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EXACT WEAK LAWS OF LARGE NUMBERS WITH APPLICATIONS TO RATIOS OF RANDOM VARIABLES

Abstract. We study convergence in probability of weighted sums of independent random variables which are not necessarily identically distributed. The results obtained are applied to ratios of independent random variables and ratios of smallest order statistics.

1. Introduction. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with the same distribution as the random variable R. It is well known (see [1], [3] and [6] for details) that if $\mathbb{E}R = 0$ or $\mathbb{E}R = \infty$, there are no sequences $(M_n)_{n\in\mathbb{N}}$ such that $\frac{1}{M_n}\sum_{k=1}^{n} R_k \to 1$ almost surely as $n \to \infty$. Therefore it is a natural problem to find sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of real numbers such that $\frac{1}{b_n}\sum_{k=1}^{n} a_k R_k \to 1$ almost surely as $n \to \infty$. Theorems of this kind are called *exact strong law of large numbers*, or *weak exact laws of large numbers* if we consider convergence in probability instead of almost sure convergence. We refer the reader to [1], [3] and [6] for details and further references on this topic.

Random variables with infinite mean arise in a natural way when we study the ratios of independent random variables or ratios of smallest order statistics. Exact strong laws of large numbers in such cases have recently been studied in [7] and [8].

Weak exact laws of large numbers for ratios of uniform random variables and order statistics from the uniform distribution were explored in [2]. The most recent results on weak exact laws may be found in [9] and [10]. Our goal is to prove a general weak exact law for independent random variables which are not necessarily identically distributed, and apply this result to ratios

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of independent random variables with arbitrary distribution satisfying some mild conditions and to ratios of smallest order statistics.

2. Main result. Throughout, $F_X(x) = \mathbb{P}(X \le x)$ will denote the distribution function of a random variable X and $\overline{F}_X(x) = 1 - F_X(x)$ its survival function. We shall also use the standard notation $\lg(x) = \log(\max(e, x))$ where log is the logarithm to the natural base.

We shall use the notion of slowly varying function; let us recall the definition (see [4] for details).

DEFINITION 2.1. Let $a \in \mathbb{R}$. A positive measurable function $L : [a, +\infty) \to \mathbb{R}$ is said to be *slowly varying at infinity* if

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for all } t > 0.$$

The main examples and properties of slowly varying functions may be found in [4, p. 16].

Let us state the main result of this section.

THEOREM 2.1. *Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of independent nonnegative random variables with distribution functions $F_{R_n}(x)$ such that $\overline{F}_{R_n}(x)/\overline{F}_R(x) \to 1$ as $n \to \infty$, uniformly on $[x_0, \infty)$ for some $x_0 \ge 0$. Here $F_R(x)$ is the distribution of some nonnegative random variable R such that $x\overline{F}_R(x) \to M > 0$ as $x \to \infty$. Then for all $\alpha > -1$ and any slowly varying function L we have*

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k R_k \xrightarrow{P} \frac{M}{\alpha + 1} \quad \textit{as } n \to \infty,$$

where $a_n = n^\alpha L(n)$ and $b_n = n^{\alpha+1} L(n) \lg n$.

Proof. The proof is based on [5, Theorem 3.3, p. 274] and the ideas of [2].

Let $\varepsilon > 0$. From the convergence $x\overline{F}_{R_n}(x) \to M$ it follows that there exists $x(\varepsilon)$ such that for $x \ge x(\varepsilon)$,

$$\frac{M - \varepsilon}{x} \le \overline{F}_R(x) \le \frac{M + \varepsilon}{x}.$$

From the uniform convergence $\overline{F}_{R_n}(x)/\overline{F}_R(x) \to 1$ there exists $n_0 = n_0(\varepsilon)$ such that for $n \ge n_0$,

$$\left| \frac{\overline{F}_{R_n}(x)}{\overline{F}_R(x)} - 1 \right| \le \varepsilon \quad \text{for all } x \in [x_0, \infty).$$

Finally, for $n \ge n_0$ and $x \ge x_1(\varepsilon) = \max(x_0, x(\varepsilon))$,

$$\frac{(1 - \varepsilon)(M - \varepsilon)}{x} \le \overline{F}_{R_n}(x) \le \frac{(1 + \varepsilon)(M + \varepsilon)}{x}. \tag{2.1}$$

By [4, Theorem 1.5.3],

$$\frac{\sup_{1\le k\le n} a_k}{b_n} \le \frac{n^\alpha \sup_{1\le k\le n} L(k)}{n^{\alpha+1}L(n)\lg n} \to 0 \quad \text{as } n\to\infty.$$

Therefore the n_0 may be chosen in such a way that $b_n/a_k \ge x_1(\varepsilon)$ for all $1\le k\le n$ and $n\ge n_0$.

First we have to check that

$$\sum_{k=1}^n \mathbb{P}\left(R_k > \frac{b_n}{a_k}\right) \to 0 \quad \text{as } n\to\infty.$$

Since $b_n\to\infty$, we have $\sum_{k=1}^{n_0-1}\mathbb{P}(R_k > b_n/a_k)\to 0$. By the version of the Karamata theorem for sequences (see [2], [4]), we have

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n k^\alpha L(k)}{n^{\alpha+1}L(n)} = \frac{1}{\alpha+1}. \tag{2.2}$$

Therefore, from (2.1) we get

$$\begin{aligned}\sum_{k=n_0}^n \mathbb{P}(R_k > b_n/a_k) &= \sum_{k=n_0}^n \overline{F}_{R_k}(b_n/a_k) \le \frac{(1+\varepsilon)(M+\varepsilon)}{b_n}\sum_{k=n_0}^n a_k\\ &\le \frac{(1+\varepsilon)(M+\varepsilon)\sum_{k=1}^n k^\alpha L(k)}{n^{\alpha+1}L(n)\lg n}\to 0.\end{aligned}$$

Now, we have to check the variance term in [5, Theorem 3.3, p. 274], i.e.

$$\sum_{k=1}^n \operatorname{Var}\left(\frac{a_kR_k}{b_n}\mathbb{I}\left(\frac{a_kR_k}{b_n}\le 1\right)\right)\to 0 \quad \text{as } n\to\infty. \tag{2.3}$$

First observe that for fixed $k\le n_0-1$ the random variables

$$\frac{a_kR_k}{b_n}\mathbb{I}\left(\frac{a_kR_k}{b_n}\le 1\right)$$

are bounded by 1 and almost surely convergent to 0 as $n\to\infty$. Thus

$$\mathbb{E}\left(\frac{a_kR_k}{b_n}\right)^2\mathbb{I}\left(\frac{a_kR_k}{b_n}\le 1\right)\to 0 \tag{2.4}$$

and in consequence

$$\sum_{k=1}^{n_0-1}\operatorname{Var}\left(\frac{a_kR_k}{b_n}\mathbb{I}((a_kR_k/b_n)\le 1)\right)\to 0 \quad \text{as } n\to\infty.$$

Moreover, by (2.1) we have

$$\frac{\sum_{k=n_0}^{n} a_k^2 \mathbb{E} R_k^2 \mathbb{I}(R_k \le b_n/a_k)}{b_n^2}$$
$$= \frac{2}{b_n^2} \sum_{k=n_0}^{n} a_k^2 \Big(\int_0^{x_1(\varepsilon)} t\overline{F}_{R_k}(t)\, dt + \int_{x_1(\varepsilon)}^{b_n/a_k} t\overline{F}_{R_k}(t)\, dt \Big)$$
$$\le \frac{2}{b_n^2} \sum_{k=1}^{n} a_k^2 \big[(x_1(\varepsilon))^2 + (b_n/a_k - x_1(\varepsilon))(1+\varepsilon)(M+\varepsilon) \big]$$
$$\le C \bigg(\frac{1}{b_n^2} \sum_{k=1}^{n} a_k^2 + \frac{1}{b_n} \sum_{k=1}^{n} a_k \bigg) \to 0,$$

and thus by (2.4) we get (2.3).

Now it remains to prove that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} a_k \mathbb{E} R_k \mathbb{I}(R_k \le b_n/a_k) = \frac{M}{\alpha+1}. \tag{2.5}$$

By the same argument as before,

$$\sum_{k=1}^{n_0-1} \mathbb{E}\bigg(\frac{a_k R_k}{b_n} \mathbb{I}\Big(\frac{a_k R_k}{b_n} \le 1 \Big) \bigg) \to 0 \quad \text{as } n \to \infty.$$

Observe that for $k \ge n_0$ we have

$$\frac{\sum_{k=n_0}^{n} a_k \mathbb{E} R_k \mathbb{I}(R_k \le b_n/a_k)}{b_n}$$
$$= \frac{1}{b_n} \sum_{k=n_0}^{n} a_k \Big(\int_0^{x_1(\varepsilon)} \overline{F}_{R_k}(t) dt + \int_{x_1(\varepsilon)}^{b_n/a_k} \overline{F}_{R_k}(t) dt \Big)$$
$$\le \frac{1}{b_n} \sum_{k=n_0}^{n} a_k [x_1(\varepsilon) + (1+\varepsilon)(M+\varepsilon)(\lg(b_n/a_k) - \lg x_1(\varepsilon))].$$

Note that $\frac{1}{b_n} \sum_{k=n_0}^{n} a_k \to 0$ and

$$\lg(b_n/a_k) = (\alpha+1)\lg n + \lg L(n) + \lg\lg n - \alpha \lg k - \lg L(k).$$

We now examine the five terms of $\frac{1}{b_n} \sum_{k=n_0}^{n} \lg(b_n/a_k)$. By using (2.2), we get

$$\frac{(\alpha+1)\lg n}{b_n} \sum_{k=n_0}^{n} a_k = \frac{(\alpha+1)\lg n}{n^{\alpha+1} L(n) \lg n} \sum_{k=n_0}^{n} k^\alpha L(k) \to 1,$$
$$\frac{\lg L(n)}{b_n} \sum_{k=n_0}^{n} a_k = \frac{\lg L(n)}{n^{\alpha+1} L(n) \lg n} \sum_{k=n_0}^{n} k^\alpha L(k) \to 0,$$

$$\frac{\lg\lg n}{b_n}\sum_{k=n_0}^{n} a_k = \frac{\lg\lg n}{n^{\alpha+1}L(n)\lg n}\sum_{k=n_0}^{n} k^{\alpha}L(k) \to 0,$$
$$\frac{\alpha}{b_n}\sum_{k=n_0}^{n} a_k \lg k = \frac{\alpha}{n^{\alpha+1}L(n)\lg n}\sum_{k=n_0}^{n} k^{\alpha}L(k)\lg k \to \frac{\alpha}{\alpha+1},$$

and finally

$$\begin{aligned}\frac{\sum_{k=n_0}^{n} a_k \lg L(k)}{b_n} &= \frac{\sum_{k=n_0}^{n} k^{\alpha}L(k)\lg L(k)}{n^{\alpha+1}L(n)\lg n}\\ &= \frac{\lg L(n)}{\lg n}\,\frac{\sum_{k=n_0}^{n} k^{\alpha}L(k)\lg L(k)}{n^{\alpha+1}L(n)\lg L(n)} \to 0.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{b_n}\sum_{k=n_0}^{n} a_k \mathbb{E}R_k\mathbb{I}(R_k \le b_n/a_k) \le \frac{(1+\varepsilon)(M+\varepsilon)}{\alpha+1}.$$

Similarly we prove

$$\liminf_{n\to\infty}\frac{1}{b_n}\sum_{k=n_0}^{n} a_k \mathbb{E}R_k\mathbb{I}(R_k \le b_n/a_k) \ge \frac{(1-\varepsilon)(M-\varepsilon)}{\alpha+1}.$$

Since ε was arbitrary, we get (2.5) and the proof is complete. ∎

3. Applications and examples. In this section we present some applications of our weak exact law of large numbers.

Let us begin with ratios of independent random variables. Exact strong laws of large numbers, in this case, were studied in detail in [7], where further references may be found. Let X and Y be independent random variables with the same distribution as a nonnegative, integrable random variable ξ with density f_ξ. We shall assume that f_ξ is bounded and continuous on $[0,+\infty)$, in particular $\lim_{x\to 0^+} f_\xi(x) = f_\xi(0)$. Observe that the survival function of the random variable $R = X/Y$ is

$$\begin{aligned}\overline{F}_R(r) &= \iint_{x/y\ge r} f_\xi(x)f_\xi(y)\,dx\,dy = \int_0^\infty\Big(\int_0^{x/r} f_\xi(y)\,dy\Big)f_\xi(x)\,dx\\ &= \frac{1}{r}\int_0^\infty\bigg(\int_0^x f_\xi\Big(\frac{s}{r}\Big)ds\bigg)f_\xi(x)\,dx.\end{aligned}$$

Therefore, by our assumptions concerning ξ, we get

$$r\overline{F}_R(r) \to \int_0^\infty\Big(\int_0^x f_\xi(0)\,ds\Big)f_\xi(x)\,dx = f_\xi(0)\mathbb{E}\xi,$$

and the following theorem follows from Theorem 2.1.

Theorem 3.1. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be sequences of i.i.d. random variables which are independent of each other and all the variables have the same distribution as a nonnegative, integrable random variable ξ with density f_ξ. Assume that f_ξ is bounded and continuous on $[0,+\infty)$. Then for all $\alpha > -1$ and any slowly varying function L we have*

$$\frac{1}{n^{\alpha+1}L(n)\lg n}\sum_{k=1}^{n} k^\alpha L(k)\frac{X_k}{Y_k} \xrightarrow{P} \frac{f_\xi(0)\mathbb{E}\xi}{\alpha+1} \qquad \text{as } n\to\infty.$$

By applying the above result to the case when ξ has the uniform distribution on $[0,p]$ we get the following corollary, which is [2, Theorem 2.2].

Corollary 3.1. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be sequences of i.i.d. random variables which are independent of each other and all the variables have the same uniform distribution $U(0,p)$. Then*

$$\frac{1}{n^{\alpha+1}L(n)\lg n}\sum_{k=1}^{n} k^\alpha L(k)\frac{X_k}{Y_k} \xrightarrow{P} \frac{1}{2(\alpha+1)} \qquad \text{as } n\to\infty.$$

Another example of an immediate application of Theorem 3.1 is the exponential case.

Corollary 3.2. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be sequences of i.i.d. random variables which are independent of each other and all the variables have the same exponential distribution* $\mathrm{Exp}(\lambda)$ *with density* $f_\xi(x) = \frac{1}{\lambda}\exp(-\frac{x}{\lambda})\mathbb{I}_{\langle 0,\infty)}(x)$. *Then*

$$\frac{1}{n^{\alpha+1}L(n)\lg n}\sum_{k=1}^{n} k^\alpha L(k)\frac{X_k}{Y_k} \xrightarrow{P} \frac{1}{\alpha+1} \qquad \text{as } n\to\infty.$$

If $\xi = |\mathcal{N}(0,1)|$ is the absolute value of the standard normal law, then $f_\xi(x) = \frac{2}{\sqrt{2\pi}}\exp(-x^2/2)\mathbb{I}_{\langle 0,\infty)}(x)$ and $\mathbb{E}\xi = \sqrt{2/\pi}$, so we get the following corollary.

Corollary 3.3. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be sequences of i.i.d. random variables which are independent of each other and all the variables have the same standard normal distribution $\mathcal{N}(0,1)$. Then*

$$\frac{1}{n^{\alpha+1}L(n)\lg n}\sum_{k=1}^{n} k^\alpha L(k)\left|\frac{X_k}{Y_k}\right| \xrightarrow{P} \frac{2}{\pi(\alpha+1)} \qquad \text{as } n\to\infty.$$

Another application of Theorem 2.1 is convergence of ratios of minimal order statistics (see [8] for details on exact strong laws in this case). Let us consider an array of random variables:

$$\begin{gathered} X_{1,1}, X_{1,2}, \ldots, X_{1,k_1} \\ \vdots \\ X_{n,1}, X_{n,2}, \ldots, X_{n,k_n} \\ \vdots \end{gathered}$$

which are independent and identically distributed with density f and distribution function F.

Denote by $X_{n,(1)} = \min_{i=1,\ldots,k_n} X_{n,i}$ the first order statistics (minimum) in the nth row and by $X_{n,(2)}$ the second order statistics in this row. Consider the ratios $R_n = X_{n,(2)}/X_{n,(1)}$. The survival function of R_n takes the form (see [8])

$$\overline{F}_{R_n}(r) = 1 - F_{R_n}(r) = \frac{k_n}{r} \int_0^\infty (1-F(t))^{k_n-1} f\left(\frac{t}{r}\right) dt. \tag{3.1}$$

It is easy to calculate that if F is the standard exponential distribution $\mathrm{Exp}(1)$, then

$$\overline{F}_{R_n}(r) = \frac{k_n}{r(k_n-1)+1}, \qquad r \geq 1.$$

And for $\overline{F}_R(r) = 1/r$, $r \geq 1$, we have

$$\left| \frac{\overline{F}_{R_n}(r)}{\overline{F}_R(r)} - 1 \right| = \left| \frac{r-1}{r(k_n-1)+1} \right| \leq \frac{1}{k_n-1}.$$

Therefore if $k_n \to \infty$, then $\overline{F}_{R_n}(r)/\overline{F}_R(r) \to 1$ uniformly for $r \geq 1$. Thus we have proved the following corollary.

Corollary 3.4. *Let $(X_{n,k})_{n\in\mathbb{N},\,1\leq k\leq k_n}$ be an array of independent random variables with the same standard exponential distribution. If $k_n \to \infty$, then for any $\alpha > -1$ and any slowly varying function L we have*

$$\frac{1}{n^{\alpha+1}L(n)\lg n} \sum_{k=1}^{n} k^\alpha L(k) \frac{X_{n,(2)}}{X_{n,(1)}} \xrightarrow{P} \frac{1}{\alpha+1} \qquad \text{as } n \to \infty.$$

When $k_n = K \geq 2$ is fixed and the random variables have the same uniform distribution on $[0,p]$ denoted by $U(0,p)$, then $\overline{F}_{R_n}(r) = 1/r$, $r \geq 1$, and a direct application of Theorem 2.1 yields [2, Theorem 3.2].

Corollary 3.5. *Let $(X_{n,k})_{n\in\mathbb{N},\,1\leq k\leq K}$ be an array of independent random variables with the same uniform distribution $U(0,p)$. Then for any $\alpha > -1$ and any slowly varying function L we have*

$$\frac{1}{n^{\alpha+1}L(n)\lg n} \sum_{k=1}^{n} k^\alpha L(k) \frac{X_{k,(2)}}{X_{k,(1)}} \xrightarrow{P} \frac{1}{\alpha+1} \qquad \text{as } n \to \infty.$$

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