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Values at non-positive integers of generalized Euler–Zagier multiple zeta-functions

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1. Introduction and the statement of the main result. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ be such that $\Re(\gamma_j) > 0$ and $\Re(b_j) > -\Re(\gamma_1)$ for all $j = 1, \dots, n$.

The generalized Euler–Zagier multiple zeta-function is defined formally for n-tuples of complex variables $\mathbf{s} = (s_1, \dots, s_n)$ by

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}}. \tag{1}$$

If $b_1 = 0$, $b_j = j - 1$ for all $j = 2, \dots, n$ and $\gamma_j = 1$ for all $j = 1, \dots, n$, then $\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b})$ coincides with the classical Euler–Zagier multiple zeta-function (see [19] and [10])

$$\sum_{1 \leq m_1 < \dots < m_n} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}.$$

The generalized Euler–Zagier multiple zeta-function $\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b})$ converges absolutely in the domain

$$\mathcal{D}_n := \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \Re(s_j + \dots + s_n) > n + 1 - j \text{ for all } j = 1, \dots, n\} \tag{2}$$

(see [12]), and has a meromorphic continuation to $\mathbb{C}^n$, whose poles are located in the union of the hyperplanes

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n,\ k_1, \dots, k_n \in \mathbb{N}_0),$$

where $\mathbb{N}_0$ denotes the set of all non-negative integers. Moreover, it is known that for $n \geq 2$, almost all n-tuples of non-positive integers lie on the singular

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locus above and are points of discontinuity (see [1], [4], [5], [11], [13]). The evaluation of (limit) values of multiple zeta-functions at those points was first considered by S. Akiyama, S. Egami and Y. Tanigawa [1], and then studied further in [2], [16], [17], [11], [15], and [14].

In [11], Y. Komori proved that for $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$ such that $\theta_j + \cdots + \theta_n \neq 0$ for all $j = 1, \ldots, n$, the limit

$$\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \to 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \tag{3}$$

exists, and expressed it in terms of $\mathbf{N}$, $\boldsymbol{\theta}$ and generalized multiple Bernoulli numbers defined implicitly as coefficients of some multiple series.

Our main result (i.e. Theorem 1) gives a closed explicit formula for $\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$ in terms of $\mathbf{N}$, $\boldsymbol{\theta}$ and *the classical Bernoulli numbers* B_k ($k \in \mathbb{N}_0$) defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \tag{4}$$

Before giving our result let us introduce a few notations:

- For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we write $|\mathbf{x}| = x_1 + \cdots + x_n$.
- For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, we write

$$\mathbf{x}^{\mathbf{k}} = \prod_{i=1}^n x_i^{k_i}, \qquad \binom{\mathbf{x}}{\mathbf{k}} = \prod_{i=1}^n \binom{x_i}{k_i};$$

- For $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we define

$$K(\mathbf{N}, \boldsymbol{\alpha}) := \Big\{ j \in \{1, \ldots, n\} \;\Big|\; (n + 1 - j) + \sum_{i=j}^n N_i = \sum_{i=j}^n \alpha_i \Big\}, \tag{5}$$

$$L(\mathbf{N}, \boldsymbol{\alpha}) := \{ j \in \{1, \ldots, n\} \mid \alpha_j \geq N_j + 1 \}. \tag{6}$$

- For $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ and $I \subset \{1, \ldots, n\}$, we define

$$\mathcal{J}(I, \mathbf{N}) := \{ \boldsymbol{\alpha} \in \mathbb{N}_0^n \mid K(\mathbf{N}, \boldsymbol{\alpha}) = I \text{ and } |L(\mathbf{N}, \boldsymbol{\alpha})| = |I| \}. \tag{7}$$

Remark. $\mathcal{J}(I, \mathbf{N})$ is a finite set and $\mathcal{J}(I, \mathbf{N}) \subset \{0, \ldots, |\mathbf{N}| + n\}^n$ (see Lemma 2 for a proof).

- For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n$ we define the polynomials (in $\mathbf{b}$) $c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k})$ (where $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n, |\mathbf{k}| \leq |\boldsymbol{\alpha}|$) as the coefficients of the polynomial $\prod_{j=1}^n (\sum_{i=1}^j X_i + b_j)^{\alpha_j}$, that is,

(8)
$$\prod_{j=1}^n \Big(\sum_{i=1}^j X_i + b_j \Big)^{\alpha_j} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k}) \mathbf{X}^{\mathbf{k}} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k}) X_1^{k_1} \cdots X_n^{k_n}.$$

With these notations our main result is the following:

THEOREM 1. *Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n$ be such that $\Re(\gamma_j) > 0$ and $\Re(b_j) > -\Re(\gamma_1)$ for all $j = 1, \ldots, n$. Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$ be such that $\theta_j + \cdots + \theta_n \neq 0$ for all $j = 1, \ldots, n$. Then, for any $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$, the limit*

$$\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \to 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b})$$

exists, and is explicitly given by

$$(9) \quad \zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) =$$

$$\sum_{I \subset \{1,\ldots,n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I,\mathbf{N})} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n \\ |\mathbf{k}| \le |\boldsymbol{\alpha}|}} \frac{c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k})(-1)^{n-|I|+\sum_{j \in L(\mathbf{N},\boldsymbol{\alpha})}(\alpha_j - N_j)} \prod_{j \notin L(\mathbf{N},\boldsymbol{\alpha})} \binom{N_j}{\alpha_j}}{\prod_{j \in L(\mathbf{N},\boldsymbol{\alpha})} \alpha_j \binom{\alpha_j - 1}{N_j} \prod_{j \notin I} (n + 1 - j + \sum_{i=j}^n N_i - \sum_{i=j}^n \alpha_i)}$$

$$\times \Big(\gamma_1^{|\mathbf{N}| - |\boldsymbol{\alpha}| + n + k_1 - 1} \prod_{j=2}^n \gamma_j^{k_j - 1}\Big) \Big(\frac{\prod_{j \in L(\mathbf{N},\boldsymbol{\alpha})} \theta_j}{\prod_{j \in I}(\theta_j + \cdots + \theta_n)}\Big) \Big(\prod_{j=1}^n B_{k_j}\Big).$$

An essential idea in our proof of Theorem 1 is to prove these formulas first for a small convenient class of multiple zeta-functions and then use the analyticity of the values on the parameters defining the multiple zeta-functions to deduce the formulas in the general case. We also prove an extension of a lemma of "Raabe type" due to E. Friedman and A. Pereira [8, Lemma 2.4] and use it in the proof.

In a forthcoming work [6] we will study a more general form of multiple zeta-functions, whose denominators are given by (not necessarily linear) polynomials.

2. Some useful lemmas

LEMMA 1. *Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ be such that $\Re(\gamma_j) > 0$ for any $j = 1, \ldots, n$. For $\mathbf{s} = (s_1, \ldots, s_n) \in \mathcal{D}_n$ (see (2)) define*

$$(10) \qquad Y_n(\mathbf{s}; \boldsymbol{\gamma}) := \int_{(1,\infty) \times (0,\infty)^{n-1}} \prod_{j=1}^n \Big(\sum_{i=1}^j \gamma_i x_i\Big)^{-s_j} \, dx_n \ldots dx_1.$$

Then, for $\mathbf{s} = (s_1, \ldots, s_n) \in \mathcal{D}_n$, $Y_n(\mathbf{s}; \boldsymbol{\gamma})$ is absolutely convergent and

$$Y_n(\mathbf{s}; \boldsymbol{\gamma}) = \frac{\gamma_1^{-s_1 - \cdots - s_n + n}}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^n (s_j + \cdots + s_n + j - n - 1)}.$$

In particular, $Y_n(\mathbf{s}; \boldsymbol{\gamma})$ has a meromorphic continuation to $\mathbb{C}^n$ and its polar locus is the set

$$\bigcup_{j=1}^n \{\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n \mid s_j + \cdots + s_n = n + 1 - j\}.$$

Proof. Just integrate first with respect to x_n, then with respect to x_{n-1} etc. ∎

LEMMA 2. *Let* $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ *and* $I \subset \{1, \ldots, n\}$. *The set* $\mathcal{J}(I, \mathbf{N})$ *defined by* (7) *is finite and contained in* $\{0, \ldots, |\mathbf{N}| + n\}^n$.

Proof. Denote by $j_1, \ldots, j_q$ the elements of I, where $q = |I|$. We can assume that $j_1 < \cdots < j_q$.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J}(I, \mathbf{N})$. It follows that for any $k = 2, \ldots, q$,

$$\begin{aligned}\sum_{j=j_{k-1}}^{j_k-1} \alpha_j &= \sum_{j=j_{k-1}}^{n} \alpha_j - \sum_{j=j_k}^{n} \alpha_j \\ &= (n+1-j_{k-1}) + \sum_{j=j_{k-1}}^{n} N_j - (n+1-j_k) - \sum_{j=j_k}^{n} N_j \\ &= (j_k - j_{k-1}) + \sum_{j=j_{k-1}}^{j_k-1} N_j \geq 1 + \sum_{j=j_{k-1}}^{j_k-1} N_j,\end{aligned}$$

hence $[j_{k-1}, j_k) \cap L(\mathbf{N}, \boldsymbol{\alpha}) \neq \emptyset$ for all $k = 2, \ldots, q$. Moreover, the inequality

$$\sum_{j=j_q}^{n} \alpha_j = (n+1-j_q) + \sum_{j=j_q}^{n} N_j \geq 1 + \sum_{j=j_q}^{n} N_j$$

implies also that $[j_q, n] \cap L(\mathbf{N}, \boldsymbol{\alpha}) \neq \emptyset$. Since $|L(\mathbf{N}, \boldsymbol{\alpha})| = q$, the above observation implies that $\min L(\mathbf{N}, \boldsymbol{\alpha}) \geq j_1$. We deduce that for $j \in L(\mathbf{N}, \boldsymbol{\alpha})$,

$$\alpha_j \leq \sum_{j=j_1}^{n} \alpha_j = (n+1-j_1) + \sum_{j=j_1}^{n} N_j \leq |\mathbf{N}| + n.$$

If $j \notin L(\mathbf{N}, \boldsymbol{\alpha})$, obviously $\alpha_j < N_j + 1 \leq |\mathbf{N}| + n$. This ends the proof. ∎

The following lemma is crucial for our proof of Theorem 1.

LEMMA 3. *Let* $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ *and* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$ *be such that* $\theta_j + \cdots + \theta_n \neq 0$ *for all* $j = 1, \ldots, n$. *Set*

$$\delta := \tfrac{1}{2} \min\{(1 + |\theta_j|)^{-1}, |\theta_j + \cdots + \theta_n|^{-1} \mid j = 1, \ldots, n\} \in (0, 1/2).$$

Let $U_\delta := \{t \in \mathbb{C} \mid |t| < \delta\}$. *For* $t \in U_\delta \setminus \{0\}$ *define*

$$G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t) := \frac{\prod_{j=1}^{n} \binom{N_j - t\theta_j}{\alpha_j}}{\prod_{j=1}^{n} \bigl(t - \frac{(N_j + \cdots + N_n) + (n+1-j) - (\alpha_j + \cdots + \alpha_n)}{\theta_j + \cdots + \theta_n}\bigr)}. \tag{11}$$

Let $q = q(\mathbf{N}, \boldsymbol{\alpha}) := |K(\mathbf{N}, \boldsymbol{\alpha})|$ *and* $q' = q'(\mathbf{N}, \boldsymbol{\alpha}) := |L(\mathbf{N}, \boldsymbol{\alpha})|$, *where* $K(\mathbf{N}, \boldsymbol{\alpha})$, $L(\mathbf{N}, \boldsymbol{\alpha})$ *are defined by* (5), (6), *respectively. Then*

(i) $q' \geq q$;

(ii) $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)$ *is analytic in* U_δ *and there exists a constant* $C = C(\mathbf{N},\theta) > 0$ (*independent of* $\boldsymbol{\alpha}$) *such that*

$$|G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)| \le C|t|^{q'-q} \quad \textit{for all } t \in U_\delta;$$

(iii) *if* $q' > q$, *then* $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0) = 0$;

(iv) *if* $q' = q$, *then*

(a) $$G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0) = \frac{(-1)^{n-q}\Big(\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})} \frac{(-1)^{\alpha_j - N_j}\theta_j}{\alpha_j\binom{\alpha_j-1}{N_j}}\Big)\prod_{j\notin L(\mathbf{N},\boldsymbol{\alpha})}\binom{N_j}{\alpha_j}}{\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}\frac{(N_j+\dots+N_n)+(n+1-j)-(\alpha_j+\dots+\alpha_n)}{\theta_j+\dots+\theta_n}};$$

(b) $\boldsymbol{\alpha} \in \mathcal{J}(K(\mathbf{N},\boldsymbol{\alpha}), N) \subset \{0,\dots,|\mathbf{N}|+n\}^n$ (*see Lemma* 2).

Proof. (i) Repeating the argument of the proof of the previous lemma with $I = K(\mathbf{N},\boldsymbol{\alpha})$ shows that $q' = |L(\mathbf{N},\boldsymbol{\alpha})| \ge q$.

(ii) First it is easy to see that $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)$ is analytic in the pointed disk $U_\delta \setminus \{0\}$. (If the integer $(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)$ is not zero, then $\big|\frac{(N_j+\dots+N_n)+(n+1-j)-(\alpha_j+\dots+\alpha_n)}{\theta_j+\dots+\theta_n}\big| \ge 2\delta$.) Moreover, for $j = 1,\dots,n$ and $t \in U_\delta$,

$$\binom{N_j - t\theta_j}{\alpha_j} = \frac{1}{\alpha_j!}\prod_{k=0}^{\alpha_j-1}(N_j - t\theta_j - k). \tag{12}$$

It follows that

- if $\alpha_j \le N_j$, then
$$\binom{N_j - t\theta_j}{\alpha_j}\bigg|_{t=0} = \binom{N_j}{\alpha_j}$$
and for $t \in U_\delta$,
$$\bigg|\binom{N_j - t\theta_j}{\alpha_j}\bigg| \le \frac{1}{\alpha_j!}\prod_{k=0}^{\alpha_j-1}(N_j + 1 - k) \le (N_j+1)!;$$
- if $\alpha_j \ge N_j + 1$, then
$$\binom{N_j - t\theta_j}{\alpha_j}\bigg|_{t=0} = \binom{N_j}{\alpha_j} = 0$$
and for $t \in U_\delta$,
$$\begin{aligned}\bigg|\binom{N_j - t\theta_j}{\alpha_j}\bigg| &\le \frac{|t\theta_j|}{\alpha_j!}\prod_{k=0}^{N_j-1}(N_j - k + 1)\prod_{k=N_j+1}^{\alpha_j-1}(k - N_j + 1)\\ &= |t\theta_j|\frac{N_j!}{\alpha_j(\alpha_j-1)\cdots(\alpha_j-N_j+1)}(N_j+1) \le (N_j+1)|t\theta_j|.\end{aligned}$$

We deduce that for $t \in U_\delta \setminus \{0\}$,

$$
\begin{aligned}
&|G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)| \\
&\quad \ll_{\mathbf{N},\boldsymbol{\theta}} \frac{\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})}|t|}{\prod_{j\in K(\mathbf{N},\boldsymbol{\alpha})}|t|\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}\big|t-\frac{(N_j+\cdots+N_n)+(n+1-j)-(\alpha_j+\cdots+\alpha_n)}{\theta_j+\cdots+\theta_n}\big|} \\
&\quad \ll_{\mathbf{N},\boldsymbol{\theta}} \frac{|t|^{q'-q}}{\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}(|t(\theta_j+\cdots+\theta_n)-(N_j+\cdots+N_n) - (n+1-j)+(\alpha_j+\cdots+\alpha_n)|)} \\
&\quad \ll_{\mathbf{N},\boldsymbol{\theta}} \frac{|t|^{q'-q}}{\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}(1/2)} \ll_{\mathbf{N},\boldsymbol{\theta}} |t|^{q'-q}.
\end{aligned}
$$

It follows that $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)$ is analytic in the whole disk U_δ and satisfies in it the uniform estimate $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t) \ll_{\mathbf{N},\boldsymbol{\theta}} |t|^{q'-q}$.

(iii) follows from (ii).

(iv) The identity (12) implies that if $\alpha_j \geq N_j+1$, then

$$
\binom{N_j-t\theta_j}{\alpha_j} \sim_{t\to 0} \frac{-t\theta_j}{\alpha_j!}\prod_{k=0}^{N_j-1}(N_j-k)\prod_{k=N_j+1}^{\alpha_j-1}(N_j-k) \sim_{t\to 0} \frac{t\theta_j(-1)^{\alpha_j-N_j}}{\alpha_j\binom{\alpha_j-1}{N_j}}.
$$

It follows that

$$
G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t) \sim_{t\to 0} \frac{(-1)^{n-q}\big(\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})}\frac{(-1)^{\alpha_j-N_j}\theta_j}{\alpha_j\binom{\alpha_j-1}{N_j}}\big)\prod_{j\notin L(\mathbf{N},\boldsymbol{\alpha})}\binom{N_j}{\alpha_j}}{\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}\frac{(N_j+\cdots+N_n)+(n+1-j)-(\alpha_j+\cdots+\alpha_n)}{\theta_j+\cdots+\theta_n}}. \ \blacksquare
$$

3. The key propositions. Now we introduce a class of multivariate zeta-functions which are slightly more general than those considered in Theorem 1. We will be working in this slightly more general class because it is more suitable for induction arguments.

Let $\mathbf{q}=(q_1,\ldots,q_n)\in\mathbb{N}^n$. Set $q=|\mathbf{q}|=q_1+\cdots+q_n$. We will use the notation $\mathbf{s}=(s_{1,1},\ldots,s_{1,q_1},\ldots,s_{j,1},\ldots,s_{j,q_j},\ldots,s_{n,1},\ldots,s_{n,q_n})$ for elements of $\mathbb{C}^q$, and denote $|\mathbf{s}|=s_{1,1}+\cdots+s_{1,q_1}+\cdots+s_{j,1}+\cdots+s_{j,q_j}+\cdots+s_{n,1}+\cdots+s_{n,q_n}$. Let $\varepsilon\geq 0$ (notice that we admit $\varepsilon=0$), $\boldsymbol{\gamma}\in\mathbb{C}^n$, and define

$$
\begin{aligned}
(13)\qquad W_\varepsilon(\mathbf{q},n) &:= \{(\mathbf{u},\boldsymbol{\gamma})\in\mathbb{C}^q\times\mathbb{C}^n \mid \Re(\gamma_j)>\varepsilon \text{ and } \Re(u_{j,k}+\gamma_1)>\varepsilon \\
&\qquad\qquad \text{for all } j=1,\ldots,n \text{ and } k=1,\ldots,q_j\},
\end{aligned}
$$

$$
V_{\varepsilon,\mathbf{q}}(\boldsymbol{\gamma}) := \{\mathbf{u}\in\mathbb{C}^q \mid \Re(u_{j,k}+\gamma_1)>\varepsilon \text{ for all } j=1,\ldots,n \text{ and } k=1,\ldots,q_j\},
$$

and

$$
\mathcal{D}_{n,\mathbf{q}} := \Big\{\mathbf{s}\in\mathbb{C}^q \,\Big|\, \Re\Big(\sum_{i=j}^{n}\sum_{k=1}^{q_i}s_{i,k}\Big)>n+1-j \text{ for all } j=1,\ldots,n\Big\}.
$$

For $\mathbf{s} \in \mathcal{D}_{n,\mathbf{q}}$ and $(\mathbf{u}, \boldsymbol{\gamma}) \in W_0(\mathbf{q}, n)$, define

(14)
$$Y_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) := \int\limits_{[1,\infty)\times[0,\infty)^{n-1}} \prod_{j=1}^{n} \prod_{k=1}^{q_j} \big(\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k}\big)^{-s_{j,k}}\, dx_n \ldots dx_1,$$

and

$$Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) := \sum_{\substack{m_1 \geq 1 \\ m_2,\ldots,m_n \geq 0}} \frac{1}{\prod_{j=1}^{n} \prod_{k=1}^{q_j} \big(\gamma_1 m_1 + \cdots + \gamma_j m_j + u_{j,k}\big)^{s_{j,k}}}. \tag{15}$$

The multiple zeta-function $Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ is absolutely convergent in $\mathcal{D}_{n,\mathbf{q}}$, and in this region

$$Y_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) = \int\limits_{[0,1]^n} Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u}_{\mathbf{q}}(\mathbf{b});\boldsymbol{\gamma})\, d\mathbf{b}. \tag{16}$$

Here, $\mathbf{u}_{\mathbf{q}}(\mathbf{b}) \in \mathbb{C}^q$ is given by

$$\mathbf{u}_{\mathbf{q}}(\mathbf{b}) = (u_{1,1}(\mathbf{b}), \ldots, u_{1,q_1}(\mathbf{b}), \ldots, u_{j,1}(\mathbf{b}), \ldots, u_{j,q_j}(\mathbf{b}), \ldots, u_{n,1}(\mathbf{b}), \ldots, u_{n,q_n}(\mathbf{b})),$$

where $\mathbf{b} = (b_1, \ldots, b_n) \in [0,1]^n$ and $u_{j,k}(\mathbf{b}) = u_{j,k} + \sum_{i=1}^{j} \gamma_i b_i$ for all $j = 1, \ldots, n$ and all $k = 1, \ldots, q_j$.

Now we state a proposition which gives several analytic properties of $Y_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ and $Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$.

Proposition 1.

(i) *The functions* $\mathbf{s} \mapsto Y_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ *and* $\mathbf{s} \mapsto Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ *can be meromorphically continued to* $\mathbb{C}^q$ *and their poles are in the set*

$$\mathcal{P}_{n,\mathbf{q}} := \bigcup_{j=1}^{n} \bigcup_{k_j \in \mathbb{N}_0} \Big\{ \mathbf{s} \in \mathbb{C}^q \,\Big|\, \sum_{i=j}^{n} \sum_{k=1}^{q_i} s_{i,k} = n + 1 - j - k_j \Big\}.$$

Therefore (16) *is valid for all* $\mathbf{s} \in \mathbb{C}^n \setminus \mathcal{P}_{n,\mathbf{q}}$.

(ii) *For fixed* $\boldsymbol{\omega} \in \mathbb{C}^q$ *and* $\boldsymbol{\theta} \in \mathbb{C}^q$ *such that* $\sum_{i=j}^{n} \sum_{k=1}^{q_i} \theta_{i,k} \neq 0$ *for all* $j = 1, \ldots, n$, *there exist* $\delta = \delta(\boldsymbol{\omega}, \boldsymbol{\theta}) > 0$ *and* $M = M(\boldsymbol{\omega}, \boldsymbol{\theta}) > 0$ *such that*

  (a) $(t, \mathbf{u}, \boldsymbol{\gamma}) \mapsto t^n\, Y_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma})$ *and* $(t, \mathbf{u}, \boldsymbol{\gamma}) \mapsto t^n\, Z_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma})$ *are analytic in* $U_\delta \times W_0(\mathbf{q}, n)$, *where* $U_\delta = \{t \in \mathbb{C} \mid |t| < \delta\}$;

  (b) *for* $\varepsilon > 0$ *and* $\boldsymbol{\gamma} \in \mathbb{C}^n$ *such that* $\Re(\gamma_j) > \varepsilon$ *for all* $j = 1, \ldots, n$, *we have*

$$|t^n\, Y_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma})| \ll_{\boldsymbol{\omega},\boldsymbol{\theta},\boldsymbol{\gamma},\varepsilon} (1 + |\mathbf{u}|)^M,$$
$$|t^n\, Z_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma})| \ll_{\boldsymbol{\omega},\boldsymbol{\theta},\boldsymbol{\gamma},\varepsilon} (1 + |\mathbf{u}|)^M,$$

  uniformly in $(t, \mathbf{u}) \in U_\delta \times V_{\varepsilon,\mathbf{q}}(\boldsymbol{\gamma})$.

Proposition 1 implies the following key result:

Corollary 1. *Let* $\boldsymbol{\omega} \in \mathbb{C}^q$ *and* $\boldsymbol{\theta} \in \mathbb{C}^q$ *be such that* $\sum_{i=j}^{n} \sum_{k=1}^{q_i} \theta_{i,k} \neq 0$ *for all* $j = 1, \dots, n$. *For* $(\mathbf{u}, \boldsymbol{\gamma}) \in W_0(\mathbf{q}, n)$ *(see* (13)*), each of the meromorphic functions*

$$t \mapsto Y_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) \quad \text{and} \quad t \mapsto Z_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma})$$

has at most a pole of order n *at* $t = 0$. *Moreover if we write*

$$Y_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{k=0}^{n} \frac{y_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma})}{t^k} + O(t) \qquad \text{as } t \to 0,$$

$$Z_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{k=0}^{n} \frac{z_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma})}{t^k} + O(t) \qquad \text{as } t \to 0,$$

then, for any $k = 0, \dots, n$,

(i) *the functions*

$$(\mathbf{u}, \boldsymbol{\gamma}) \mapsto y_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \quad \text{and} \quad (\mathbf{u}, \boldsymbol{\gamma}) \mapsto z_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma})$$

are analytic in $W_0(\mathbf{q}, n)$ *and*

$$y_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma}) = \int_{[0,1]^n} z_{-k,\mathbf{q}}(\mathbf{u}_{\mathbf{q}}(\mathbf{b}); \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma})\, d\mathbf{b} \tag{17}$$

in that domain;

(ii) *there exists* $M = M(\boldsymbol{\omega}, \boldsymbol{\theta}) > 0$ *such that for* $\varepsilon > 0$ *and* $\boldsymbol{\gamma} \in \mathbb{C}^n$ *such that* $\Re(\gamma_j) > \varepsilon$ *for all* $j = 1, \dots, n$ *we have, uniformly in* $\mathbf{u} \in V_{\varepsilon,\mathbf{q}}(\boldsymbol{\gamma})$,

$$y_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \ll_{\boldsymbol{\omega},\boldsymbol{\theta},\boldsymbol{\gamma},\varepsilon} (1 + |\mathbf{u}|)^M,$$
$$z_{-k,\mathbf{q}}(\mathbf{u}; \boldsymbol{\omega}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \ll_{\boldsymbol{\omega},\boldsymbol{\theta},\boldsymbol{\gamma},\varepsilon} (1 + |\mathbf{u}|)^M.$$

Deduction of Corollary 1 from Proposition 1. The corollary follows from Proposition 1(ii) by applying Cauchy's formula which expresses the coefficients of Laurent's expansion of a given one-variable meromorphic function in terms of its integrals on small disks around its singular point. The identity (17) follows by using in addition the equality (16). ∎

Proof of Proposition 1. The proof of the assertion for $Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ is similar to (and easier than) the proof for $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$. So we will only give the proof for $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$. We use induction on n.

• *Case* $n = 1$. For $(\mathbf{u}, \gamma) = ((u_1, \dots, u_q), \gamma) \in W_0(q, 1)$ and $\mathbf{s} = (s_1, \dots, s_q) \in \mathcal{D}_{1,q}$, we have

$$Z_{1,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) = \sum_{m \geq 1} \frac{1}{\prod_{k=1}^{q} (\gamma m + u_k)^{s_k}}.$$

Let $K \in \mathbb{N}_0$. For $m \geq 1$ define

$$\psi_m(z) := \prod_{k=1}^{q} \left(1 + \frac{u_k}{\gamma m} z\right)^{-s_k}.$$

Since

$$\prod_{k=1}^{q} \left(\gamma m + u_k\right)^{-s_k} = \prod_{k=1}^{q} (\gamma m)^{-s_k} \psi_m(1),$$

applying Taylor's formula with remainder [7, (3.4)] to the function $\psi_m(z)$ we obtain for $m \geq 1$,

$$\prod_{k=1}^{q} (\gamma m + u_k)^{-s_k} = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| \leq K}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \gamma^{-|\mathbf{s}|-|\boldsymbol{\alpha}|} m^{-|\mathbf{s}|-|\boldsymbol{\alpha}|}$$
$$+ (K+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| = K+1}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \int_0^1 (1-y)^K \prod_{k=1}^{q} \left(\gamma m + u_k y\right)^{-s_k - \alpha_k} dy.$$

It follows that for $(\mathbf{u}, \boldsymbol{\gamma}) \in W_0(\mathbf{q}, 1)$ and $\mathbf{s} \in \mathcal{D}_{1,q}$,

$$Z_{1,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| \leq K}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \gamma^{-|\mathbf{s}|-|\boldsymbol{\alpha}|} \zeta(|\mathbf{s}| + |\boldsymbol{\alpha}|)$$
$$+ (K+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| = K+1}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \mathcal{R}_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha})$$

where

$$R_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha}) = \sum_{m \geq 1} \int_0^1 (1-y)^K \prod_{k=1}^{q} (\gamma m + u_k y)^{-s_k - \alpha_k} dy.$$

Let $\varepsilon > 0$. Uniformly in $m \in \mathbb{N}$, $(\mathbf{u}, \gamma) \in W_\varepsilon(\mathbf{q}, 1)$, $y \in [0,1]$ and $k \in \{1, \ldots, q\}$, we have

$$|\gamma m + u_k y| \geq \Re(\gamma) m + \Re(u_k) y = \varepsilon m + (\Re(\gamma) - \varepsilon) m + \Re(u_k) y$$
$$\geq \varepsilon m + (\Re(\gamma) - \varepsilon + \Re(u_k)) y \geq \varepsilon m$$

and

$$|\gamma m + u_k y| \leq |\gamma| m + |u_k| \leq (|\gamma| + |u_k|) m.$$

The theorem of analyticity under the integral sign then implies that

$$(\mathbf{s}, \mathbf{u}, \gamma) \mapsto R_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha})$$

is holomorphic in $\{\mathbf{s} \in \mathbb{C}^q \mid \Re(s_1+\cdots+s_q) > -K\} \times W_\varepsilon(q,1)$ and the estimate

$$R_K(\mathbf{s};\mathbf{u};\gamma;\boldsymbol{\alpha}) \ll_{s,K,\varepsilon} (1+|\mathbf{u}|+|\gamma|)^{|\mathbf{s}|+K+1}$$

holds there uniformly in $(\mathbf{u},\gamma) \in W_\varepsilon(q,1)$. By using in addition the classical properties of the Riemann zeta-function, we deduce that $\mathbf{s} \mapsto Z_{1,\mathbf{q}}(\mathbf{s};\mathbf{u};\gamma)$ has a meromorphic continuation to $\{\mathbf{s} \in \mathbb{C}^q \mid \Re(s_1+\cdots+s_q) > -K\}$ with poles in $\mathcal{P}_{1,\mathbf{q}}$, and (ii) holds for any $\boldsymbol{\omega} \in \mathbb{C}^q$ such that $\Re(\omega_1+\cdots+\omega_q) > -K$.

By letting $K \to \infty$ and $\varepsilon \to 0$, we end the proof of Proposition 1 in the case $n = 1$.

• *Induction step $n-1 \to n$.* Let $(\mathbf{u},\boldsymbol{\gamma}) \in W_0(\mathbf{q},n)$ and $\mathbf{s} \in \mathcal{D}_{n,\mathbf{q}}$. Fix $m_1 \geq 1$ and $m_2,\ldots,m_{n-1} \geq 0$. The function $\varphi(x) := \prod_{i=1}^{q_n}(\gamma_1 m_1+\cdots+\gamma_{n-1}m_{n-1}+\gamma_n x+u_{n,i})^{-s_{n,i}}$ belongs to $\mathcal{C}^\infty[0,\infty)$ and for all $k \in \mathbb{N}_0$ and all $x \in [0,\infty)$,

$$\varphi^{(k)}(x) = k! \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|=k}} \gamma_n^{|\boldsymbol{\alpha}|} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} (\gamma_1 m_1+\cdots+\gamma_{n-1}m_{n-1}+\gamma_n x+u_{n,i})^{-s_{n,i}-\alpha_i}.$$

Let $K \in \mathbb{N}_0$, and let $\tilde{B}_k$ $(k \geq 0)$ be the modified Bernoulli numbers defined by $\tilde{B}_k := B_k$ for all $k \neq 1$ and $\tilde{B}_1 := -B_1 = 1/2$. (In some references $\tilde{B}_k$ is written as B_k.) By applying the Euler–Maclaurin formula to $\varphi(x)$, we obtain

$$\begin{aligned}
(18)\qquad &\sum_{m_n=0}^{\infty} \prod_{i=1}^{q_n} \Big(\sum_{j=1}^{n} \gamma_j m_j + u_{n,i}\Big)^{-s_{n,i}} \\
&= \int_0^\infty \prod_{i=1}^{q_n} \Big(\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i}\Big)^{-s_{n,i}} dx \\
&\quad + \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|\leq K}} \frac{(-1)^{|\boldsymbol{\alpha}|}\tilde{B}_{|\boldsymbol{\alpha}|+1}\gamma_n^{|\boldsymbol{\alpha}|}}{|\boldsymbol{\alpha}|+1} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \Big(\sum_{j=1}^{n-1} \gamma_j m_j + u_{n,i}\Big)^{-s_{n,i}-\alpha_i} \\
&\quad + (-1)^K \gamma_n^{K+1} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|=K+1}} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \int_0^\infty B_{K+1}(x) \\
&\qquad\qquad \times \prod_{i=1}^{q_n} \Big(\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i}\Big)^{-s_{n,i}-\alpha_i} dx,
\end{aligned}$$

where $B_{K+1}(x)$ is the $(K+1)$th periodic Bernoulli polynomial. For the first integrand in (18), again using Taylor's formula with remainder we have

$$\begin{aligned}(19)\quad &\prod_{i=1}^{q_n}\Big(\sum_{j=1}^{n-1}\gamma_j m_j+\gamma_n x+u_{n,i}\Big)^{-s_{n,i}}\\ &=\Big(\sum_{j=1}^{n-1}\gamma_j m_j+\gamma_n x\Big)^{-\sum_{i=1}^{q_n}s_{n,i}}\prod_{i=1}^{q_n}\bigg(1+\frac{u_{n,i}}{\sum_{j=1}^{n-1}\gamma_j m_j+\gamma_n x}\bigg)^{-s_{n,i}}\\ &=\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|\le K}}\bigg(\prod_{i=1}^{q_n}\binom{-s_{n,i}}{\alpha_i}u_{n,i}^{\alpha_i}\bigg)\Big(\sum_{j=1}^{n-1}\gamma_j m_j+\gamma_n x\Big)^{-\sum_{i=1}^{q_n}(s_{n,i}+\alpha_i)}\\ &\quad+(K+1)\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|=K+1}}\bigg(\prod_{i=1}^{q_n}\binom{-s_{n,i}}{\alpha_i}u_{n,i}^{\alpha_i}\bigg)\\ &\quad\times\int_0^1(1-y)^K\prod_{i=1}^{q_n}\Big(\sum_{j=1}^{n-1}\gamma_j m_j+\gamma_n x+yu_{n,i}\Big)^{-s_{n,i}-\alpha_i}\,dy.\end{aligned}$$

Substituting (18) and (19) into (15), and carrying out the first integration, we find that, for $K\in\mathbb{N}_0$, $(\mathbf{u},\boldsymbol{\gamma})\in W_0(\mathbf{q},n)$ and $\mathbf{s}\in\mathcal{D}_{n,\mathbf{q}}$,

$$\begin{aligned}(20)\quad &Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})\\ &=\sum_{\substack{m_1\ge 1\\ m_2,\dots,m_{n-1}\ge 0}}\frac{1}{\prod_{j=1}^{n-1}\prod_{k=1}^{q_j}(\gamma_1 m_1+\cdots+\gamma_j m_j+u_{j,k})^{s_{j,k}}}\\ &\quad\times\sum_{m_n=0}^{\infty}\frac{1}{\prod_{i=1}^{q_n}(\gamma_1 m_1+\cdots+\gamma_n m_n+u_{n,i})^{s_{n,i}}}\\ &=\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|\le K}}\frac{\prod_{i=1}^{q_n}\binom{-s_{n,i}}{\alpha_i}u_{n,i}^{\alpha_i}}{\gamma_n(-1+\sum_{i=1}^{q_n}(s_{n,i}+\alpha_i))}\\ &\quad\times{\sum}'\frac{1}{[\prod_{j=1}^{n-1}\prod_{k=1}^{q_j}(\sum_{i=1}^{j}\gamma_i m_i+u_{j,k})^{s_{j,k}}](\sum_{j=1}^{n-1}\gamma_j m_j)^{\sum_{i=1}^{q_n}(s_{n,i}+\alpha_i)-1}}\\ &\quad+\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|\le K}}\frac{(-1)^{|\boldsymbol{\alpha}|}\tilde{B}_{|\boldsymbol{\alpha}|+1}\gamma_n^{|\boldsymbol{\alpha}|}}{|\boldsymbol{\alpha}|+1}\prod_{i=1}^{q_n}\binom{-s_{n,i}}{\alpha_i}\\ &\quad\times{\sum}'\frac{1}{[\prod_{j=1}^{n-1}\prod_{k=1}^{q_j}(\sum_{i=1}^{j}\gamma_i m_i+u_{j,k})^{s_{j,k}}](\prod_{i=1}^{q_n}(\sum_{j=1}^{n-1}\gamma_j m_j+u_{n,i})^{s_{n,i}+\alpha_i})}\\ &\quad+\mathcal{R}^1_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})+\mathcal{R}^2_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}),\end{aligned}$$

where $\sum' = \sum_{m_1\geq 1,\, m_2,\dots,m_{n-1}\geq 0}$ and

$$\mathcal{R}^1_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) = (K+1) \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|=K+1}} \left(\prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} u_{n,i}^{\alpha_i}\right)$$
$$\times \sum{}' \frac{\int_0^\infty \int_0^1 (1-y)^K \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + y u_{n,i})^{-s_{n,i}-\alpha_i}\, dy\, dx}{\prod_{j=1}^{n-1}\prod_{k=1}^{q_j} (\gamma_1 m_1 + \cdots + \gamma_j m_j + u_{j,k})^{s_{j,k}}},$$

and

$$\mathcal{R}^2_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) = (-1)^K \gamma_n^{K+1} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^{q_n}\\ |\boldsymbol{\alpha}|=K+1}} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i}$$
$$\times \sum{}' \frac{\int_0^\infty B_{K+1}(x)\ \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i})^{-s_{n,i}-\alpha_i} dx}{\prod_{j=1}^{n-1}\prod_{k=1}^{q_j} (\gamma_1 m_1 + \cdots + \gamma_j m_j + u_{j,k})^{s_{j,k}}}.$$

The formula (20) is the key to the induction process. In fact, the induction hypothesis implies that the first two terms on the right-hand side of (20) can be continued meromorphically to the whole space, and their poles are in $\mathcal{P}_{n,\mathbf{q}}$.

The remaining task is to evaluate $\mathcal{R}^1_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ and $\mathcal{R}^2_{K,n}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$. Define

$$\mathcal{D}_{n,\mathbf{q}}(K) := \Big\{\mathbf{s}\in\mathbb{C}^q \;\Big|\; \Re\Big(\sum_{i=j}^{n}\sum_{k=1}^{q_i} s_{i,k}\Big) > n+1-j-K \text{ for all } j=1,\dots,n\Big\}.$$

Let $\varepsilon>0$. Uniformly in $x_1\geq 1$, $x_2,\dots,x_n\geq 0$, $(\mathbf{u},\boldsymbol{\gamma})\in W_\varepsilon(\mathbf{q},n)$, $y\in[0,1]$, $j\in\{1,\dots,n\}$ and $k\in\{1,\dots,q_j\}$, we have

$$\begin{aligned}(21)\qquad |\gamma_1 x_1+\cdots+\gamma_j x_j + u_{j,k} y| &\geq \Re(\gamma_1)x_1 + \Re(u_{j,k})y + \sum_{i=2}^{j} \Re(\gamma_i)x_i\\ &= \varepsilon x_1 + (\Re(\gamma_1)-\varepsilon)x_1 + \Re(u_{j,k})y + \sum_{i=2}^{j}\Re(\gamma_i)x_i\\ &\geq \varepsilon x_1 + (\Re(\gamma_1)-\varepsilon+\Re(u_{j,k}))y + \sum_{i=2}^{j}\Re(\gamma_i)x_i\\ &\geq \varepsilon x_1 + \sum_{i=2}^{j}\Re(\gamma_i)x_i \geq \varepsilon\Big(\sum_{i=1}^{j} x_i\Big),\end{aligned}$$

and

$$\begin{aligned}(22)\qquad |\gamma_1 x_1+\cdots+\gamma_j x_j + u_{j,k} y| &\leq |\gamma_1|x_1+\cdots+|\gamma_j|x_j + |u_{j,k}|\\ &\leq (1+|\mathbf{u}|+|\boldsymbol{\gamma}|)(x_1+\cdots+x_j).\end{aligned}$$

Combining (21) and (22) we see that for any $\varepsilon \in (0,1)$ and any compact subset H of $\mathbb{C}$ we have, uniformly in $x_1 \geq 1$, $x_2, \ldots, x_n \geq 0$, $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$, $y \in [0,1]$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, q_j\}$ and $s \in H$,

$$
\begin{aligned}
(23) \qquad & |(\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y)^{-s}| \\
&= |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} e^{\Im(s) \arg(\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y)} \\
&\leq |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} e^{\frac{\pi}{2}|\Im(s)|} \\
&\ll_H |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} \\
&\ll_H \begin{cases} \varepsilon^{-\Re(s)} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) \geq 0 \\ (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{-\Re(s)} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) < 0 \end{cases} \\
&\ll_{H,\varepsilon} \begin{cases} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) \geq 0 \\ (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|s|} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) < 0 \end{cases} \\
&\ll_{H,\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|s|} (x_1 + \cdots + x_j)^{-\Re(s)}.
\end{aligned}
$$

We deduce that for $K \in \mathbb{N}_0$, $\varepsilon > 0$, $\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n}$ such that $|\boldsymbol{\alpha}| = K + 1$, and any compact subset $\mathcal{K}$ of $\mathcal{D}_{n,\mathbf{q}}(K)$ we have, uniformly in $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$, $\mathbf{s} \in \mathcal{K}$, $m_1 \geq 1$ and $m_2, \ldots, m_n \geq 0$,

$$
\begin{aligned}
&\left| \frac{\int_0^\infty \int_0^1 (1-y)^K \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + y u_{n,i})^{-s_{n,i} - \alpha_i} \, dy \, dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\gamma_1 m_1 + \cdots + \gamma_j m_j + u_{j,k})^{s_{j,k}}} \right| \\
&\ll_{K,\mathcal{K},\varepsilon} (1+|\mathbf{u}|+|\boldsymbol{\gamma}|)^{|\mathbf{s}|+|\boldsymbol{\alpha}|} \frac{\int_0^\infty \int_0^1 (1-y)^K (\sum_{j=1}^{n-1} m_j + x)^{-\Re \sum_{i=1}^{q_n} (s_{n,i} + \alpha_i)} \, dy \, dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (m_1 + \cdots + m_j)^{\Re(s_{j,k})}} \\
&\ll_{K,\mathcal{K},\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+|\boldsymbol{\alpha}|} \frac{\int_0^\infty (\sum_{j=1}^{n-1} m_j + x)^{-\Re \sum_{i=1}^{q_n} (s_{n,i} + \alpha_i)} dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (m_1 + \cdots + m_j)^{\Re(s_{j,k})}} \\
&\ll_{K,\mathcal{K},\varepsilon} \frac{(1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+K+1}}{(\prod_{j=1}^{n-1} (m_1 + \cdots + m_j)^{\Re(\sum_{i=1}^{q_j} s_{j,i})}) (m_1 + \cdots + m_{n-1})^{\Re(\sum_{i=1}^{q_n} s_{j,i}) + K}}.
\end{aligned}
$$

In view of (2), the theorem of analyticity under the integral sign implies that $(\mathbf{s}, \mathbf{u}, \boldsymbol{\gamma}) \mapsto \mathcal{R}^1_{K,n}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ is holomorphic in $\mathcal{D}_{n,\mathbf{q}}(K) \times W_\varepsilon(\mathbf{q}, n)$ and satisfies there the estimate

$$
\mathcal{R}^1_{K,n}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) \ll_{\mathbf{s},K,\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+K+1} \text{ uniformly in } (\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n).
$$

A similar argument shows that $(\mathbf{s}, \mathbf{u}, \boldsymbol{\gamma}) \mapsto \mathcal{R}^2_{K,n}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ is holomorphic in $\mathcal{D}_{n,\mathbf{q}}(K) \times W_\varepsilon(\mathbf{q}, n)$ and the estimate

$$
\mathcal{R}^2_{K,n}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) \ll_{\mathbf{s},K,\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+K+1}
$$

holds there uniformly in $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$.

Now we can conclude from (20) that $\mathbf{s} \mapsto Z_{n,\mathbf{q}}(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ has meromorphic continuation to $\mathcal{D}_{n,q}(K)$ with poles in $\mathcal{P}_{n,\mathbf{q}}$ and that Proposition 1(ii) holds for any $\boldsymbol{\omega} \in \mathcal{D}_{n,\mathbf{q}}(K)$.

By letting $K \to \infty$ and $\varepsilon \to 0$, we get the assertion of Proposition 1 in case n. This finishes the proof of Proposition 1. ■

Now we can prove the following necessary result:

PROPOSITION 2. *Let $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_n) \in \mathbb{C}^n$ be such that $\Re(\gamma_j) > 0$ for all $j = 1,\dots,n$. Define*

$$V_0(\boldsymbol{\gamma}) := \{\mathbf{u} \in \mathbb{C}^n \mid \Re(u_j + \gamma_1) > 0 \text{ for all } j = 1,\dots,n\}.$$

For $\mathbf{u} \in V_0(\boldsymbol{\gamma})$ and $\mathbf{s} \in \mathcal{D}_n$, define

$$Y_n(\mathbf{s};\mathbf{u};\boldsymbol{\gamma}) := \int_{(1,\infty)\times(0,\infty)^{n-1}} \prod_{j=1}^{n} \Big(\sum_{i=1}^{j} \gamma_i x_i + u_j\Big)^{-s_j} dx_n \dots dx_1. \tag{24}$$

Let $\boldsymbol{\theta} = (\theta_1,\dots,\theta_n) \in \mathbb{C}^n$ be such that $\theta_j + \dots + \theta_n \neq 0$ for all $j = 1,\dots,n$. Then, for any $\mathbf{N} = (N_1,\dots,N_n) \in \mathbb{N}_0^n$, the limit

$$Y_n^{\boldsymbol{\theta}}(-\mathbf{N};\mathbf{u};\boldsymbol{\gamma}) := \lim_{t\to 0} Y_n(-\mathbf{N} + t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma})$$

exists, and we have

$$Y_n^{\boldsymbol{\theta}}(-\mathbf{N};\mathbf{u};\boldsymbol{\gamma}) = \sum_{I\subset\{1,\dots,n\}} \sum_{\boldsymbol{\alpha}\in\mathcal{J}(I,\mathbf{N})} A(\mathbf{N},I,\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\gamma})\mathbf{u}^{\boldsymbol{\alpha}}$$

where

(25)
$$A(\mathbf{N},I,\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\gamma}) = \frac{(-1)^{n-|I|+\sum_{j\in L(\mathbf{N},\boldsymbol{\alpha})}(\alpha_j-N_j)}\prod_{j\notin L(\mathbf{N},\boldsymbol{\alpha})}\binom{N_j}{\alpha_j}}{\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})}\alpha_j\binom{\alpha_j-1}{N_j}\prod_{j\notin I}(n+1-j+\sum_{i=j}^n N_i-\sum_{i=j}^n \alpha_i)} \times \Big(\gamma_1^{|\mathbf{N}|-|\boldsymbol{\alpha}|+n}\prod_{j=1}^{n}\gamma_j^{-1}\Big)\frac{\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})}\theta_j}{\prod_{j\in I}(\theta_j+\dots+\theta_n)}.$$

Proof. First we recall from Proposition 1 that $Y_n(\mathbf{s};\mathbf{u};\boldsymbol{\gamma})$ has a meromorphic continuation to the whole complex space $\mathbb{C}^n$ and its poles lie in

$$\mathcal{P}_n := \bigcup_{j=1}^{n} \bigcup_{k_j\in\mathbb{N}_0} \{\mathbf{s} = (s_1,\dots,s_n) \in \mathbb{C}^n \mid s_j + \dots + s_n = n+1-j-k_j\}.$$

Define

$$V_1(\gamma_1) := \{\mathbf{u} \in \mathbb{C}^n \mid |u_j| < \Re(\gamma_1) \text{ for all } j = 1,\dots,n\}. \tag{26}$$

Let $\mathbf{s} = (s_1,\dots,s_n) \in \mathcal{D}_n$ and first assume that $\mathbf{u} \in V_1(\gamma_1)$. Then, uniformly in $\mathbf{x} = (x_1, \ldots, x_n) \in [1, \infty) \times [0, \infty)^{n-1}$,

$$\left|\frac{u_j}{\sum_{i=1}^{j} \gamma_i x_i}\right| \le \frac{|u_j|}{\sum_{i=1}^{j} \Re(\gamma_i) x_i} \le \frac{|u_j|}{\Re(\gamma_1)} < 1.$$

Therefore

$$\prod_{j=1}^{n}\Big(\sum_{i=1}^{j} \gamma_i x_i + u_j\Big)^{-s_j} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \prod_{j=1}^{n}\Big(\sum_{i=1}^{j} \gamma_i x_i\Big)^{-s_j-\alpha_j},$$

where the right-hand side converges uniformly in $\mathbf{x} = (x_1, \ldots, x_n) \in [1, \infty) \times [0, \infty)^{n-1}$. This implies that for any $\mathbf{s} = (s_1, \ldots, s_n) \in \mathcal{D}_n$,

$$Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} Y_n(\mathbf{s} + \boldsymbol{\alpha}; \boldsymbol{\gamma}),$$

where $Y_n(\mathbf{s}; \boldsymbol{\gamma})$ is defined by (10). Applying Lemma 1 we find that for any $\mathbf{s} = (s_1, \ldots, s_n) \in \mathcal{D}_n$,

$$(27) \qquad Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \frac{\gamma_1^{-|\mathbf{s}|-|\boldsymbol{\alpha}|+n} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}}}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^{n} (s_j + \cdots + s_n + \alpha_j + \cdots + \alpha_n + j - n - 1)}.$$

Moreover, since $\mathbf{u} \in V_1(\gamma_1)$, the right-hand side of (27) is uniformly convergent in any compact subset of $\mathbb{C}^n \setminus \mathcal{P}_n$. It follows that the meromorphic continuation of $Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ is given by (27) for any $\mathbf{s} \in \mathbb{C}^n \setminus \mathcal{P}_n$.

Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$ be such that $\sum_{i=j}^{n} \theta_i \ne 0$ for all j and let $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$. Set $\delta := \frac{1}{2} \min\{(1 + |\theta_j|)^{-1}, |\theta_j + \cdots + \theta_n|^{-1} \mid j = 1, \ldots, n\} \in (0, 1/2)$ and $U_\delta = \{t \in \mathbb{C} \mid |t| < \delta\}$. From (27) we obtain

$$Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \frac{\gamma_1^{|\mathbf{N}|+n-|\boldsymbol{\alpha}|-t|\boldsymbol{\theta}|} \mathbf{u}^{\boldsymbol{\alpha}}}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^{n} (\theta_j + \cdots + \theta_n)} G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)$$

for any $t \in U_\delta \setminus \{0\}$, where $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(t)$ is defined by (11). By Lemma 3(ii) and Lebesgue's dominated convergence theorem, the limit $Y_n^{\boldsymbol{\theta}}(-\mathbf{N}; \mathbf{u}; \boldsymbol{\gamma}) := \lim_{t\to 0} Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma})$ exists and

$$(28) \qquad Y_n^{\boldsymbol{\theta}}(-\mathbf{N}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \frac{\gamma_1^{|\mathbf{N}|+n-|\boldsymbol{\alpha}|} \mathbf{u}^{\boldsymbol{\alpha}}}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^{n} (\theta_j + \cdots + \theta_n)} \; G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0),$$

where $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0)$ is defined in Lemma 3. Moreover, Lemma 3(iii, iv(b)) implies that $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0) = 0$ if $\boldsymbol{\alpha} \notin \{0, |\mathbf{N}| + n\}^n$. It follows that the sum on the right-hand side of (28) is finite.

Therefore by using the expression of $G_{\mathbf{N},\boldsymbol{\alpha},\boldsymbol{\theta}}(0)$ given by Lemma 3 and rearranging terms we obtain

$$(29)\qquad Y_n^{\boldsymbol{\theta}}(-\mathbf{N};\mathbf{u};\boldsymbol{\gamma})$$
$$= \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^n \\ |K(\mathbf{N},\boldsymbol{\alpha})|=|L(\mathbf{N},\boldsymbol{\alpha})|}} \frac{\gamma_1^{|\mathbf{N}|+n-|\boldsymbol{\alpha}|}\mathbf{u}^{\boldsymbol{\alpha}}}{(\gamma_1\cdots\gamma_n)\prod_{j=1}^n(\theta_j+\cdots+\theta_n)}$$
$$\times \frac{(-1)^{n-|K(\mathbf{N},\boldsymbol{\alpha})|}\Big(\prod_{j\in L(\mathbf{N},\boldsymbol{\alpha})}\frac{(-1)^{\alpha_j-N_j}\theta_j}{\alpha_j\binom{\alpha_j-1}{N_j}}\Big)\prod_{j\notin L(\mathbf{N},\boldsymbol{\alpha})}\binom{N_j}{\alpha_j}}{\prod_{j\notin K(\mathbf{N},\boldsymbol{\alpha})}\frac{(N_j+\cdots+N_n)+(n+1-j)-(\alpha_j+\cdots+\alpha_n)}{\theta_j+\cdots+\theta_n}}$$
$$= \sum_{I\subset\{1,\dots,n\}}\sum_{\boldsymbol{\alpha}\in\mathcal{J}(I,\mathbf{N})} A(\mathbf{N},I,\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\gamma})\mathbf{u}^{\boldsymbol{\alpha}},$$

where the coefficients $A(\mathbf{N},I,\boldsymbol{\alpha},\boldsymbol{\theta},\boldsymbol{\gamma})$ are defined by (25).

Fix $\mathbf{N}\in\mathbb{N}_0^n$. We will now extend the region of $\mathbf{u}$ for which the proposition holds. Denote the last member of (29) by $\psi(\mathbf{u})$. Since $\psi(\mathbf{u})$ is polynomial in $\mathbf{u}$, it is analytic on $V_0(\boldsymbol{\gamma})$. Moreover, Corollary 1 implies that for any $\mathbf{u}\in V_0(\boldsymbol{\gamma})$,

$$Y_n^{\boldsymbol{\theta}}(-\mathbf{N}+t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma}) = \sum_{k=0}^n \frac{y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta},\boldsymbol{\gamma})}{t^k} + O(t) \quad \text{as } t\to 0,$$

where for any $k=0,\dots,n$, $\mathbf{u}\mapsto y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta},\boldsymbol{\gamma})$ is analytic in $V_0(\boldsymbol{\gamma})$. On the other hand, (29) implies that for any $\mathbf{u}\in V_1(\gamma_1)$,

$$(30)\qquad y_0(\mathbf{u};-\mathbf{N},\boldsymbol{\theta},\boldsymbol{\gamma}) = \psi(\mathbf{u}),\ y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta},\boldsymbol{\gamma}) = 0 \quad \text{for all } k=1,\dots,n.$$

Since $V_1(\gamma_1)$ is a non-empty open subset of the convex (and hence connected) open set $V_0(\boldsymbol{\gamma})$, it follows by analytic continuation that (30) holds for any $\mathbf{u}\in V_0(\boldsymbol{\gamma})$. This ends the proof of Proposition 2. ■

4. An extension of Raabe's lemma. For any $\delta\in\mathbb{R}$, define

$$\mathcal{H}_n(\delta) := \{\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n \mid \Re(z_i)>\delta \text{ for all } i=1,\dots,n\}.$$

Lemma 4 (An extension of Raabe's lemma). *Let $\delta>0$. Let $g:\mathcal{H}_n(-\delta)\to\mathbb{C}$ be an analytic function in $\mathcal{H}_n(-\delta)$ such that there exist constants $K>0$ and $c\in(0,\pi)$ such that*

$$(31)\qquad |g(\mathbf{z})|\le Ke^{c(|z_1|+\cdots+|z_n|)} \quad \textit{for any } \mathbf{z}\in\mathcal{H}_n(-\delta).$$

For any $\mathbf{x}=(x_1,\dots,x_n)\in\mathcal{H}_n(-\delta)$, define

$$(32)\qquad f(\mathbf{x}) = \int_{[0,1]^n} g(\mathbf{x}+\mathbf{y})\,d\mathbf{y}.$$

Assume that f is a polynomial of degree at most d. Then so is g. Moreover, if we write $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \prod_{i=1}^n x_i^{\alpha_i}$ (with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$), then

$$g(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \prod_{i=1}^{n} B_{\alpha_i}(x_i), \tag{33}$$

where the $B_k(x)$ are the classical Bernoulli polynomials.

Remark. Raabe's transform (32) is an important operator which makes it possible to derive several properties of a Dirichlet series from its associated Dirichlet integral. For the history of Raabe's formula, see E. Friedman and S. Ruijsenaars [9, p. 367]. E. Friedman and A. Pereira [8] proved this lemma under the assumption that both f and g are polynomials. In the present paper, we only assume in Lemma 4 that g is analytic in a suitable domain satisfying (31), which is necessary for Carlson's theorem that we use in our proof. A question that deserves more investigation is to find the optimal constant c in (31) for which Lemma 4 remains valid.

Proof of Lemma 4. We proceed by induction on n.

• *Case $n = 1$.* The theorem on differentiation under the integral sign implies that for any $x \in \mathcal{H}_1(-\delta)$,

$$0 = f^{(d+1)}(x) = \int_{[0,1]} g^{(d+1)}(x+y)\, dy = g^{(d)}(x+1) - g^{(d)}(x).$$

It follows that

$$g^{(d)}(k) = g^{(d)}(0) \quad \text{for all } k \in \mathbb{N}_0.$$

Let $z \in \mathbb{C}$ be such that $\Re(z) \geq 0$. The Cauchy formula and (31) imply that

$$|g^{(d)}(z)| = \left| \frac{d!}{2\pi i} \int_{|t-z|=\delta/2} \frac{g(t)}{(t-z)^{d+1}}\, dt \right| \leq K' e^{c|z|},$$

where $K' = K d! (\delta/2)^{-d} e^{c\delta/2} > 0$.

Then it follows from Carlson's classical theorem (F. Carlson [3]; see [18, 5.81, p. 186]) that

$$g^{(d)}(z) = g^{(d)}(0) \quad \text{for all } z \in \mathbb{C} \text{ such that } \Re(z) \geq 0.$$

Thus, g is a polynomial of degree at most d.

Now since we know that f and g are both polynomials, (33) is a consequence of the lemma of Friedman and Pereira [8, Lemma 2.4] of Raabe type. This ends the proof in the case $n = 1$.

• *Induction step $n - 1 \to n$.* Let $\delta > 0$ and let $g : \mathcal{H}_n(-\delta) \to \mathbb{C}$ be an analytic function satisfying the assumptions of Lemma 4. Let $\boldsymbol{\beta} \in \mathbb{N}_0^n$ be such that $|\boldsymbol{\beta}| > d$. The Cauchy formula and (31) imply that there exist

$K' > 0$ and $c \in (0, \pi)$ such that

$$|\partial^{\boldsymbol\beta} g(\mathbf{z})| \le K' e^{c(|z_1|+\cdots+|z_n|)} \quad \text{for all } \mathbf{z} \in \mathcal{H}_n(-\delta/2). \tag{34}$$

Fix $\mathbf{z}' = (z_1, \ldots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2)$ and define $h : \mathcal{H}_1(-\delta/2) \to \mathbb{C}$ by

$$h(z_n) := \int_{[0,1]^{n-1}} \partial^{\boldsymbol\beta} g(z_1 + a_1, \ldots, z_{n-1} + a_{n-1}, z_n)\, da_1 \ldots da_{n-1}.$$

It is easy to see that h is analytic in $\mathcal{H}_1(-\delta/2)$ and that (34) implies that

$$|h(z_n)| \le K'(\mathbf{z}')\, e^{c|z_n|} \quad \text{for all } z_n \in \mathcal{H}_1(-\delta/2),$$

where $K'(\mathbf{z}') = K'\big(\frac{e^c-1}{c}\big)^{n-1} e^{c(|z_1|+\cdots+|z_{n-1}|)} > 0$.

On the other hand, since $|\boldsymbol\beta| > d$, for any $z_n \in \mathcal{H}_1(-\delta/2)$ we have

$$\begin{aligned}\int_{[0,1]} h(z_n + a_n)\, da_n &= \int_{[0,1]^n} \partial^{\boldsymbol\beta} g(z_1 + a_1, \ldots, z_n + a_n)\, da_1 \ldots da_n \\ &= \partial^{\boldsymbol\beta}\Big(\int_{[0,1]^n} g(z_1 + a_1, \ldots, z_n + a_n)\, da_1 \ldots da_n\Big) \\ &= \partial^{\boldsymbol\beta} f(\mathbf{z}) = 0.\end{aligned}$$

The case $n = 1$ then implies that $h(z_n) = 0$ for any $z_n \in \mathcal{H}_1(-\delta/2)$. As a consequence,

$$\int_{[0,1]^{n-1}} \partial^{\boldsymbol\beta} g(z_1 + a_1, \ldots, z_{n-1} + a_{n-1}, z_n)\, da_1 \ldots da_{n-1} = 0 \tag{35}$$

for all $\mathbf{z} = (z_1, \ldots, z_n) \in \mathcal{H}_n(-\delta/2)$.

Now fix $z_n \in \mathcal{H}_1(-\delta/2)$ and define $\ell : \mathcal{H}_{n-1}(-\delta/2) \to \mathbb{C}$ by

$$\ell(z_1, \ldots, z_{n-1}) = \partial^{\boldsymbol\beta} g(z_1, \ldots, z_{n-1}, z_n).$$

It is easy to see that ℓ is analytic in $\mathcal{H}_{n-1}(-\delta/2)$ and that (34) implies that

$$|\ell(\mathbf{z}')| \le K''(z_n)\, e^{c(|z_1|+\cdots+|z_{n-1}|)} \quad \text{for all } \mathbf{z}' = (z_1, \ldots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2),$$

where $K''(z_n) = K' e^{c|z_n|} > 0$.

It then follows from our induction hypothesis and (35) that

$$\ell(z_1, \ldots, z_{n-1}) = 0 \quad \text{for all } \mathbf{z}' = (z_1, \ldots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2)$$

and hence that for any $\boldsymbol\beta \in \mathbb{N}_0^n$ with $|\boldsymbol\beta| > d$ we have

$$\partial^{\boldsymbol\beta} g(z_1, \ldots, z_{n-1}, z_n) = 0 \quad \text{for all } \mathbf{z} = (z_1, \ldots, z_n) \in \mathcal{H}_n(-\delta/2).$$

Thus g is a polynomial of degree at most d. Now since we know that both f and g are polynomials, (33) is again a consequence of Raabe's lemma of Friedman and Pereira. This ends the induction argument and the proof of Lemma 4. ∎

We end this section with the following useful lemma. This lemma is maybe not new, but we give a proof of it in order to be self-contained.

For $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$, define

$$\mathcal{H}_n(\boldsymbol{\delta}) := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \Re(z_j) > \delta_j \text{ for all } j = 1, \ldots, n\}.$$

LEMMA 5. *Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ be such that $\mu_j \geq \delta_j$ for all $j = 1, \ldots, n$. Let $f : \mathcal{H}_n(\boldsymbol{\delta}) \to \mathbb{C}$ be an analytic function. Assume that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \prod_{j=1}^n (\mu_j, \infty)$. Then $f(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathcal{H}_n(\boldsymbol{\delta})$.*

Proof. We use induction on n.

If $n = 1$ the lemma is clear. Let $n \geq 2$. Assume that the conclusion is true for functions of $n - 1$ variables. Fix $x_1, \ldots, x_{n-1} \in \mathbb{R}$ such that $x_i > \mu_i$ for all $i = 1, \ldots, n - 1$. Define $F : \mathcal{H}_1(\mu_n) \to \mathbb{C}$ by $F(z) = f(x_1, \ldots, x_{n-1}, z)$. It follows from our assumptions that F is an analytic function in $\mathcal{H}_1(\mu_n)$ and $F(x) = 0$ for all $x \in (\mu_n, \infty)$. Hence $F(z) = 0$ for all $z \in \mathcal{H}_1(\mu_n)$, that is,

(36)
$$f(x_1, \ldots, x_{n-1}, z_n) = 0 \text{ for all } (x_1, \ldots, x_{n-1}, z_n) \in \Big(\prod_{j=1}^{n-1} (\mu_j, \infty)\Big) \times \mathcal{H}_1(\mu_n).$$

Now fix $z_n \in \mathbb{C}$ such that $\Re(z_n) > \mu_n$. Let $\boldsymbol{\mu}' = (\mu_1, \ldots, \mu_{n-1})$ and define $g : \mathcal{H}_{n-1}(\boldsymbol{\mu}') \to \mathbb{C}$ by $g(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, z_n)$. Then g is analytic in $\mathcal{H}_{n-1}(\boldsymbol{\mu}')$ and (36) implies that $g(x_1, \ldots, x_{n-1}) = 0$ for all $(x_1, \ldots, x_{n-1}) \in \prod_{j=1}^{n-1} (\mu_j, \infty)$. The induction hypothesis then implies that

$$g(z_1, \ldots, z_{n-1}) = 0 \quad \text{for all } (z_1, \ldots, z_{n-1}) \in \mathcal{H}_{n-1}(\boldsymbol{\mu}').$$

We deduce that

$$f(z_1, \ldots, z_n) = 0 \quad \text{for all } \mathbf{z} \in \mathcal{H}_n(\boldsymbol{\mu}).$$

This ends the proof of Lemma 5 since $\mathcal{H}_n(\boldsymbol{\mu})$ is a non-empty open subset of the domain $\mathcal{H}_n(\boldsymbol{\delta})$. ■

5. Completion of the proof of Theorem 1. Fix $\mathbf{N} = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$. Assume that

$$\theta_j + \cdots + \theta_n \neq 0 \quad \text{for all } j = 1, \ldots, n.$$

Set

$$W := \{(\mathbf{u}, \boldsymbol{\gamma}) \in \mathbb{C}^n \times \mathbb{C}^n \mid \Re(\gamma_j) > 0 \text{ and } \Re(u_j + \gamma_1) > 0 \text{ for all } j = 1, \ldots, n\}.$$

For $(\mathbf{u}, \boldsymbol{\gamma}) \in W$ and $\mathbf{s} \in \mathcal{D}_n$, we consider $Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ defined by (24) and

$$Z_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \ldots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \cdots + \gamma_j m_j + u_j)^{s_j}}. \tag{37}$$

We know from Corollary 1 that for any $(\mathbf{u}, \boldsymbol{\gamma}) \in W$, the functions

$$t \mapsto Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) \quad \text{and} \quad t \mapsto Z_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma})$$

are meromorphic and have at most a pole of order n at $t=0$. Write

$$Y_n(-\mathbf{N}+t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma}) = \sum_{k=0}^{n} \frac{y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma})}{t^k} + O(t) \quad \text{as } t\to 0,$$

$$Z_n(-\mathbf{N}+t\boldsymbol{\theta};\mathbf{u};\boldsymbol{\gamma}) = \sum_{k=0}^{n} \frac{z_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma})}{t^k} + O(t) \quad \text{as } t\to 0.$$

Corollary 1 then implies that for any $k=0,\dots,n$, the functions

$$(38) \qquad (\mathbf{u},\boldsymbol{\gamma}) \mapsto y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma}) \quad \text{and} \quad (\mathbf{u},\boldsymbol{\gamma}) \mapsto z_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma})$$

are analytic in W and

$$(39) \qquad y_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma}) = \int_{[0,1]^n} z_{-k}(\mathbf{u}(\mathbf{b});-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma})\, d\mathbf{b}$$

in that domain, where $\mathbf{u}(\mathbf{b}) = (u_1(\mathbf{b}),\dots,u_n(\mathbf{b})) \in \mathbb{C}^n$ with $\mathbf{b}=(b_1,\dots,b_n)$, $u_j(\mathbf{b}) = u_j + \sum_{i=1}^{j} \gamma_i b_i$ for all j.

For $\boldsymbol{\gamma}\in\mathbb{C}^n$ such that $\Re(\gamma_j)>0$ for all $j=1,\dots,n$, define

$$\mathcal{V}(\boldsymbol{\gamma}) := \Big\{\mathbf{u}\in\mathbb{C}^n \;\Big|\; \Re(u_j+\gamma_1) > \Re\Big(\sum_{i=1}^{j}\gamma_i\Big) + 1 \text{ for all } j=1,\dots,n\Big\}.$$

Temporarily we assume that $\boldsymbol{\gamma}\in(1,\infty)^n$ and $\mathbf{u}\in\mathcal{V}(\boldsymbol{\gamma})$. It is easy to see that for all $\mathbf{a}\in\mathcal{H}_n(-1)$ and all $j=1,\dots,n$,

$$\Re(\mathbf{u}_j(\mathbf{a})+\gamma_1) = \Re(u_j+\gamma_1) + \sum_{i=1}^{j}\gamma_i\Re(a_i) > \Re(u_j+\gamma_1) - \sum_{i=1}^{j}\gamma_i > 1,$$

that is, for all $\mathbf{a}\in\mathcal{H}_n(-1)$,

$$\mathbf{u}(\mathbf{a}) \in V_1(\boldsymbol{\gamma}) = \{\mathbf{z}\in\mathbb{C}^n \mid \Re(z_j+\gamma_1) > 1 \text{ for all } j=1,\dots,n\}.$$

For $k=0,\dots,n$ and $\mathbf{a}\in\mathcal{H}_n(-1)$, define

$$f_k(\mathbf{a}) := y_{-k}(\mathbf{u}(\mathbf{a});-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma}) \quad \text{and} \quad g_k(\mathbf{a}) := z_{-k}(\mathbf{u}(\mathbf{a});-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma}).$$

Corollary 1 implies that for $k=0,\dots,n$ the following hold:

- f_k and g_k are analytic functions in $\mathcal{H}_n(-1)$;
- $f_k(\mathbf{x}) = \int_{[0,1]^n} g_k(\mathbf{x}+\mathbf{y})\, d\mathbf{y}$ for all $\mathbf{x}\in\mathcal{H}_n(-1)$;
- there exists a constant $M=M(\mathbf{N},\boldsymbol{\theta})>0$ such that, uniformly in $\mathbf{a}\in\mathcal{H}_n(-1)$, we have $g_k(\mathbf{a}) \ll_{\mathbf{N},\boldsymbol{\theta},\mathbf{u},\boldsymbol{\gamma}} (1+|\mathbf{a}|)^M$.

On the other hand, Proposition 2 implies that for all $\mathbf{a}\in\mathcal{H}_n(-1)$,

$$f_k(\mathbf{a}) = 0 \quad \text{for all } k=1,\dots,n$$

and

$$f_0(\mathbf{a}) = \sum_{I\subset\{1,\ldots,n\}} \sum_{\boldsymbol{\alpha}\in\mathcal{J}(I,\mathbf{N})} A(\mathbf{N}, I, \boldsymbol{\alpha}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \prod_{j=1}^{n} \Big(u_j + \sum_{i=1}^{j} \gamma_i a_i\Big)^{\alpha_j},$$

where the coefficients $A(\mathbf{N}, I, \boldsymbol{\alpha}, \boldsymbol{\theta}; \boldsymbol{\gamma})$ are defined by (25). In particular, $f_0(\mathbf{a})$ is a polynomial in $\mathbf{a}$.

We then deduce from Lemma 4 that for all $\mathbf{a} \in \mathcal{H}_n(-1)$,

$$g_k(\mathbf{a}) = 0 \quad \text{for all } k = 1, \ldots, n$$

and

$$g_0(\mathbf{a}) = \sum_{I\subset\{1,\ldots,n\}} \sum_{\boldsymbol{\alpha}\in\mathcal{J}(I,\mathbf{N})} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n \\ |\mathbf{k}|\le|\boldsymbol{\alpha}|}} \widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) A(\mathbf{N}, I, \boldsymbol{\alpha}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \prod_{j=1}^{n} B_{k_j}(a_j),$$

where the polynomials $\widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k})$ are defined by

$$\prod_{j=1}^{n} \Big(\sum_{i=1}^{j} \gamma_i X_i + u_j\Big)^{\alpha_j} = \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n \\ |\mathbf{k}|\le|\boldsymbol{\alpha}|}} \widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) \mathbf{X}^{\mathbf{k}} = \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n \\ |\mathbf{k}|\le|\boldsymbol{\alpha}|}} \widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) X_1^{k_1} \ldots X_n^{k_n}. \tag{40}$$

By taking $\mathbf{a} = 0$, we find that, for all $\boldsymbol{\gamma} \in (1,\infty)^n$ and all $\mathbf{u} \in \mathcal{V}(\boldsymbol{\gamma})$,

$$z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) = 0 \quad \text{for all } k = 1, \ldots, n \tag{41}$$

and

$$z_0(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) = \sum_{I\subset\{1,\ldots,n\}} \sum_{\boldsymbol{\alpha}\in\mathcal{J}(I,\mathbf{N})} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n \\ |\mathbf{k}|\le|\boldsymbol{\alpha}|}} \widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) A(\mathbf{N}, I, \boldsymbol{\alpha}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \prod_{j=1}^{n} B_{k_j}. \tag{42}$$

Since $\widetilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) = c_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) \gamma_1^{k_1} \cdots \gamma_n^{k_n}$ (where $c_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k})$ is defined by (8)), the right-hand side of (42) coincides with the right-hand side of (9).

Moreover, for any fixed $\boldsymbol{\gamma} \in (1,\infty)^n$, $\mathcal{V}(\boldsymbol{\gamma})$ is a non-empty open subset of the domain $V_0(\boldsymbol{\gamma})$ and we know that for all $k = 0, \ldots, n$, $\mathbf{u} \mapsto z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma})$ is analytic in $V_0(\boldsymbol{\gamma})$. It follows then by analytic continuation that for any $\boldsymbol{\gamma} \in (1,\infty)^n$ the identities (41) and (42) hold for any $\mathbf{u}$ in the whole domain $V_0(\boldsymbol{\gamma})$.

Now fix $\mathbf{u} \in \mathbb{C}^n$ and set $\eta(\mathbf{u}) := \max\{0, -\Re(u_1), \ldots, -\Re(u_n)\}$. Define

$$\mathcal{G}(\mathbf{u}) := \{\boldsymbol{\gamma} \in \mathbb{C}^n \mid \Re(\gamma_1) > \eta(\mathbf{u}) \text{ and } \Re(\gamma_j) > 0 \text{ for all } j = 2, \ldots, n\}.$$

From the definitions of $V_0(\boldsymbol{\gamma})$ and W, it is easy to see that $\{\mathbf{u}\}\times\mathcal{G}(\mathbf{u})\subset W$. It then follows from (38) that

$$\boldsymbol{\gamma}\mapsto z_{-k}(\mathbf{u};-\mathbf{N},\boldsymbol{\theta};\boldsymbol{\gamma}) \text{ is analytic in the domain } \mathcal{G}(\mathbf{u}).$$

We already know that the identities (41) and (42) hold for $\boldsymbol{\gamma}\in(1,\infty)^n\cap\mathcal{G}(\mathbf{u})$. Lemma 5 then implies that for any $\mathbf{u}\in\mathbb{C}^n$ these identities hold for any $\boldsymbol{\gamma}$ in the whole domain $\mathcal{G}(\mathbf{u})$. This ends the proof of Theorem 1. ∎

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