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STOCHASTIC APPROXIMATION PROCEDURE IN SEMI-MARKOV ENVIRONMENT APPLIED TO ALCOHOL CONSUMPTION MODEL

Abstract. In this paper, we consider a stochastic approximation procedure with semi-Markov switchings in an averaging scheme with a small parameter.

1. Introduction. Due to the wide use of stochastic diffusion problems, conditions for stability and control of such systems seem to be important. In [6] sufficient conditions for stability of stochastic systems via Lyapunov function properties are given and estimates of large deviations of linear diffusion systems are obtained. Problems of optimal control of diffusion processes described by stochastic differential equations with acceptable control are described in [20]. On the other hand, asymptotic behaviour is considered in [25] and [26].

For conditions of weak convergence of random processes in [11, 16, 12] the method of small parameter and a singular perturbation problem solution was used for the construction of the generator limiting process. This method is applied to schemes of averaging of diffusion approximation and to schemes of averaging of asymptotically small diffusion. In particular in [16] the cases of random evolution of Markov and semi-Markov switchings were examined.

Construction of semi-Markov processes and investigation of asymptotic properties of random processes with semi-Markov switchings are considered in [1, 2, 3, 4]. For these processes, weak convergence to the solution of ap-

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propriate partial differential equations and an averaging scheme of diffusion processes in diffusion approximation is presented in [3, 5].

In [19] asymptotic properties of semi-Markov processes with linearly perturbed operator maintainer a Markov process were analysed via the semi-group property. These results were then developed in [14]. A classification of solutions of a singular perturbation problem for random processes with the use of semi-Markov switchings is described in [16] and in [15] with the use of the compensating operator (see [27]). Using the compensating operator [10] one obtains sufficient conditions for convergence of a random evolution with semi-Markov switchings to the diffusion process in the averaging scheme (see also [13]).

The results of these studies have found various applications [8, 9, 18, 17].

In [21] convergence of stochastic procedures is established using properties of Lyapunov type functions. Stochastic Approximation Procedure (SAP) by a regression function with semi-Markov switchings was considered in [7].

2. Problem. In this paper, we consider a dynamical system with semi-Markov switchings using a small parameter. $x(t)$, $t \geq 0$, is a semi-Markov process in the standard phase space $(X, \mathcal{E})$, generated by the renewal Markov process x_n, τ_n, $n \geq 0$, defined by the semi-Markov kernel

$$Q(t, x, B) = P(x, B)G_x(t),$$

where the stochastic kernel

$$P(x, B) := P\{x_{n+1} \in B \mid x_n = x\}, \qquad B \in \mathcal{E},$$

defines an embedded Markov chain $x_n = x(\tau_n)$ at the renewal moments,

$$\tau_n = \sum_{k=1}^{n} \theta_k, \quad n \geq 0, \qquad \tau_0 = 0,$$

with intervals $\theta_{k+1} = \tau_{k+1} - \tau_k$ between the renewal moments. The θ_n are defined by the distribution functions

$$G_x(t) = P\{\theta_{n+1} \leq t \mid x_n = x\} =: P\{\theta_x \leq t\}.$$

Define

$$g(x) = \int_0^\infty \overline{G}_x(ds), \qquad \overline{G}_x(s) = 1 - G_x(s).$$

The semi-Markov process is defined by

$$x(t) = x_{\nu(t)}, \qquad t \geq 0,$$

where the counting process $\nu(t)$ is defined by

$$\nu(t) := \max\{n : \tau_n \leq t\}, \qquad t \geq 0.$$

We shall assume that the semi-Markov process $x(t)$, $t \geq 0$, is regular (the probability of reaching any state is positive) and uniformly ergodic [16, p. 33] with stationary distribution $\pi(B)$, $B \in \mathcal{E}$:

$$\pi(dx) = \rho(dx)g(x)/m, \quad \text{where} \quad m = \int_X g(x)\,\rho(dx).$$

Here $\rho(B), B \in \mathcal{E}$, is a stationary distribution of the embedded Markov chain (x_n). Note that the process $x(t)$ has a generator $\mathbf{Q}$,

$$\mathbf{Q}\phi(x) = \frac{1}{g(x)} \int_X P(x, dy)[\phi(y) - \phi(x)],$$

which acts in the Banach space $\mathbb{B}$ of all bounded real-valued measurable functions on X, with the sup-norm $\|\phi\| = \sup_{x\in X} |\phi(x)|$ for $\phi \in \mathbb{B}$. We have

$$\mathbb{B} = N_{\mathbf{Q}} \oplus R_{\mathbf{Q}} \tag{1}$$

where $N_{\mathbf{Q}} := \{\phi : \mathbf{Q}\phi = 0\}$ and $R_{\mathbf{Q}} := \{\mathbf{Q}\psi : \psi \in \mathbb{B}\}$. Given $\mathbf{Q}$, we can define the *potential operator* or simply the *potential* of $\mathbf{Q}$ by

$$\mathbf{R}_0 = \mathit{\Pi} - (\mathbf{Q} + \mathit{\Pi})^{-1} \quad \text{where} \quad \mathit{\Pi}\phi(x) := \int_X \pi(dx)\,\phi(x).$$

SAP for a diffusion process $u^\varepsilon(t) \in \mathbb{R}^d$ in an averaging scheme with a small parameter $\varepsilon > 0$ is defined by a stochastic differential equation

$$du^\varepsilon(t) = a(t)\left[C\left(u^\varepsilon(t); x\left(\frac{t}{\varepsilon}\right)\right) dt + \sigma\left(u^\varepsilon(t); x\left(\frac{t}{\varepsilon}\right)\right) dw(t)\right] \tag{2}$$

where

- $u^\varepsilon(t)$, $t \geq 0$, is a random evolution in a diffusion process [16, 2, 15, 27];
- $x(t)$, $t \geq 0$, is a semi-Markov process [16, 1, 19, 14];
- $w(t)$ is the Wiener process [25, 26, 11],
- $a(t)$ satisfies $\int_{t_0}^\infty a(t)\,dt = \infty$, $\int_{t_0}^\infty a^2(t)\,dt < \infty$.

The semigroup $\mathbf{T}_x(t)$, $t \geq 0$, $x \in X$, associated to the system

$$du_x(t) = a(t)[C(u_x(t); x)dt + \sigma(u_x(t); x)dw(t)], \qquad u_x(0) = u, \tag{3}$$

is defined by

$$\mathbf{T}_x(t)\phi(u) = \phi(u_x(t, u)) \tag{4}$$

where

$$u_x(t, u) := u_x(t), \qquad u_x(0) = u.$$

Notice that $u_x(t + s, u) = u_x(s, u_x(t, u))$, which is the semigroup property for the trajectories $u_x(t, u)$ [16, p. 44].

The generating operator $\mathbf{A}_x(t)$ of the semigroup $\mathbf{T}_x(t)$ is defined by

$$\mathbf{A}_x(t)\phi(u) = a(t)C(u, x)\phi'(u) + a^2(t)\frac{1}{2}\sigma^2(u, x)\phi''(u), \tag{5}$$

where $\phi(\cdot) \in C^2(\mathbb{R}^d)$ and $a(\cdot) \in C^1(\mathbb{R})$. Note that a solution of (3) exists when the following conditions are satisfied:

$$\|C(u_1,x) - C(u_2,x)\| + \|\sigma(u_1,x) - \sigma(u_2,x)\| < l(\|u_1 - u_2\|), \quad \forall x \in X,$$
$$\|C(u,x)\| + \|\sigma(u,x)\| < l(1 + \|u\|), \quad \forall x \in X,$$

for some $l > 0$ [21, Chapter 3, Sec. 4].

3. Main result. The main result of the paper can be formulated as follows:

THEOREM 3.1. *Let $C(\cdot,x) \in C^3(\mathbb{R}^d)$ for all $x \in X$ and let $V(u)$ be the Lyapunov function for the dynamical system $\frac{du}{dt} = C(u)$, where $C(u) = \int_X C(u,x)\,\pi(dx)$, which satisfies the following conditions:*

(C1) $C(u)V'(u) \le -c_0 V(u)$, $c_0 > 0$,
(C2) $|R_0 C(u,x)V'(u)| \le c_1(1 + V(u))$,
(C3) $|C(u,x)R_0[C(u,x)V'(u)]'| \le c_2(1 + V(u))$,
(C4) $|C(u,x)[C(u,x)[C(u,x)V'(u)]']'| \le c_3(1 + V(u))$,
(C5) (Cramer condition) $\sup_{x\in X} \int_0^\infty e^{ht} G_x(t)\,dt \le H < \infty, h > 0$,
(C6) $\int_0^\infty a(t)\,dt = \infty$, $\int_0^\infty a^2(t)\,dt < \infty$,
(C7) $b_1^\varepsilon(t,s) = \left|\frac{a(t+\varepsilon s)}{a(t)}\right| \le A_1 < \infty$, $a(\cdot) \in C^1(\mathbb{R})$ *and* $b_2^\varepsilon(t,s) = \left|\frac{a'(t+\varepsilon s)}{a(t)}\right| < A_2 < \infty$.

Then the solution $u^\varepsilon(t)$, $t \ge 0$, of (2) *converges to some point $u^* \in \mathbb{R}$ almost surely.*

We introduce an advanced Markov renewal process (MRP) (see [16]), using the given sequence:

$$u_n^\varepsilon = u^\varepsilon(\tau_n^\varepsilon), \quad x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon), \quad \tau_n^\varepsilon = \varepsilon\tau_n, \tag{6}$$

where $\tau_n = \sum_{k=1}^n \theta_k$, $n \ge 0$, $\tau_0 = 0$ are the renewal times of the semi-Markov process $x(t)$, $t \ge 0$, determined by the distribution function of the time spent in state x.

DEFINITION 1 ([16, 10]). The *compensating operator* of the advanced MRP (6) is defined by

$$\begin{aligned} &\mathbf{L}_t^\varepsilon(x)\phi(u,x) \\ &\quad = \varepsilon^{-1}[E\{\phi(u_{n+1}^\varepsilon, x_{n+1}^\varepsilon) \mid u_n^\varepsilon = u,\, x_n^\varepsilon = x,\, \tau_n^\varepsilon = t\} - \phi(u,x)]/g(x). \end{aligned} \tag{7}$$

To simplify notation we let

$$q(x) = \frac{1}{g(x)} \quad \text{and} \quad \mathbf{P}\phi(u,x) = \int_X \phi(u,y)\,P(x,dy).$$

We have

LEMMA 3.2. *The compensating operator* $\mathbf{L}_t^\varepsilon(x)$ *is of the following form:*

$$\mathbf{L}_t^\varepsilon(x)\phi(u,x) = \varepsilon^{-1}\mathbf{Q}\phi(u,x) + \varepsilon^{-1}[\mathbf{G}_t^\varepsilon(x) - I]\mathbf{Q}_0\phi(u,x), \tag{8}$$

where

$$\mathbf{G}_t^\varepsilon(x) = \int_0^\infty G_x(ds)\mathbf{T}_x(t+\varepsilon s), \qquad \mathbf{Q}_0\phi(x) = \frac{1}{g(x)}\mathbf{P}\phi(u,x).$$

Proof. From [16, 27, 13] we have

$$\begin{aligned} E[\phi(u_{n+1}^\varepsilon, x_{n+1}^\varepsilon) \mid u_n^\varepsilon = u,\, x_n^\varepsilon = x,\, \tau_n^\varepsilon = t] &= E_{u,x,t}\phi(u_{n+1}^\varepsilon, x_{n+1}^\varepsilon) \\ &= \int_0^\infty G_x(ds)\,\mathbf{T}_x(t+\varepsilon s)\int_X P(x,dy)\,\phi(u,y). \end{aligned}$$

Hence

$$\begin{aligned} &\mathbf{L}_t^\varepsilon(x)\phi(u,x) \\ &\quad= \varepsilon^{-1}q(x)\Big[\int_0^\infty G_x(ds)\,\mathbf{T}_x(t+\varepsilon s)\int_X P(x,dy)\,\phi(u,y) - \phi(u,x)\Big] \\ &\quad= \varepsilon^{-1}q(x)\int_X P(x,dy)\,[\phi(u,y)-\phi(u,x)] \\ &\qquad+ \varepsilon^{-1}q(x)\int_0^\infty G_x(ds)\,[\mathbf{T}_x(t+\varepsilon s) - I]\int_X P(x,dy)\,\phi(u,y). \end{aligned}$$

Thus we obtain (8).

LEMMA 3.3. *The compensating operator* $\mathbf{L}_t^\varepsilon(x)$ *has the asymptotic representations*

$$\mathbf{L}_t^\varepsilon(x)\phi(u,x) = \varepsilon^{-1}\mathbf{Q}\phi(u,x) + q(x)\theta_1^\varepsilon(x)\mathbf{P}\phi(u,x), \tag{9}$$

$$\mathbf{L}_t^\varepsilon(x)\phi(u,x) = \varepsilon^{-1}\mathbf{Q}\phi(u,x) + \mathbf{A}_x(t)\mathbf{P}\phi(u,x) + \varepsilon^2 a^2(t)\theta_2^\varepsilon(x)\mathbf{P}\phi(u,x), \tag{10}$$

where

$$\theta_1^\varepsilon(x) = \int_0^\infty \overline{G}_x(s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds,$$

$$\theta_2^\varepsilon(x) = q(x)\int_0^\infty \overline{G}_x^{(2)}(s)\widetilde{\mathbf{A}}_x^1(t+\varepsilon s)\widetilde{\mathbf{A}}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds,$$

where

$$\begin{aligned} \widetilde{\mathbf{A}}_x^1(t+\varepsilon s)\phi(u) &= b_2^\varepsilon(t,s)C(u,x)\phi'(u) + a(t+\varepsilon s)b_2^\varepsilon(t,s)\sigma^2(u,x)\phi''(u) \\ &= \frac{1}{a(t)}\mathbf{A}_x'(t+\varepsilon s)\phi(u), \end{aligned}$$

$$\begin{aligned}\widetilde{\mathbf{A}}_x(t+\varepsilon s)\phi(u) &= b_1^\varepsilon(t,s)C(u,x)\phi'(u) + \tfrac{1}{2}a(t+\varepsilon s)b_1^\varepsilon(t,s)\sigma^2(u,x)\phi''(u)\\ &= \frac{1}{a(t)}\mathbf{A}_x(t+\varepsilon s)\phi(u),\\ b_1^\varepsilon(t,s) &= \frac{a(t+\varepsilon s)}{a(t)}, \qquad b_2^\varepsilon(t,s) = \frac{a'(t+\varepsilon s)}{a(t)}\end{aligned}$$

Proof. For the semigroup $\mathbf{T}_x(t+\varepsilon s)$, $t \geq 0$, $x \in X$, we have,

$$d\mathbf{T}_x(t+\varepsilon s) = \varepsilon \mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)ds.$$

Integrating by parts we obtain (see (8))

$$\begin{aligned}\mathbf{G}_t^\varepsilon(x) - I &= \int_0^\infty G_x(ds)\,[\mathbf{T}_x(t+\varepsilon s) - I]\\ &= -\overline{G}_x(s)[\mathbf{T}_x(t+\varepsilon s) - I]|_0^\infty + \varepsilon \int_0^\infty \overline{G}_x(s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds.\end{aligned}$$

Given the Cramer condition we have $\lim_{s\to\infty}\overline{G}_x(s) = 0$. Moreover $T_x(t) = I$, so

$$\mathbf{G}_t^\varepsilon(x) - I = \varepsilon \int_0^\infty \overline{G}_x(s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds = \varepsilon\theta_1^\varepsilon(x).$$

Hence we obtain (9).

For

$$\mathbf{G}_{t,1}^\varepsilon(x) = \int_0^\infty \overline{G}_x(s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds$$

integrating by parts using the substitutions $u = \mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)$ and $dv = \overline{G}_x(ds)$ with $du = (\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s))'ds$ and $v = -\overline{G}_x^{(2)}(s)$ we get

$$\begin{aligned}\mathbf{G}_{t,1}^\varepsilon(s) &= \int_0^\infty \overline{G}_x(s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\,ds\\ &= -\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\cdot\overline{G}_x^{(2)}(s)|_0^\infty\\ &\quad + \varepsilon\int_0^\infty (\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s))'\cdot\overline{G}_x^{(2)}(s)\,ds\\ &= g(x)\mathbf{A}_x(t)I + \varepsilon^2\int_0^\infty \mathbf{A}_x'(t+\varepsilon s)\mathbf{A}_x(t+\varepsilon s)\mathbf{T}_x(t+\varepsilon s)\cdot\overline{G}_x^{(2)}(s)\,ds\\ &= g(x)\mathbf{A}_x(t)I + \varepsilon^2 a^2(t)g(x)\theta_2^\varepsilon(x),\end{aligned}$$

where

$$\overline{G}_x^{(2)}(t) = \int_t^\infty \overline{G}_x(s)\,ds.$$

Substituting this result into (9), we obtain (10).

LEMMA 3.4. *The compensating operator* $\mathbf{L}_t^\varepsilon(x)$ *satisfies, with* $\phi^\varepsilon(u,x) = \phi(u) + \varepsilon\phi_1(u,x)$,

$$\mathbf{L}_t^\varepsilon(x)\phi^\varepsilon(u,x) = \mathbf{L}_t\phi(u,x) + \varepsilon\theta_t^\varepsilon(x)\phi(u)$$

where $\theta_t^\varepsilon(x)\phi(u) = q(x)\theta_1(x)\mathbf{P}\mathbf{R}_0\widetilde{\mathbf{L}}_t(x)\phi(u) + \varepsilon a^2(t)\theta_2^\varepsilon(x)\phi(u)$, $\widetilde{\mathbf{L}}_t(x) = \mathbf{A}_x(t) - \mathbf{L}_t$ *and* $\phi(\cdot) \in C^3(\mathbb{R})$.

Proof. We have

$$\mathbf{L}_t^\varepsilon(x)[\phi(u) + \varepsilon\phi_1(u,x)] =$$
$$\varepsilon^{-1}\mathbf{Q}\phi(u)+\mathbf{A}_x(t)\phi(u)+\mathbf{Q}\phi_1(u,x)+\varepsilon q(x)\theta_1^\varepsilon(x)\mathbf{P}\phi_1(u,x)+\varepsilon a^2(t)\theta_2^\varepsilon(x)\mathbf{P}\phi(u).$$

Now

$$\mathbf{L}_t\phi(u) = \mathit{\Pi}\mathbf{A}_x(t)\mathit{\Pi}\phi(u) = a(t)C(u)\phi'(u) + \frac{a^2(t)\sigma^2(u)}{2}\phi''(u)$$

where

$$C(u) = \int_X C(u,x)\,\pi(dx), \qquad \sigma^2(u) = \int_X \sigma^2(u,x)\,\pi(dx).$$

From $\phi(\cdot) \in N_{\mathbf{Q}}$ we have

$$\mathbf{A}_x(t)\phi(u) + \mathbf{Q}\phi_1(u,x) = \mathbf{L}_t\phi(u),$$

Hence

$$\mathbf{Q}\phi_1(u,x) = (\mathbf{A}_x(t) - \mathbf{L}_t)\phi(u) = \widetilde{\mathbf{L}}_t(x)\phi(u).$$

Using this result we obtain

$$\phi_1(u,x) = \mathbf{R}_0\widetilde{\mathbf{L}}_t(x)\phi(u).$$

Finally,

$$\mathbf{L}_t^\varepsilon(x)\phi^\varepsilon(u,x) = \mathbf{L}_t\phi(u) + \varepsilon\theta_t^\varepsilon(x)\phi(u)$$

where

$$\theta_t^\varepsilon(x) = q(x)\theta_1(x)\mathbf{P}\mathbf{R}_0\widetilde{\mathbf{L}}_t(x) + a^2(t)\theta_2^\varepsilon(x). \ \blacksquare$$

Consider a Lyapunov function $V(u)$ for the equation

$$\frac{du}{dt} = C(u).$$

The next lemma follows immediately from the previous one:

COROLLARY 3.5. *For the Lyapunov function* $V^\varepsilon(u,x) = V(u)+\varepsilon V_1(u,x)$ *we have*

$$\mathbf{L}_t^\varepsilon(x)V^\varepsilon(u,x) = \mathbf{L}_tV(u) + \varepsilon\theta_t^\varepsilon(x)V(u).$$

Finally, we are ready to prove our main Theorem 3.1.

Proof of Theorem 3.1. Using conditions (C4)–(C7) we have

$$|\theta_t^\varepsilon(x)V(u)| \le A_3(1 + V(u)) < \infty.$$

Next, using conditions (C1)–(C3) we find that

$$\mathbf{L}_t V(u) \leq -c_0 a(t) V(u) + a^2(t) c_1 (1 + V(u)).$$

Now from the Korolyuk theorem (see Theorem 6.1 or [16]) and the Nevelson–Hasminskii theorem (see Theorem 6.2 or [21]) we deduce that

$$P\Big(\lim_{\varepsilon \to 0} u^\varepsilon(t) = u^\star\Big) = 1,$$

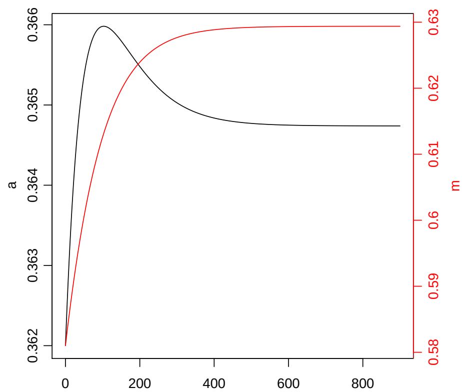
which completes the proof. ∎

4. Application. This result can be used in controlling the process in an averaging scheme with semi-Markov switchings [22].

A mathematical model for alcohol consumption is considered [23]. In [24], an equilibrium point was obtained in the case of deterministic values of coefficients in the form

$$\begin{cases} a' = \mu + \gamma - \gamma m + \beta a^2 - a[\beta + \mu + \gamma - (d - d_A)(1 - a)], \\ m' = \beta a - \beta a^2 - m[\alpha + \mu + a(d - d_A)], \end{cases} \tag{11}$$

where $\alpha = 0.000110247$ (rate at which a nonrisk consumer moves to the risk consumption subpopulation), $\beta = 0.0284534$ (transmission rate due to social pressure to increase alcohol consumption: family, friends, marketing, TV, etc.), $\mu = 0.01$ (birth rate in Spain), $d_A = 0.008$ (death rate in Spain), $d = 0.009$ (augmented death rate due to alcohol consumption), $\gamma = 0.00144$ (rate at which a risk consumer becomes a nonconsumer), $a(t)$ is the rate of non-consumers, and $m(t)$ the rate of non-risk consumers. It is the same model as presented in [23], but without delay (assuming $I(a_t) \approx a_t$).

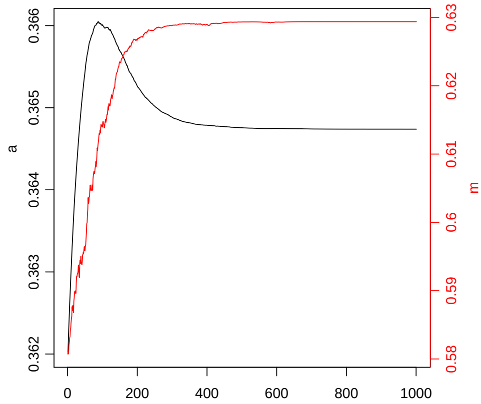


The solution of system (11) converges to the equilibrium point: $a^* = 0.3647389407$, $m^* = 0.6293831151$ (see [23]). We propose below a modification of the above system considering a semi-Markov model with switchings. More precisely, we consider two modifications studied in the following subsections.

4.1. Model with stochastic drift. Consider the model

$$
\left\{
\begin{aligned}
a' &= \mu + \gamma - \gamma m + \beta a^2 - a[\beta + \mu + \gamma - (d - d_A)(1 - a)] \\
&\qquad + \sigma_1 (a - a^\star) w_1', \\
m' &= \beta a - \beta a^2 - m[\alpha + \mu + a(d - d_A)] + \sigma_2 (m - m^\star) w_2',
\end{aligned}
\right. \tag{12}
$$

where $\alpha = 0.000110247$, $\beta = 0.0284534$, $\mu = 0.01$, $d_A = 0.008$, $d = 0.009$, $\gamma = 0.00144$, $a_0 = 0.362$, $m_0 = 0.581$, $a(t)$ is the nonconsumers rate, and $m(t)$ the rate of nonrisk consumers. It is the same model as in [24], $I(a_t) \approx a_t$. Here w_1', w_2' denote mutually independent standard Wiener processes.



The solution of (12) converges to the equilibrium point $a^* = 0.3647389407$, $m^* = 0.6293831151$ [24].

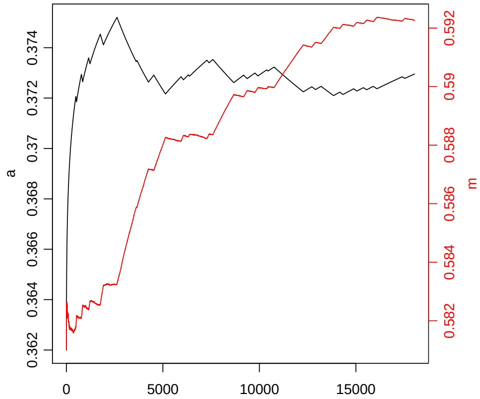
4.2. Stochastic approximation procedure applied to stochastic drift with semi-Markov switchings. The next model involves nonconstant coefficients. In this case, we use a stochastic approximation procedure for a diffusion process with semi-Markov switchings,

$$
\left\{
\begin{aligned}
a' &= c\big[\mu(X) + \gamma - \gamma m + \beta a^2 - a[\beta + \mu(X) + \gamma - (d - d_A)(1 - a)] \\
&\qquad + \sigma_1 (a - a^\star) w_1'\big], \\
m' &= c\big[\beta a - \beta a^2 - m[\alpha + \mu(X) + a(d - d_A)] + \sigma_2 (m - m^\star) w_2'\big],
\end{aligned}
\right. \tag{13}
$$

where $\alpha = 0.000110247$, $\beta = 0.0284534$, $\mu = 0.01$, $d_A = 0.008$, $d = 0.009$, $\gamma = 0.00144$, $a_0 = 0.362$, $m_0 = 0.581$, $a(t)$ is the rate of nonconsumers, and $m(t)$ the rate of nonrisk consumers. It is the same model as in [24], $I(a_t) \approx a_t$, $a^\star = 0.3647389407$, $m^\star = 0.6293831151$ (the solution of the simple model (12)). Here w_1', w_2' are mutually independent standard Wiener processes and

$$\mu(X) = \mu + X$$

where X is a semi-Markov process with states ± 0.005, and with $G_x(t) \sim$ unif$[0, 50]$. $G_x(t)$ could have another distribution, but should satisfy the Cramer condition (see (C6) of Theorem 3.1).

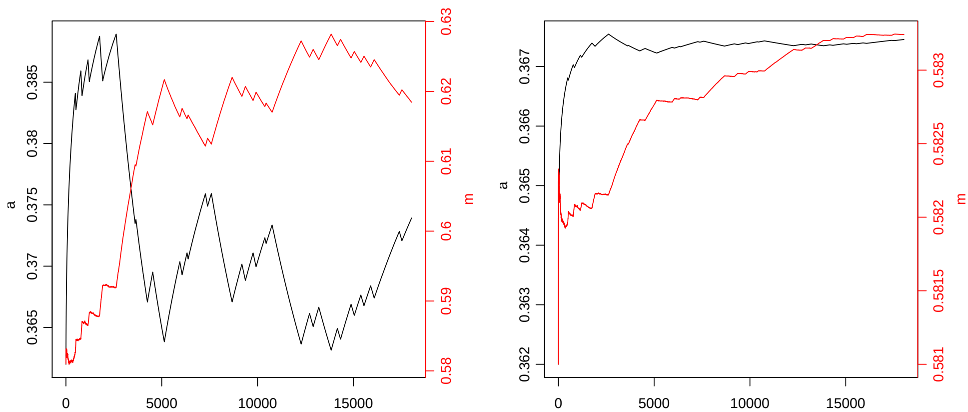


$c(t) = \frac{1}{t^{3/4}+1}$

5. Conclusion. Considering the seventh condition of the theorem ($\int_0^\infty c(t)\,dt = \infty, \int_0^\infty c^2(t)\,dt < \infty$), the following *control* functions are available:

$$c(t) = \frac{1}{t^\alpha + 1}$$

where $\alpha \in (1/2, 1]$. In our example, when $\alpha = 1$ we get a slight local variation but slow convergence to the solution (figure (b)). In contrast, when $\alpha = 1/2$, we get faster convergence to the solution, but more local variations (figure (a)). The selection of the appropriate α can be the subject of a separate study; in our opinion, it is reasonable to take $\alpha = 3/4$, which combines the advantages and disadvantages of extreme values in this context.



(a) $c(t) = \frac{1}{\sqrt{t}+1}$ (b) $c(t) = \frac{1}{t+1}$

6. Appendix

THEOREM 6.1 (Pattern limit theorem [16, p. 202]). *Suppose the following conditions hold:*

(D1) *The family of stochastic processes* $\xi^\varepsilon(t)$, $t \geq 0$, $\varepsilon \geq 0$, *is relatively compact.*

(D2) *There exists a family of test functions* $\phi^\varepsilon(\cdot,\cdot)$ *in* $C_0^3(\mathbb{R} \times E)$ *such that*

$$\lim_{\varepsilon\to 0} \phi^\varepsilon(u,x) = \phi(u) \quad \text{uniformly in } u, x.$$

(D3) *We have*

$$\lim_{\varepsilon\to 0} \mathbf{L}_t^\varepsilon \phi^\varepsilon(u,x) = \mathbf{L}_t\phi(u) \quad \text{uniformly on } u, x.$$

The family of functions $\mathbf{L}_t^\varepsilon \phi^\varepsilon$, $\varepsilon > 0$, *is uniformly bounded, and* $\mathbf{L}_t^\varepsilon \phi^\varepsilon$ *and* $\mathbf{L}_t\phi$ *belong to* $C(\mathbb{R}^d \times E)$.

(D4) *The convergence of the initial values holds, that is,*

$$\xi^\varepsilon(0) \to \xi(0), \quad \varepsilon \to 0,$$

and

$$\sup_{\varepsilon>0} \mathbb{E}|\xi^\varepsilon(0)| \leq C < +\infty.$$

Then we have the weak convergence

$$\xi^\varepsilon(t) \rightharpoonup \xi(t), \quad \varepsilon \to 0.$$

The limit process $\xi(t)$, $t \geq 0$, *is given by the solution of*

$$\frac{d}{dt}\phi(\xi(t)) = \mathbf{L}_t\phi(\xi).$$

We used conditions (D2) and (D3) in (3.4). Condition (D1) has been proved in [27].

Let

$$\mathbf{L}_t\phi(u) = a(t)C(u)\phi'(u) + \frac{1}{2}a^2(t)\sigma^2(u)\phi''(u),$$

where $\phi(\cdot) \in C^2(\mathbb{R}^d)$.

THEOREM 6.2 ([21, Chapter 3, formula (8.5)]). *Let* $V(u)$ *be a Lyapunov function such that*

$$V(u) \to \infty, \quad |u| \to \infty,$$

and

$$\mathbf{L}_t V(u) \leq -a(t)c_0 V(u) + c_1 a^2(u)(1 + V(u))$$

where $\int_{t_0}^\infty a(t)\,dt = \infty$ *and* $\int_{t_1}^\infty a^2(t)\,dt < \infty$, *and* $c_0 > 0, c_1 > 0$. *Then*

$$P\Big(\lim_{t\to\infty} u(t) = u^\star\Big) = 1,$$

where $C(u^\star) = 0$.

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