mathbf g)$ that $\boldsymbol\lambda$ is indeed a solution to Problem 4.2. ∎

COROLLARY 8.3. *Under the hypotheses of Theorem* 8.2*, assume moreover that κ is continuous on $\breve A^+ \times \breve A^-$ and Case* I *holds. Then* (8.6) *and* (8.8) *in* (ii) *are equivalent to the following apparently stronger relations:*

$$W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}(x) \le w^i_{\boldsymbol\lambda} g_i(x) \quad \text{for all } x \in S(\lambda^i) \text{ and all } i \in I_0,$$

$$W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}(x) = w^i_{\boldsymbol\lambda} g_i(x) \quad \text{for nearly all } x \in S(\lambda^i) \text{ and all } i \in I \setminus I_0.$$

Proof. This will follow once we have proven that for any $i \in I$, $W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}|_{\breve A_i}$ is l.s.c., which in turn holds if $\kappa(\cdot, R\boldsymbol\lambda^-)|_{\breve A^+}$ and $\kappa(\cdot, R\boldsymbol\lambda^+)|_{\breve A^-}$ are continuous. To establish the latter, write

$$(8.18)\qquad \kappa^*(x,y) := -\kappa(x,y) + \sup_{(x',y') \in \breve A^+ \times \breve A^-} \kappa(x',y'), \qquad (x,y) \in \breve A^+ \times \breve A^-.$$

Under the stated assumptions, κ^* is nonnegative and continuous, and hence

$$\kappa^*(x, R\boldsymbol\lambda^-) := \int \kappa^*(x,y)\, dR\boldsymbol\lambda^-(y), \qquad x \in \breve A^+,$$

is l.s.c. In the same manner as in the proof of Lemma 7.1, we observe from (1.3) that $|R\boldsymbol\lambda|(X) \le C < \infty$. Integrating (8.18) with respect to $R\boldsymbol\lambda^-$, we therefore see that $\kappa^*(x, R\boldsymbol\lambda^-)$, $x \in \breve A^+$, coincides up to a finite summand with the restriction of $-\kappa(x, R\boldsymbol\lambda^-)$ to $\breve A^+$. It follows that $\kappa(x, R\boldsymbol\lambda^-)|_{\breve A^+}$ must be upper semicontinuous. Being also l.s.c., $\kappa(x, R\boldsymbol{\lambda}^-)|_{\breve{A}^+}$ is actually continuous as desired. The same holds with the indices + and − reversed. ■

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