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On a reduction map for Drinfeld modules

by

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1. Introduction. The local-to-global principle we investigate is of the following type. Assume we are given an object A (of a small category $\mathcal{C}$) associated with a ring of integers $\mathcal{O}_R$ of some ring R. Assume that for any prime ideal $\mathfrak{p} \lhd R$ there exists a reduced object $A_{\mathfrak{p}}$. Assume further that we are given a property $PROP$ of A such that the corresponding property $PROP_{\mathfrak{p}}$ for $A_{\mathfrak{p}}$ makes sense. Then one can raise the following question:

Question 1.1 (Local-to-global principle). *Assume that $PROP_{\mathfrak{p}}$ holds for almost all prime ideals $\mathfrak{p} \lhd R$. Does it follow that $PROP$ holds for A?*

In 1975 A. Schinzel [Sch75] generalized the work of Skolem [Sk37] from 1937 and proved the following theorem concerning exponential equations.

Theorem 1.2 (A. Schinzel). *If $\alpha_1, \ldots, \alpha_k, \beta$ are nonzero elements of a number field K and the congruence*

$$\alpha_1^{x_1}\alpha_2^{x_2}\ldots\alpha_k^{x_k} \equiv \beta \bmod \mathfrak{p}$$

is soluble for almost all prime ideals $\mathfrak{p}$ of K then the corresponding equation is soluble in rational integers, that is, there exist $n_1, \ldots, n_k \in \mathbb{Z}$ such that $\beta = \alpha_1^{n_1}\ldots\alpha_k^{n_k}$.

This theorem is in fact the *detecting linear dependence* problem for number fields and of the kind considered in Question 1.1. The reduction map is the usual reduction modulo a non-Archimedean prime in a number field. It is well known that some questions concerning number fields can be translated to the context of abelian varieties. An analogous question for abelian varieties was raised by W. Gajda (see [We03]) and has been studied extensively [Ko03], [We03], [BGK05], [GG09], [Jo13]. A related problem, called *the support problem*, originated from a question of P. Erdős about integers

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and was solved in [CRS97] for number fields and elliptic curves. The support problem for some abelian varieties and intermediate Jacobians was treated in [BGK03] and solved for all abelian varieties by M. Larsen [La03]. In the case of Drinfeld modules, the support problem was studied by A. Li [Li06]. For the history of the detecting linear dependence problem for abelian varieties, as well as a generalization of it to linear algebraic groups, see [FK17].

In [BK11] G. Banaszak and the second named author proved the following theorem:

THEOREM 1.3. *Let $\mathbb{A}/F$ be an abelian variety defined over a number field F. Assume that $\mathbb{A}$ is isogenous to $\mathbb{A}_1^{e_1}\times\cdots\times\mathbb{A}_t^{e_t}$ with $\mathbb{A}_i$ simple, pairwise nonisogenous abelian varieties such that*

$$\dim_{\mathrm{End}_{F'}(\mathbb{A}_i)^0} H_1(\mathbb{A}_i(\mathbb{C});\mathbb{Q}) \geq e_i$$

for each $1\leq i\leq t$, where $\mathrm{End}_{F'}(\mathbb{A}_i)^0 := \mathrm{End}_{F'}(\mathbb{A}_i)\otimes\mathbb{Q}$ and F'/F is a finite extension such that the isogeny is defined over F'. Let $P\in\mathbb{A}(F)$ and let Λ be a subgroup of $\mathbb{A}(F)$. If $\mathrm{red}_v(P)\in\mathrm{red}_v(\Lambda)$ for almost all primes v of $\mathcal{O}_F$ then $P\in\Lambda+\mathbb{A}(F)_{\mathrm{tor}}$. Moreover if $\mathbb{A}(F)_{\mathrm{tor}}\subset\Lambda$, then the following conditions are equivalent:

(1) $P\in\Lambda$.
(2) $\mathrm{red}_v(P)\in\mathrm{red}_v(\Lambda)$ *for almost all primes v of $\mathcal{O}_F$.*

Theorem 1.3 gives a numerical criterion needed for a local-to-global principle to hold (up to torsion). Let us make a few more comments on Theorem 1.3. This principle for abelian varieties with a commutative endomorphism ring was proven in [We03]. Notice that the reduction map makes sense, since for an abelian variety $\mathbb{A}$ over a number field one has the Néron model $\mathcal{A}$ such that the Mordell–Weil group $\mathcal{A}(\mathcal{O}_F)$ equals $\mathbb{A}(F)$ (cf. [BLR90]). Notice also that for abelian varieties over global fields, one has the Poincaré decomposition theorem, i.e. any abelian variety $\mathbb{A}/F$ can be decomposed over some field $F'\supset F$ uniquely up to an isogeny as a product $\mathbb{A}=\mathbb{A}_1^{e_1}\times\cdots\times\mathbb{A}_t^{e_t}$ where $\mathbb{A}_i$, $i=1,\ldots,t$, are geometrically simple abelian varieties [M70].

The main result of this paper is a counterpart of Theorem 1.3 for Drinfeld modules, or rather Anderson $\mathbf{t}$-modules [A86], [BP09] that are products of Drinfeld modules. The general framework of the proof is similar to that of [BK11] or [BK13]. (In [BK13] the local-to-global principle for étale K-theory of curves was treated.) As in previous papers, there are significant subtle differences between the proofs. Let $A=\mathbb{F}_q[t]$ be the ring of polynomials of one variable over a finite field $\mathbb{F}_q$. Our main theorem is the following:

THEOREM 1.4. *Let $\widehat{\varphi}=\phi_1^{e_1}\times\cdots\times\phi_t^{e_t}$ be a $\mathbf{t}$-module where ϕ_i for $1\leq i\leq t$ are pairwise nonisogenous Drinfeld modules of generic characteristic defined over $\mathcal{O}_K$. Assume that $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i)=A$ for $1\leq i\leq t$. Let $N_i\subset\phi_i(\mathcal{O}_K)$*

be a finitely generated A-submodule of the Mordell–Weil group. Pick an A-submodule $\Lambda \subset N = N_1^{e_1} \times \cdots \times N_k^{e_t}$. *Assume that* $d_i = \mathrm{rank}(\phi_i) \geq e_i$ *for* $1 \leq i \leq t$. *Let* $P \in N$ *and assume that* $\mathrm{red}_{\mathcal{P}}(P) \in \mathrm{red}_{\mathcal{P}}(\Lambda)$ *for almost all primes* $\mathcal{P}$ *of* $\mathcal{O}_K$. *Then* $P \in \Lambda + N_{\mathrm{tor}}$.

REMARK 1.1. Let $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i) = A$ and let S_i denote the finite set of places of K at which ϕ_i (understood as defined over K) has bad reduction (cf. Definition 4.1). Let $S = \bigcup_{i=1}^t S_i$. Notice that our proofs work for the S-integers $\mathcal{O}_{K,S}$, i.e. we can assume in Theorem 5.1 that the Drinfeld modules $\phi_i, 1 \leq i \leq t$, are defined over $\mathcal{O}_{K,S}$ (cf. Definition 2.3).

REMARK 1.2. Notice also that for any Drinfeld module $\phi : A \to K\{\tau\}$ defined over K with endomorphism ring A there exists a minimal model over $\mathrm{Spec}(\mathcal{O}_{K,S_{\mathrm{bad}}})$, where S_{bad} is the finite set of primes of bad reduction [T93, Proposition 2.2].

The basic definitions concerning Drinfeld modules and reduction maps are given in the following sections. Let us emphasize the major differences here between the situations described in Theorems 1.3 and 1.4. Firstly, the category of Drinfeld modules is not semisimple, so we have to state our theorem for **t**-modules that are products of Drinfeld modules. Secondly, the Mordell–Weil group of a Drinfeld module of generic characteristic is not finitely generated, and we have to be careful to use a finitely generated A-submodule of that group. Finally, our proof relies on the *reduction theorem* below. In the proof of Theorem 1.5, we use the Ribet–Bashmakov method [R79], developed for Drinfeld modules in [P16], [H11]. This method works nicely for Drinfeld modules ϕ for which $\mathrm{End}_{K^{\mathrm{sep}}}(\phi) = A$.

THEOREM 1.5. *Let* $A = \mathbb{F}_q[t]$, *let* ϕ_i *for* $1 \leq i \leq t$ *be Drinfeld modules of generic characteristic defined over* K *such that* $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i) = A$, *and let* $\mathcal{P} \in \mathfrak{M}_A$ *be a maximal ideal and* π_P *its generator. Let* $x_{i,j} \in \phi_i(\mathcal{O}_K)$ *for some* s_i *and* $1 \leq j \leq s_i$ *be linearly independent elements over* A *for each* $1 \leq i \leq t$. *There is a set* W *of prime ideals* $\mathcal{W}$ *of* $\mathcal{O}_K$ *of positive density such that* $\mathrm{red}_{\mathcal{W}}(x_{i,j}) = 0$ *in* $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$ *for all* $1 \leq j \leq s_i$ *and* $1 \leq i \leq t$.

As a corollary we obtain the following:

THEOREM 1.6. *Let* $A = \mathbb{F}_q[t]$, *and let* ϕ_i *for* $1 \leq i \leq t$ *be Drinfeld modules of generic characteristic defined over* K *such that* $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i) = A$. *Let* $\mathcal{P} \in \mathfrak{M}_A$ *and* $m \in \mathbb{N} \cup \{0\}$. *Let* $x_{i,j} \in \phi_i(\mathcal{O}_K)$ *for some* s_i *and* $1 \leq j \leq s_i$ *be linearly independent elements of* A *and let* $T_{i,j} \in \phi_i[\mathcal{P}^m]$ *be arbitrary torsion points for all* $1 \leq j \leq s_i$ *and* $1 \leq i \leq t$. *Then there is a set* W *of prime ideals* $\mathcal{W}$ *of* $\mathcal{O}_K$ *of positive density such that*

$$\mathrm{red}_{\mathcal{W}'}(T_{i,j}) = \mathrm{red}_{\mathcal{W}}(x_{i,j})$$

in $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$ *for all* $1 \leq j \leq s_i$ *and* $1 \leq i \leq t$, *where* $\mathcal{W}'$ *is a prime in* $\mathcal{O}_L$

over $\mathcal{W}$, where L is the compositum of the fields $K(\phi_i[\mathcal{P}^m])$ for $1 \leq i \leq t$, $\mathrm{red}_{\mathcal{W}'} : \phi_i(\mathcal{O}_L) \to \phi_i^{\mathcal{W}}(k_{\mathcal{W}'})$ *and* $k_{\mathcal{W}'} = \mathcal{O}_L/\mathcal{W}'$*.*

Remark 1.3. Here $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}} = \{\alpha \in \phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W}) \mid \exists k \in \mathbb{N}, \pi_{\mathcal{P}}^k \alpha = 0\}$ and the equality $\mathrm{red}_{\mathcal{W}}(x_{i,j}) = 0$ (resp. $\mathrm{red}_{\mathcal{W}'}(T_{i,j}) = \mathrm{red}_{\mathcal{W}}(x_{i,j})$) in $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$ means, by slight abuse of notation, that it holds after projection $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W}) \to \phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$.

We view Theorems 1.5 and 1.6 as interesting in their own right, and not only as key steps in proving Theorem 1.4. Some other applications are described in Section 6, where we show that the recent results of S. Barańczuk can be extended to the situation we are considering.

The content of the paper is as follows. In Section 2 we review some general definitions and facts concerning Drinfeld modules. The reader is advised to consult the general sources [G96], [L96], [BP09]. Section 3 is devoted to Kummer theory [P16], [H11]. In Section 4 we prove the reduction theorems, i.e. Theorems 1.5 and 1.6. In Section 5 we give a proof of the local-to-global principle for **t**-modules that are products of Drinfeld modules. In Section 6 we state theorems analogous to S. Barańczuk's and indicate how to prove them.

2. Preliminaries on Drinfeld modules. Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements. Let F be a field of transcendence degree 1 over $\mathbb{F}_q$, i.e. a function field of a smooth projective curve X over $\mathbb{F}_q$, and let A be the ring of elements of F regular outside a fixed closed point ∞. Let K be a finitely generated field over $\mathbb{F}_q$. The ring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{\mathrm{a},K})$ of $\mathbb{F}_q$-linear endomorphisms of the additive algebraic group over K is the twisted (noncommutative) polynomial ring in one variable, $K\{\tau\}$. The endomorphism τ corresponds to $u \mapsto u^q$ and the commutation relation is $\tau u = u^q \tau$, $u \in K$.

Remark 2.1. We prove our main theorems for $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$, so in our case $X = \mathbb{P}^1_{\mathbb{F}_q}$.

Definition 2.1. An *A-field* K is a fixed morphism $\iota : A \to K$. The kernel of ι is a prime ideal $\mathcal{P}$ of A called the *characteristic*. The characteristic of ι is called *finite* if $\mathcal{P} \neq 0$, and *generic* (zero) if $\mathcal{P} = 0$.

Definition 2.2. A *Drinfeld A-module* is a homomorphism $\phi : A \to K\{\tau\}$, $a \mapsto \phi_a$, of $\mathbb{F}_q$-algebras such that

- $D \circ \phi = \iota$,
- for some $a \in A$, $\phi_a \neq \iota(a)\tau^0$,

where $D(\sum_{i=0}^{\nu} a_i \tau^i) = a_0$. The *characteristic* of a Drinfeld module is the characteristic of ι.

Definition 2.3. We say that ϕ is *defined over* $\mathcal{O}_K$ or ϕ *has integral coefficients* if $\phi : A \to \mathcal{O}_K\{\tau\}$. Similarly, if S is a finite set of places of $\mathcal{O}_K$ and $\mathcal{O}_{K,S}$ denotes the ring of S-integers in $\mathcal{O}_K$, we say that ϕ is *defined over* $\mathcal{O}_{K,S}$ if $\phi : A \to \mathcal{O}_{K,S}\{\tau\}$.

Let ϕ be a Drinfeld module over the A-field K. Let

$$\mu_\phi(a) := -\deg \phi_a(\tau), \quad \mu_\phi(0) = -\infty.$$

It is easy to prove that $\mu_\phi(a) = -d \deg a$ for any a [G96, Lemma 4.5.1]. The integer d is called the *rank* of the Drinfeld module ϕ. Assume that K is an A-field with a nontrivial discrete valuation v and $v(A) \geq 0$. Let $\mathcal{O}_v = \{\gamma \in K \mid v(\gamma) \geq 0\}$ be the valuation ring of K. Let ϕ be a Drinfeld module with integral coefficients, i.e. every ϕ_a is in $\mathcal{O}_K\{\tau\}$. There exists a reduction ϕ^v of the Drinfeld module ϕ defined over $k_v = \mathcal{O}_v/\mathfrak{m}_v$ [G96, Definition 4.10.1]. By [G96, Lemma 4.10.2], for any Drinfeld module ϕ defined over K there exists a Drinfeld module with integral coefficient at v isogenous to ϕ. For the definition of an isogeny of Drinfeld modules see [G96, Definition 4.4.3].

Definition 2.4. Let ϕ be a Drinfeld module over K, and L be an algebraic extension of K. The *Mordell–Weil group* $\phi(L)$ is the additive group of L viewed as an A-module via evaluation of the polynomials ϕ_a, $a \in A$.

The *rank* of an A-module M is the dimension of the F-vector space $M \otimes_A F$. An A-module is called *tame* if all its submodules of finite rank are finitely generated. B. Poonen [P95, Theorem 1] proved that $\phi(L)$ is the direct sum of a finite torsion submodule and a free A-module of rank $\aleph_0$. However, by [P95, Lemma 4], $\phi(L)$ is a tame A-module.

Let $I \subseteq A$ be an ideal. In general, for any A, I is generated by two elements $\{a_{i_1}, a_{i_2}\}$. Let ϕ_I be a monic polynomial which is a right greatest common divisor of $\phi_{a_{i_1}}$ and $\phi_{a_{i_2}}$. It exists since in $K\{\tau\}$ one has a right division algorithm (cf. [G96]), and it is a generator of the left ideal (in $K\{\tau\}$) generated by $\phi_{a_{i_1}}$ and $\phi_{a_{i_2}}$ [G96, Definition 4.4.4]. Let $\bar{K}$ denote an algebraic closure of K.

Definition 2.5. For an ideal I let $\phi[I] \subset \phi(\bar{K})$ be the finite subgroup of roots of ϕ_I.

Notice that since I is an ideal of A, $\phi[I]$ is stable under ϕ_a, $a \in A$, and the Galois group $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the separable closure $K^{\mathrm{sep}} \subset \bar{K}$ acts on $\phi[I]$. For any ideal I prime to the characteristic we have (cf. [Ro02, Corollary to Theorem 13.1], [G96])

$$\phi[I] \cong (A/I)^d$$

and thus we get a representation

$$\bar{\rho}_I : G_K \to \mathrm{Aut}_A(\phi[I]) \cong \mathrm{GL}_d(A/I).$$

DEFINITION 2.6. Let ϕ be a Drinfeld module and $\mathcal{P}$ be a maximal ideal different from the characteristic $\mathcal{P}_0$. The *$\mathcal{P}$-adic Tate module* is defined as

$$T_{\mathcal{P}}(\phi) = \mathrm{Hom}_A(F_{\mathcal{P}}/A_{\mathcal{P}}, \phi[\mathcal{P}^\infty]) \tag{2.1}$$

where $\phi[\mathcal{P}^\infty] = \bigcup_{m\geq 1}\phi[\mathcal{P}^m]$ and $F_{\mathcal{P}}$ (resp. $A_{\mathcal{P}}$) is the $\mathcal{P}$-adic completion of F (resp. A).

Let $\phi^n : \phi[\mathcal{P}^{n+1}] \to \phi[\mathcal{P}^n]$ be the multiplication by π_P map. Then (2.1) can be written in the following way:

$$T_{\mathcal{P}}(\phi) \cong \varprojlim \phi[\mathcal{P}^m]. \tag{2.2}$$

Notice that (2.2) becomes a free $A_{\mathcal{P}}$-module of rank d, and G_K acts on $T_{\mathcal{P}}(\phi)$ continuously. Since the action of G_K commutes with multiplication by elements of $A_{\mathcal{P}}$, we obtain a $\mathcal{P}$-adic representation

$$\rho_{\mathcal{P}} : G_K \to \mathrm{Aut}_{A_{\mathcal{P}}}(T_{\mathcal{P}}(\phi)) \cong \mathrm{GL}_d(A_{\mathcal{P}}). \tag{2.3}$$

Let $\mathrm{red}_{\mathcal{P}} : \mathrm{GL}_d(A_{\mathcal{P}}) \to \mathrm{GL}_d(k_{\mathcal{P}})$ be the projection map. Then we have $\bar{\rho}_{\mathcal{P}} = \mathrm{red}_{\mathcal{P}} \circ \rho_{\mathcal{P}}$.

Let $\mathfrak{M}_A$ be the set of all maximal ideals of A and let $\hat{A} = \prod_{\mathcal{P}\in\mathfrak{M}_A} A_{\mathcal{P}}$. Under the assumption that ϕ is of generic characteristic and $\mathrm{End}_{K^{\mathrm{sep}}}(\phi) = A$, in [PR109] it is proved that the adelic representation

$$\rho_{\mathrm{ad}} : G_K \to \mathrm{GL}_d(\hat{A}) \tag{2.4}$$

has open image.

For our purposes, especially for the proof of Theorem 5.5, we need the following general result [Wa01, Proposition 6] concerning the Mordell–Weil groups of Drinfeld modules defined over *finitely generated* (over the field of fractions F of A) fields L.

PROPOSITION 2.1 ([Wa01]). *Let $\bar{L}$ (resp. L^{sep}) be an algebraic (resp. separable) closure of L. Each of the A-modules $\phi(\bar{L})$ and $\phi(L^{\mathrm{sep}})$ is the direct sum of an F-vector space of dimension $\aleph_0$ and a torsion submodule. Furthermore, when the A-characteristic is generic, the torsion submodule of each of the above A-modules is isomorphic to $(F/A)^d$ .When the A-characteristic of L is $\mathcal{P}$, the torsion submodule of $\phi(\bar{L})$ (resp. $\phi(L^{\mathrm{sep}})$) is isomorphic to $\bigoplus_{\beta\neq\mathcal{P}}(F_\beta/A_\beta)^d \oplus (F_{\mathcal{P}}/A_{\mathcal{P}})^{d-\bar{h}}$ (resp. a submodule between $\bigoplus_{\beta\neq\mathcal{P}}(F_\beta/A_\beta)^d$ and $\bigoplus_{\beta\neq\mathcal{P}}(F_\beta/A_\beta)^d \oplus (F_{\mathcal{P}}/A_{\mathcal{P}})^{d-\bar{h}}$) where β is in* $\mathrm{Spec}(A)$, *and d and $\bar{h}$ are the rank and the height of ϕ respectively.*

3. Kummer theory. From now on we assume that $A = \mathbb{F}_q[t]$ is the ring of polynomials in one variable and $F = \mathbb{F}_q(t)$ is the field of rational functions over $\mathbb{F}_q$. In order to prove Theorem 1.5 we need Kummer theory in the context of Drinfeld modules, which corresponds to one constructed by Ribet [R79] for extensions of abelian varieties by tori. Such an extension was developed in [H11] and [P16]. Now we recall the relevant facts from these references. For a detailed exposition see the original sources.

Consider a Drinfeld module $\phi : A \to K\{\tau\}$ of generic characteristic. We will consider K^{sep} as an A-module via ϕ. We assume that $\mathrm{End}_{K^{\mathrm{sep}}}(\phi) = A$. Let $s \geq 1$ and Λ be an A-submodule of K generated by s A-linearly independent elements $x_1, \ldots, x_s$. Let $y_i \in \phi_I^{-1}(\{x_i\}) \subset K^{\mathrm{sep}}$. This is possible since $\phi_I - x_i$ is a separable polynomial. Define

$$\eta : G_K \to \phi[I]^s \cong \mathrm{Mat}_{d\times s}(A/I), \quad \eta(\sigma) = (\sigma(y_1) - y_1, \ldots, \sigma(y_s) - y_s), \tag{3.1}$$

and

$$\Phi_I : G_K \to \mathrm{Mat}_{d\times s}(A/I) \rtimes \mathrm{GL}_d(A/I), \quad \Phi_I(\sigma) = (\eta(\sigma), \bar\rho_I(\sigma)). \tag{3.2}$$

The action of GL_d on $\mathrm{Mat}_{d\times s}$ is given by matrix multiplication. Now, following [H11], we describe the map analogous to (3.2) for $A_{\mathcal{P}}$, $\mathcal{P} \in \mathfrak{M}_A$.

Let $\phi[\mathcal{P}^\infty] = \bigcup_{n\geq 0} \phi[\mathcal{P}^n] \subset K^{\mathrm{sep}}$, $\phi[\mathcal{P}^0] := 0$, $K_{\mathcal{P}^\infty} = K(\phi[\mathcal{P}^\infty])$. Let K_{ad} be the compositum of the fields $K_{\mathcal{P}^\infty}$ for all $\mathcal{P} \in \mathfrak{M}_A$, and let $\pi_{\mathcal{P}}^n$ be a generator of $\mathcal{P}^n$. For an A-submodule M of K^{sep} and $n \geq 0$ denote by $\phi_{\pi_{\mathcal{P}}^n}^{-1}(M) \subset K^{\mathrm{sep}}$ the inverse image of M under the endomorphism $\phi_{\pi_{\mathcal{P}}^n}$ (see Definition 2.2). By a slight abuse of notation, denote by ϕ^n (cf. (2.2)) the map $\phi^n : \phi_{\pi_{\mathcal{P}}^{n+1}}^{-1}(M) \to \phi_{\pi_{\mathcal{P}}^n}^{-1}(M)$ as well. The extended $\mathcal{P}$-adic Tate module is defined as $T_{\mathcal{P}}[M] = \varprojlim \phi_{\pi_{\mathcal{P}}^n}^{-1}(M)$. Let $K \subset L \subset K^{\mathrm{sep}}$ and $M \subset L$. Then the absolute Galois group G_L acts trivially on M and acts continuously on $T_{\mathcal{P}}[M]$ by $\sigma(t_n) = (\sigma(t_n))$, $(t_n) \in T_{\mathcal{P}}[M]$. One has an exact sequence of G_L-A-modules

$$0 \to T_{\mathcal{P}}(\phi) \to T_{\mathcal{P}}[M] \to M \to 0 \tag{3.3}$$

where the surjection in (3.3) is given by the map $\mathrm{pr}_M : T_{\mathcal{P}}[M] \to M$, $\mathrm{pr}_M((t_n)_{n\geq 0}) = t_0$. For any $\sigma \in G_L$ and $(t_n) \in T_{\mathcal{P}}[M]$ one has $\sigma(t_n) - t_n \in \operatorname{Ker} \phi_{\pi_{\mathcal{P}}^n}$ and therefore

$$\phi^n(\sigma(t_{n+1}) - t_{n+1}) = \sigma(\phi^n(t_{n+1})) - \phi^n(t_{n+1}) = \sigma(t_n) - t_n. \tag{3.4}$$

Formulas (3.4) show that the map

$$\xi_{L,M} : T_{\mathcal{P}}[M] \to \mathrm{Map}(G_L, T_{\mathcal{P}}(\phi)), \quad (t_n) \mapsto [\sigma \mapsto (\sigma(t_n) - t_n)_{n\geq 0}],$$

is well defined. One can specialize this construction to $M = \Lambda$ and $K = L$. For each $1 \leq i \leq s$ choose $(t_{i,n}) \in \mathrm{pr}_\Lambda^{-1}(x_i)$ and define a map

$$\eta_{\mathcal{P}} : G_K \to (T_{\mathcal{P}}(\phi))^s \cong \mathrm{Mat}_{d\times s}(A_{\mathcal{P}}) \tag{3.5}$$

by the formula $\eta_{\mathcal{P}}(\sigma) = (\xi_{K,\Lambda}((t_{1,n})_{n\geq 0})(\sigma), \ldots, \xi_{K,\Lambda}((t_{s,n})_{n\geq 0})(\sigma))$. Using [H11, Lemma 4.3] one obtains the A-module homomorphism

$$\bar\xi_{K_{\mathrm{ad}},\Lambda} : \Lambda \to \mathrm{Hom}(G_{K_{\mathrm{ad}}}, T_{\mathcal{P}}(\phi)), \quad (t_n)_{n\geq 0} \mapsto [\sigma \mapsto (\sigma(t_n) - t_n)_{n\geq 0}].$$

Let $\psi_i = \bar{\xi}_{K_{\mathrm{ad}},\Lambda}(x_i) \in \mathrm{Hom}(G_{K_{\mathrm{ad}}}, T_{\mathcal{P}}(\phi))$ for the fixed generators $x_1, \ldots, x_s$ of Λ and let

$$\Psi_{\mathcal{P}} : G_{K_{\mathrm{ad}}} \to (T_{\mathcal{P}}(\phi))^s \cong \mathrm{Mat}_{d\times s}(A_{\mathcal{P}}), \qquad \sigma \mapsto (\psi_1(\sigma), \ldots, \psi_s(\sigma)). \tag{3.6}$$

For $T \subset \mathfrak{M}_A$ we have the homomorphism

$$\Psi_T : G_{K_{\mathrm{ad}}} \to \prod_{\mathcal{P}\in T} \mathrm{Mat}_{d\times s}(A_{\mathcal{P}}), \qquad \sigma \mapsto (\Psi_{\mathcal{P}}(\sigma))_{\mathcal{P}\in T}.$$

Denote $\Psi_{\mathrm{ad}} := \Psi_{\mathfrak{M}_A}$. The main results in [H11] are the following:

Proposition 3.1 ([H11, Proposition 4.6]). *The image of $\Psi_{\mathcal{P}}$ is equal to $\mathrm{Mat}_{d\times s}(A_{\mathcal{P}})$ for almost all $\mathcal{P} \in \mathfrak{M}_A$ and is open for all $\mathcal{P} \in \mathfrak{M}_A$.*

Theorem 3.2 ([H11, Theorem 4.4]). *The image of Ψ_{ad} is open.*

For prime ideals $\mathcal{P} \in \mathfrak{M}_A$ we shall use the modules

$$V_{\mathcal{P}}(\phi) = T_{\mathcal{P}}(\phi) \otimes_{A_{\mathcal{P}}} F_{\mathcal{P}}.$$

Remark 3.1. Notice that in view of (3.6) we have $T_{\mathcal{P}}(\phi) \cong \mathrm{Mat}_{d\times 1}(A_{\mathcal{P}})$ and $\dim_{F_{\mathcal{P}}} V_{\mathcal{P}}(\phi) = d$.

4. Proof of the reduction theorem. In what follows we assume that the fields of definition for the Drinfeld modules involved are finite extensions of $F = \mathbb{F}_q(t)$.

Let $\phi : A \to \mathcal{O}_K\{\tau\}$ be a Drinfeld module of generic characteristic defined over $\mathcal{O}_K$. For a maximal ideal $\mathcal{P} \subset \mathcal{O}_K$ we can reduce the coefficients of ϕ_a, for every $a \in A$, modulo $\mathcal{P}$ and obtain a Drinfeld module over the finite field $\mathcal{O}_K/\mathcal{P}$. This Drinfeld module, denoted by $\phi^{\mathcal{P}}$, has the special characteristic $\mathcal{P}$.

Definition 4.1. Let ϕ be a fixed Drinfeld module of rank d defined over K. A prime $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_K)$ is called *good* if

(1) there exists $\alpha \in K^\times$ such that $\phi_1 = \alpha\phi\alpha^{-1}$ has $\mathcal{P}$-integral coefficients,
(2) the reduced map $\phi_1^{\mathcal{P}} : A \to \mathcal{O}_K/\mathcal{P}\{\tau\}$ is a Drinfeld module of rank d.

Primes that are not good are called *bad*.

It is well known (because A is a finitely generated ring over $\mathbb{F}_q$) that almost all primes $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_K)$ are good. Moreover, for almost all primes one can take $\alpha = 1$ ([G94, p. 320], [Th04]). There also exists an analogue of the classical *Néron–Ogg–Shafarevich* criterion for abelian varieties [Ta82], [G94].

Definition 4.2. The *reduction map mod $\mathcal{P}$ for Mordell–Weil groups* is the A-module homomorphism

$$\mathrm{red}_{\mathcal{P}} : \phi(\mathcal{O}_K) \to \phi^{\mathcal{P}}(\mathcal{O}_K/\mathcal{P}).$$

Let L be a finite extension of K.

PROPOSITION 4.1 ([BKo95, Proposition 1.3]). *Let ϕ be a Drinfeld module over $\mathcal{O}_L$ of rank d, let $\mathcal{P}$ be a maximal ideal of $\mathcal{O}_L$, and let $\mathfrak{p} = \mathrm{Ker}(A \to \mathcal{O}_L \to \mathcal{O}_L/\mathcal{P})$ be the special characteristic. Let I be prime to $\mathfrak{p}$. Then*

(1) $\mathrm{Tor}_\phi(\mathcal{P}) = \{x \in \mathcal{P} \mid \phi_a(x) = 0$ *for some* $a \in A\}$ *has no nontrivial* I*-torsion,*
(2) *the reduction map is an injection on the* I*-torsion between* $\phi(\mathcal{O}_L)[I]$ *and* $\phi^{\mathcal{P}}(O_L/\mathcal{P})[I]$.

REMARK 4.1. In [BKo95] the authors consider Drinfeld modules of rank 2 but their proof works for arbitrary rank.

THEOREM 4.2. *Let $A = \mathbb{F}_q[t]$, let ϕ_i for $1 \leq i \leq t$ be Drinfeld modules of generic characteristic defined over K such that $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i) = A$, and let $\mathcal{P} \in \mathfrak{M}_A$ be a maximal ideal and π_P its generator. Let $x_{i,j} \in \phi_i(\mathcal{O}_K)$ for some s_i and $1 \leq j \leq s_i$ be linearly independent elements over A for each $1 \leq i \leq t$. There is a set W of prime ideals $\mathcal{W}$ of $\mathcal{O}_K$ of positive density such that $\mathrm{red}_{\mathcal{W}}(x_{i,j}) = 0$ in $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$ for all $1 \leq j \leq s_i$ and $1 \leq i \leq t$.*

Proof. By Proposition 3.1 there exists $m \in \mathbb{N}$ such that for any $\mathcal{P}$ one has

$$\mathcal{P}^m \prod_{i=1}^{t} T_{\mathcal{P}}(\phi_i)^{s_i} \subset \Psi_{\mathcal{P}}(G_{K_{\mathrm{ad}}}) \subset \prod_{i=1}^{t} T_{\mathcal{P}}(\phi_i)^{s_i}.$$

Let $\Gamma = \sum_{i=1}^{t}\sum_{j=1}^{s_i} A x_{i,j}$, denote $K_{\mathcal{P}^\infty} := K(\widehat{\varphi}[\mathcal{P}^\infty])$, where $\widehat{\varphi}[\mathcal{P}^\infty] = \bigcup_{i=1}^{t} \phi_i[\mathcal{P}^\infty]$ and let

$$\frac{1}{\pi_{\mathcal{P}}^{\infty}}\Gamma := \{x \in K^{\mathrm{sep}} \mid \exists m,\, \pi_{\mathcal{P}}^m x \in \Gamma\}.$$

Similarly, define $K_{\mathcal{P}^k} := K(\widehat{\varphi}[\mathcal{P}^k])$ and

$$\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma := \{x \in K^{\mathrm{sep}} \mid \pi_{\mathcal{P}}^k x \in \Gamma\}.$$

Let $H_{\mathcal{P}^k} = G(K^{\mathrm{sep}}/K_{\mathcal{P}^k})$ and $H_{\mathcal{P}^\infty} = G(K^{\mathrm{sep}}/K_{\mathcal{P}^\infty})$. The Kummer map (3.5) yields the maps

$$\psi_{i,j}^{(k)} : H_{\mathcal{P}^k} \to \phi_i[\mathcal{P}^k], \qquad \psi_{i,j}^{(k)}(\sigma) := \sigma\left(\frac{1}{\pi_{\mathcal{P}}^k} x_{i,j}\right) - \frac{1}{\pi_{\mathcal{P}}{}^k} x_{i,j}. \tag{4.1}$$

Let $\psi_{i,j}$ be the inverse limit of $\psi_{i,j}^{(k)}$. In the commutative diagram below, $\overline{\Psi_{\mathcal{P}}^k} = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{s_i} \overline{\psi}_{i,j}^{(k)}$ and $\overline{\Psi_{\mathcal{P}}} = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{s_i} \overline{\psi}_{i,j}$, where $\overline{\psi}_{i,j}^{(k)}$ and $\overline{\psi}_{i,j}$ are the maps induced by $\psi_{i,j}^{(k)}$ and $\psi_{i,j}$. Notice that by construction the maps

$\overline{\psi}_{i,j}^{(k)}$ and $\overline{\Psi_{\mathcal{P}}^{k}}$ are injective for all $k \geq 1$.

(4.2)

$$\begin{tikzcd}
G\big(K_{\mathcal{P}^\infty}\big(\frac{1}{\pi_{\mathcal{P}}^{\infty}}\Gamma\big)/K_{\mathcal{P}^\infty}\big) \arrow[r, "\overline{\Psi_{\mathcal{P}}}"] \arrow[d] & \bigoplus_{i=1}^{t} T_{\mathcal{P}}(\phi_i)^{s_i}/\mathcal{P}^m \bigoplus_{i=1}^{t} T_{\mathcal{P}}(\phi_i)^{s_i} \arrow[d, "\cong"] \\
G\big(K_{\mathcal{P}^{k+1}}\big(\frac{1}{\pi_{\mathcal{P}}^{k+1}}\Gamma\big)/K_{\mathcal{P}^{k+1}}\big) \arrow[r, "\overline{\Psi_{\mathcal{P}}^{k+1}}"] \arrow[d] & \bigoplus_{i=1}^{t}(\phi_i[\mathcal{P}^{k+1}])^{s_i}/\mathcal{P}^m \bigoplus_{i=1}^{t}(\phi_i[\mathcal{P}^{k+1}])^{s_i} \arrow[d, "\cong"] \\
G\big(K_{\mathcal{P}^k}\big(\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma\big)/K_{\mathcal{P}^k}\big) \arrow[r, "\overline{\Psi_{\mathcal{P}}^{k}}"] & \bigoplus_{i=1}^{t}(\phi_i[\mathcal{P}^{k}])^{s_i}/\mathcal{P}^m \bigoplus_{i=1}^{t}(\phi_i[\mathcal{P}^{k}])^{s_i}
\end{tikzcd}$$

Since the maps $\overline{\Psi_{\mathcal{P}}^{k}}$ and $\overline{\Psi_{\mathcal{P}}^{k+1}}$ are injective and the bottom right vertical arrow is an isomorphism, we see that the bottom left arrow is an injection. This shows the following inequality for orders of Galois groups:

$$\left|G\left(K_{\mathcal{P}^{k+1}}\left(\frac{1}{\pi_{\mathcal{P}}^{k+1}}\Gamma\right)/K_{\mathcal{P}^{k+1}}\right)\right| \leq \left|G\left(K_{\mathcal{P}^{k}}\left(\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma\right)/K_{\mathcal{P}^{k}}\right)\right|.$$

As these Galois groups are finite, equality is achieved for sufficiently large k. Therefore the bottom left vertical arrow is also a surjection and the images of $\overline{\Psi_{\mathcal{P}}^{k+1}}$ and $\overline{\Psi_{\mathcal{P}}^{k}}$ in the diagram (4.2) are isomorphic for large enough k.

Let us consider the diagram of fields

(4.3)

$$\begin{tikzcd}
 & K_{\mathcal{P}^{k+1}}\big(\frac{1}{\pi_{\mathcal{P}}^{k+1}}\Gamma\big) \arrow[d, dash] & \\
 & K_{\mathcal{P}^{k+1}}\big(\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma\big) \arrow[dl, dash] \arrow[dr, dash] & \\
K_{\mathcal{P}^{k}}\big(\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma\big) \arrow[dr, dash] & & K_{\mathcal{P}^{k+1}} \arrow[dl, dash] \\
 & K_{\mathcal{P}^{k}} &
\end{tikzcd}$$

The above mentioned surjection of Galois groups yields, for large enough k,

$$K_{\mathcal{P}^k}\left(\frac{1}{\pi_{\mathcal{P}}^{k}}\Gamma\right) \cap K_{\mathcal{P}^{k+1}} = K_{\mathcal{P}^k}. \tag{4.4}$$

Let $\tilde{h} \in G(K_{\mathcal{P}^\infty}/K_{\mathcal{P}^k})$ be the homothety $1+\pi_{\mathcal{P}}^{k}u$, $u \in \mathbb{F}_q^*$, acting on $T_{\mathcal{P}}(\widehat{\phi})$. For $k \gg 0$ such a homothety exists according to the open image theorem of Pink and Rütsche [PR09] (cf. (2.4)) and the fact that $K_{\mathcal{P}^k}$ is a finite extension of K. Let $h \in G(K_{\mathcal{P}^{k+1}}/K_{\mathcal{P}^k})$ be a projection of $\tilde{h}$. By (4.4) there exists $\sigma \in G\big(K_{\mathcal{P}^{k+1}}\big(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma\big)/K_{\mathcal{P}^k}\big)$ such that $\sigma|_{K_{\mathcal{P}^k}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)} = \mathrm{id}$ and $\sigma|_{K_{\mathcal{P}^{k+1}}} = h$. By the Chebotarev density theorem for global fields [FJ08, Theorem 6.3.1] there is a set of primes $\mathcal{W} \in \mathcal{O}_K$ of positive density such that there exists a prime $\mathcal{W}_1 \in \mathcal{O}_{K_{\mathcal{P}^{k+1}}(\frac{1}{\pi_{\mathcal{P}}^{k+1}}\Gamma)}$ with Frobenius in $G\big(K_{\mathcal{P}^{k+1}}\big(\frac{1}{\pi_{\mathcal{P}}^{k+1}}\Gamma\big)\big)$ equal to σ. Assume that $\mathcal{W} \nmid \mathcal{P}$. Let $\mathcal{W}$ and $\mathcal{W}_1$ be such primes and let $\mathcal{W}_2 \in \mathcal{O}_{K_{\mathcal{P}^k}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)}$ be the prime below $\mathcal{W}_1$. Consider the commutative diagram

$$
\begin{tikzcd}
\phi_i(\mathcal{O}_K) \arrow[r, "\mathrm{red}_{\mathcal{W}}"] \arrow[d] & \phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}} \arrow[d, "="] \\
\phi_i(\mathcal{O}_{K_{\mathcal{P}^k}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)}) \arrow[r, "\mathrm{red}_{\mathcal{W}_2}"] \arrow[d] & \phi_i^{\mathcal{W}_2}(k_{\mathcal{W}_2})_{\pi_{\mathcal{P}}} \arrow[d] \\
\phi_i(\mathcal{O}_{K_{\mathcal{P}^{k+1}}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)}) \arrow[r, "\mathrm{red}_{\mathcal{W}_1}"] & \phi_i^{\mathcal{W}_1}(k_{\mathcal{W}_1})_{\pi_{\mathcal{P}}}
\end{tikzcd}
\tag{4.5}
$$

where for brevity we denote $k_{\mathcal{W}} := \mathcal{O}_K/\mathcal{W}$ and similarly for $k_{\mathcal{W}_1}$ and $k_{\mathcal{W}_2}$. The subscript $\pi_{\mathcal{P}}$ for the Drinfeld modules with finite coefficients denotes the $\pi_{\mathcal{P}}$-torsion e.g. $\phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}} = \{\alpha \in \phi_i^{\mathcal{W}}(k_{\mathcal{W}}) \mid \exists k \in \mathbb{N},\ \pi_{\mathcal{P}}^k\alpha = 0\}$. In the diagram (4.5) reduction maps are by slight abuse of notation the compositions of the reduction maps with the projections on the $\pi_{\mathcal{P}}$-torsion part (cf. Remark 1.3).

Definition 4.3. By the $\pi_{\mathcal{P}}$-*order* of a point $x \in \phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}}$ we mean the least positive integer m such that $\pi_{\mathcal{P}}^m x = 0$.

Remark 4.2. Notice that by our choice of $\mathcal{W}$, $\mathcal{W}_1$ and $\mathcal{W}_2$ the $\mathcal{W}_1$-Frobenius element σ restricts to the identity on $K_{\mathcal{P}^k}\big(\frac{1}{\pi^k}\Gamma\big)$. Hence $k_{\mathcal{W}_2} = k_{\mathcal{W}}$ and $\phi_i^{\mathcal{W}_2}(k_{\mathcal{W}_2})_{\pi_{\mathcal{P}}} = \phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}}$.

Let $c_{i,j}$ be the $\pi_{\mathcal{P}}$-order of $\mathrm{red}_{\mathcal{W}}(x_{i,j}) \in \phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}}$. All the vertical arrows in the diagram (4.5) are injections. Let $y_{i,j} = \frac{1}{\pi_{\mathcal{P}}^k}x_{i,j} \in \phi_i(\mathcal{O}_{K_{\mathcal{P}^k}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)}) \subset \phi_i(\mathcal{O}_{K_{\mathcal{P}^{k+1}}(\frac{1}{\pi_{\mathcal{P}}^k}\Gamma)})$. One readily verifies that the $\pi_{\mathcal{P}}$-order of $\mathrm{red}_{\mathcal{W}_1}(y_{i,j})$ equals $k + c_{i,j}$. By Remark 4.2 the element $\mathrm{red}_{\mathcal{W}_1}(y_{i,j})$ comes from an element of $\phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}}$. Assume $c_{i,j} \geq 1$. We have

$$\sigma(\pi_{\mathcal{P}}^{c_{i,j}-1}\,\mathrm{red}_{\mathcal{W}_1}(y_{i,j})) = (1 + \pi_{\mathcal{P}}^k u)\pi_{\mathcal{P}}^{c_{i,j}-1}\,\mathrm{red}_{\mathcal{W}_1}(y_{i,j}). \tag{4.6}$$

This is because the $\pi_{\mathcal{P}}$-order of $\pi_{\mathcal{P}}^{c_{i,j}-1}\,\mathrm{red}_{\mathcal{W}_1}(y_{i,j})$ in $\phi_i^{\mathcal{W}}(k_{\mathcal{W}})_{\pi_{\mathcal{P}}}$ is equal to $k+1$. Notice that we have chosen $\mathcal{W}$ such that $\mathcal{W} \nmid \mathcal{P}$ and since the reduction map is injective on a torsion subgroup prime to $\mathcal{W}$ (cf. Proposition 4.1), we see, by definition of the field $K_{\mathcal{P}^{k+1}}$, that there exists a torsion element of the bottom left Mordell–Weil group which maps onto $\pi_{\mathcal{P}}^{c_{i,j}-1}\operatorname{red}_{\mathcal{W}_1}(y_{i,j})$. Since on this torsion element, σ acts via the homothety h, we have (4.6).

On the other hand, by our choice of $\mathcal{W}$ the Frobenius at $\mathcal{W}_1$ acts on $\pi_{\mathcal{P}}^{c_{i,j}-1}\operatorname{red}_{\mathcal{W}_1}(y_{i,j})$ via σ. Thus since $\operatorname{red}_{\mathcal{W}_1}(y_{i,j})\in\phi_i^{\mathcal{W}_2}(k_{\mathcal{W}_2})_{\pi_{\mathcal{P}}}$, we have

$$\sigma(\pi_{\mathcal{P}}^{c_{i,j}-1}\operatorname{red}_{\mathcal{W}_1}(y_{i,j}))=\pi_{\mathcal{P}}^{c_{i,j}-1}\operatorname{red}_{\mathcal{W}_1}(y_{i,j}). \tag{4.7}$$

Comparing (4.6) and (4.7) we obtain

$$\pi_{\mathcal{P}}^{c_{i,j}-1+k}u\operatorname{red}_{\mathcal{W}_1}(y_{i,j})=\pi_{\mathcal{P}}^{c_{i,j}-1}u\operatorname{red}_{\mathcal{W}_1}(x_{i,j})=0.$$

But this contradicts the assumption that the $\pi_{\mathcal{P}}$-order of $\operatorname{red}_{\mathcal{W}}(x_{i,j})$ is equal to $c_{i,j}$. ∎

We also have the following

Theorem 4.3. *Let $A=\mathbb{F}_q[t]$, and let ϕ_i for $1\le i\le t$ be Drinfeld modules of generic characteristic defined over K such that* $\operatorname{End}_{K^{\mathrm{sep}}}(\phi_i)=A$. *Let $\mathcal{P}\in\mathfrak{M}_A$ and $m\in\mathbb{N}\cup\{0\}$. Let $x_{i,j}\in\phi_i(\mathcal{O}_K)$ for some s_i and $1\le j\le s_i$ be linearly independent elements over A and let $T_{i,j}\in\phi_i[\mathcal{P}^m]$ be arbitrary torsion points for all $1\le j\le s_i$ and $1\le i\le t$. Then there is a set W of prime ideals $\mathcal{W}$ of $\mathcal{O}_K$ of positive density such that*

$$\operatorname{red}_{\mathcal{W}'}(T_{i,j})=\operatorname{red}_{\mathcal{W}}(x_{i,j})$$

in $\phi_i^{\mathcal{W}}(\mathcal{O}_K/\mathcal{W})_{\pi_{\mathcal{P}}}$ for all $1\le j\le s_i$ and $1\le i\le t$, where $\mathcal{W}'$ is a prime in $\mathcal{O}_L$ over $\mathcal{W}$, where L is the compositum of the fields $K(\phi_i[\mathcal{P}^m])$ for $1\le i\le t$, $\operatorname{red}_{\mathcal{W}'}:\phi_i(\mathcal{O}_L)\to\phi_i^{\mathcal{W}}(k_{\mathcal{W}'})$ and $k_{\mathcal{W}'}=\mathcal{O}_L/\mathcal{W}'$.

Proof. Put $x_{i,j}=P_{i,j}-T_{i,j}$, take L instead of K and apply Theorem 4.2. ∎

5. Local-to-global principle for t-modules associated to direct sums of Drinfeld modules. In what follows we work in the category of Anderson **t**-modules that are direct products of Drinfeld modules. This is the setting of [PT06]. So, we assume that the **t**-module $\widehat{\varphi}$ equals $\phi_1^{e_1}\times\cdots\times\phi_t^{e_t}$, where ϕ_i is a Drinfeld module of generic characteristic of rank d_i. We assume that for every $1\le i\le t$, $\operatorname{End}_{K^{\mathrm{sep}}}(\phi_i)=A$ and all modules are defined over the same ring of integers $\mathcal{O}_K$. We further assume that we are given a finitely generated A-submodule $N=N_1^{e_1}\times\cdots\times N_t^{e_t}$ of the Mordell–Weil group $\widehat{\varphi}(\mathcal{O}_K)=\phi_1(\mathcal{O}_K)^{e_1}\times\cdots\times\phi_t(\mathcal{O}_K)^{e_t}$, where $N_i\subset\phi_i(\mathcal{O}_K)$.

Remark 5.1. According to the result of Poonen [P95] the Mordell–Weil group is a direct sum of a free A-module on $\aleph_0$ generators and a finite torsion module.

In view of Remark 5.1, the following theorem is analogous to [BK11, Theorem 4.1]:

Theorem 5.1. *Let $\widehat{\varphi} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t}$ be a* $\mathbf{t}$*-module where ϕ_i for $1 \le i \le t$ are pairwise nonisogenous Drinfeld modules of generic characteristic defined over $\mathcal{O}_K$. Assume that $\mathrm{End}_{K^{\mathrm{sep}}}(\phi_i) = A$ for each $1 \le i \le t$. Let $N_i \subset \phi_i(\mathcal{O}_K)$ be a finitely generated A-submodule of the Mordell–Weil group. Pick an A-submodule $\Lambda \subset N = N_1^{e_1} \times \cdots \times N_k^{e_t}$. Assume that $d_i = \mathrm{rank}\,\phi_i \ge e_i$ for each $1 \le i \le t$. Let $P \in N$ and assume that $\mathrm{red}_{\mathcal{P}}(P) \in \mathrm{red}_{\mathcal{P}}(\Lambda)$ for almost all primes $\mathcal{P}$ of $\mathcal{O}_K$. Then $P \in \Lambda + N_{\mathrm{tor}}$.*

Let $F = \mathbb{F}_q(t)$ be the field of fractions of A. Notice that $d_i = \dim_{F_{\mathcal{P}}} V_{\mathcal{P}}(\phi_i)$ for $\mathcal{P} \in \mathfrak{M}_A$ (cf. Remark 3.1).

Corollary 5.2. *If $N_{\mathrm{tor}} \subset \Lambda$ then the following conditions are equivalent:*

- $P \in \Lambda$.
- $\mathrm{red}_{\mathcal{P}}(P) \in \mathrm{red}_{\mathcal{P}}(\Lambda)$ *for almost all primes $\mathcal{P}$ of $\mathcal{O}_K$.*

Recall some facts from [BK11, Section 3] concerning modules over division algebras adapted to our situation.

Let $\mathrm{Mat}_e(R)$ be the ring of square matrices of dimension e with coefficients in a ring R. Let $D_i = F = \mathrm{End}(\phi_i) \otimes_A F$. Then $\mathrm{End}(\widehat{\varphi}) = \mathrm{Mat}_{e_1}(A) \times \cdots \times \mathrm{Mat}_{e_t}(A)$ and $\mathrm{End}(\widehat{\varphi}) \otimes_A F = \mathrm{Mat}_{e_1}(F) \times \cdots \times \mathrm{Mat}_{e_t}(F)$.

Definition 5.1. Let $K_1(j)$ be the left ideal of the algebra $\mathrm{Mat}_{e_j}(F)$ consisting of the matrices $\boldsymbol{\alpha}(j)_1 = (a(j)_{l,m})$, $1 \le l, m \le e_j$, such that $a(j)_{l,m} = 0$ if $m \ne 1$.

Definition 5.2. For a D_i-vector space W_i let $\boldsymbol{\omega}(i) = (\omega(i), 0, \ldots, 0)^T \in W_i^{e_i}$ with $\omega(i) \in W_i$. Let $\mathbb{D} = \prod_{i=1}^t D_i$ and $\mathbb{M}_e(\mathbb{D}) = \prod_{i=1}^t \mathrm{Mat}_{e_i}(D_i)$, where $e = (e_1, \ldots, e_t)$.

Remark 5.2. Let W_i be a finite-dimensional F-vector space over D_i, $1 \le i \le t$. Then $W = \bigoplus_{i=1}^t W_i^{e_i}$ has an obvious $\mathbb{M}_e(\mathbb{D})$-module structure.

The following lemma is essentially a specialization of [BK11, Corollary 3.2].

Lemma 5.3. *Every nonzero simple $\mathbb{M}_e(\mathbb{D})$-submodule of $W = \bigoplus_{i=1}^t W_i^{e_i}$ is of the form*

$$K(j)_1\boldsymbol{\omega}(j) = \{(a_{1,1}\omega(j), \ldots, a_{e_j,1}\omega(j))^T \mid a_{k,1} \in D_j\},$$

where $1 \le k \le e_j$, $1 \le j \le t$ and $\omega(j) \in W_j$.

The trace homomorphism $\mathrm{tr} : \mathbb{M}_e(\mathbb{D}) \to F$ is defined as $\mathrm{tr} = \sum_{i=1}^t \mathrm{tr}_i$ where $\mathrm{tr}_i : \mathrm{Mat}_{e_i}(D_i) \to F$ is the usual trace. The following lemma corresponds to [BK11, Lemma 3.3].

LEMMA 5.4. *The induced map* $\mathrm{tr} : \mathrm{Hom}_{\mathbb{M}_e(\mathbb{D})}(W, \mathbb{M}_e(\mathbb{D})) \to \mathrm{Hom}_F(W, F)$ *is an isomorphism.*

Proof. Replace $\mathbb{Q}$ by F in [BK11, proof of Lemma 3.3]. ∎

Semisimplicity of $\mathbb{M}_e(\mathbb{D})$ implies that the module W is semisimple and therefore for any $\boldsymbol{\pi} \in \mathrm{Hom}_{\mathbb{M}_e(\mathbb{D})}(W, \mathbb{M}_e(\mathbb{D}))$ there exists $\boldsymbol{s} : \mathrm{Im}\, \boldsymbol{\pi} \to W$ such that $\boldsymbol{\pi} \circ \boldsymbol{s} = \mathrm{Id}$. One has of course the splittings $\boldsymbol{\pi} = \prod_{i=1}^{t} \mathrm{Im}\, \boldsymbol{\pi}(i)$ and $\boldsymbol{s} = \bigoplus_{i=1}^{t} \boldsymbol{s}(i)$ where $\boldsymbol{\pi}(i) \in \mathrm{Hom}_{\mathrm{Mat}_{e_i}(D_i)}(W_i^{e_i}, \mathrm{Mat}_{e_i}(D_i))$, $\boldsymbol{s}(i) \in \mathrm{Hom}_{\mathrm{Mat}_{e_i}(D_i)}(\mathrm{Im}\, \boldsymbol{\pi}(i), W_i^{e_i})$ and $\boldsymbol{\pi}(i) \circ \boldsymbol{s}(i) = \mathrm{Id}$.

5.1. Proof of Theorem 5.1. Since the A-torsion of N is finite (cf. Remark 5.1) we can consider a torsion free A-module $\Omega := cN$, where $c = g_1 \dots g_k$ is the product of generators of the A-annihilator of N_{tor}, and replace N by Ω. We can also assume that $\Lambda \subset \Omega$ and $P \in \Omega$. Let $P_1, \dots, P_s$ be an A-basis of Ω such that

$$\Lambda = Av_1P_1 + \dots + Av_sP_s, \quad P = n_1P_1 + \dots + n_sP_s$$

where $v_i, n_i \in A$ for $1 \le i \le s$. Assume that $P \notin \Lambda$. This is equivalent to $P \otimes 1 \notin \Lambda \otimes_A A_{\mathcal{U}}$ for some $\mathcal{U} \in A$ where $A_{\mathcal{U}}$ is the completion of A with respect to $\mathcal{U}$. Thus there exists $1 \le j_0 \le s$ and a natural number m_1 such that $\mathcal{U}^{m_1} \parallel n_{j_0}$ and $\mathcal{U}^{m_1+1} \mid v_{j_0}$. Define the following map of A-modules:

$$\pi : \Omega \to A, \quad \pi(R) = a_{j_0}, \quad R = \sum_{i=1}^{s} a_i P_i, \quad a_i \in A.$$

We shall also write π for the map $\pi \otimes_A \mathrm{Id}_F : \Omega \otimes_A F \to F$. By Lemma 5.4 we obtain a map $\boldsymbol{\pi} \in \mathrm{Hom}_{\mathbb{M}_e(\mathbb{D})}(\Omega \otimes_A F, \mathbb{M}_e(\mathbb{D}))$ such that $\mathrm{tr}\, \boldsymbol{\pi} = \pi$. By the above discussion we also have $\boldsymbol{s}$ such that $\boldsymbol{\pi} \circ \boldsymbol{s} = \mathrm{Id}$. We have

$$\Omega \otimes_A F \cong \mathrm{Im}\, \boldsymbol{s} \oplus \mathrm{Ker}\, \boldsymbol{\pi} \quad \text{and} \quad \Omega^{e_i} \otimes_A F \cong \mathrm{Im}\, \boldsymbol{s}(i) \oplus \mathrm{Ker}\, \boldsymbol{\pi}(i), 1 \le i \le t.$$

By Lemma 5.3 we have the decompositions

$$\mathrm{Im}\, \boldsymbol{s}(i) = \bigoplus_{k=1}^{k_i} K(i)_1 \boldsymbol{\omega}_k(i) \quad \text{and} \quad \mathrm{Ker}\, \boldsymbol{\pi}(i) = \bigoplus_{k=k_i+1}^{u_i} K(i)_1 \boldsymbol{\omega}_k(i).$$

By assumptions $k_i \le e_i \le d_i$ for every $1 \le i \le t$.

The elements $\omega_1(i), \dots, \omega_{k_i}(i), \dots, \omega_{u_i}(i)$ constitute a basis for the F-vector space $\Omega_i \otimes_A F$. (Notice that $D_i = F$ by assumption.) Without loss of generality one can assume that $\omega_{k_i+1}(i), \dots, \omega_{u_i}(i) \in \Omega_i$. The F-module $\Omega_i \otimes_A F$ is free. We have $\mathcal{R} = \mathrm{End}_A\, \Omega \subset \mathbb{M}_e(\mathbb{D}) = \mathcal{R} \otimes_A F$. Since Ω is a finitely generated A-module, there exists a polynomial $M_0 \in A$ such that the homomorphisms $M_0 \boldsymbol{\pi} : \Omega \to \mathcal{R}$ and $\boldsymbol{s} : M_0 \boldsymbol{\pi}(\Omega) \to \Omega$ are well defined.

Define the $\mathrm{Mat}_{e_i}(A)$-module

$$\Gamma(i) = \sum_{k=1}^{k_i} K(i)_1 M_0 \boldsymbol{\omega}_k(i) + \sum_{k=k_i+1}^{u_i} K(i)_1 \boldsymbol{\omega}_k(i) \subset \Omega_i^{e_i}.$$

Let $\Gamma = \bigoplus \Gamma(i) \subset \Omega$, and let $M_2, M_3 \in A$ be polynomials of minimal degrees such that $M_2 \Omega \subset \Gamma$ and $M_3 \Gamma \subset M_2 \Omega$. The choice of j_0 implies $\pi(P) \notin \pi(\Lambda \otimes_A A_{\mathcal{U}}) + \mathcal{U}^m \pi(\Omega \otimes_A A_{\mathcal{U}})$ for every $m > m_1$. Choose such an m. Then since $\mathrm{tr}\, M_0 \boldsymbol{\pi} = M_0 \pi$ we have

$$M_0 \boldsymbol{\pi}(P) \notin M_0 \boldsymbol{\pi}(\Lambda \otimes_A A_{\mathcal{U}}) + M_0 \mathcal{U}^m \boldsymbol{\pi}(\Omega \otimes_A A_{\mathcal{U}}). \tag{5.1}$$

Let $K(i)_{1,\mathcal{U}} = K(i)_1 \otimes_A A_{\mathcal{U}} \subset \mathrm{Mat}_{e_i}(A_{\mathcal{U}})$ and $Q \in \Lambda$. By the definition of $M_2 \in A$ we have

$$M_2(P \otimes 1 - Q \otimes 1) = M_0^2 \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\boldsymbol{\alpha}_k(i)_1 - \boldsymbol{\beta}_k(i)_1) \boldsymbol{\pi}(\boldsymbol{\omega}_k(i)) \tag{5.2}$$

where $\boldsymbol{\alpha}_k(i)_1, \boldsymbol{\beta}_k(i)_1 \in K(i)_{1,\mathcal{U}}$, $1 \le k \le u_i$, $1 \le i \le t$. By (5.1) and (5.2) and the choice of M_3 we have

$$M_0^2 \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\boldsymbol{\alpha}_k(i)_1 - \boldsymbol{\beta}_k(i)_1) \boldsymbol{\pi}(\boldsymbol{\omega}_k(i)) \notin \mathcal{U}^m M_0 \boldsymbol{\pi}(M_3 \Gamma).$$

This implies that for some $1 \le i \le t$ and $1 \le k \le k_i$ we have

$$\boldsymbol{\alpha}_k(i)_1 - \boldsymbol{\beta}_k(i)_1 \notin \mathcal{U}^m M_3 K(i)_{1,\mathcal{U}}. \tag{5.3}$$

Notice that for all $n' \in \mathbb{N}$ there is an isomorphism $\phi_i[\mathcal{U}^{n'}] \cong T_{\mathcal{U}}(\phi_i)/v^{n'} T_{\mathcal{U}}(\phi_i)$ where v is a generator of the ideal $\mathcal{U}$. Let $\mathcal{L}_{\mathcal{U}} = \bigoplus_{i=1}^{t} T_{\mathcal{U}}(\phi_i)$ and $\eta_1(i), \ldots, \eta_{d_i}(i)$ be a basis of $T_{\mathcal{U}}(\phi_i)$ over $A_{\mathcal{U}}$ (cf. Remark 3.1). Let m_0 and m_3 be the natural numbers such that $v^{m_0} \parallel M_0$ and $v^{m_3} \parallel M_3$. Let $V_{\mathcal{U},i} = T_{\mathcal{U}}(\phi_i) \otimes_{A_{\mathcal{U}}} F_{\mathcal{U}}$. Then $\dim_{F_{\mathcal{U}}} V_{\mathcal{U},i} = d_i$. The quotient $T_{\mathcal{U}}(\phi_i)/v^{n'} T_{\mathcal{U}}(\phi_i)$ of $A_{\mathcal{U}}$-modules is a free $A_{\mathcal{U}}/v^{n'} A_{\mathcal{U}}$-module with the basis $T_1(i), \ldots, T_{d_i}(i)$ where $T_k(i)$ is the image of $\eta_k(i)$ in $\phi_i[\mathcal{U}^{n'}]$. Let $\boldsymbol{\eta}_k(i) = (\eta_k(i), 0, \ldots, 0)^T \in T_{\mathcal{U}}(\phi_i)^{e_i}$ and $\boldsymbol{T}_k(i) = (T_k(i), 0, , \ldots, 0)^T \in \phi_i[\mathcal{U}^{n'}]^{e_i}$ (cf. Definition 5.2). Take $n' > m + m_0 + m_3$. By Theorem 4.3, applied for $\mathcal{P} = \mathcal{U}$, there exists a set of primes $\mathcal{W} \in \mathcal{O}_L$ (where L is the compositum of the fields defined in Theorem 4.3) of positive density such that

$$\mathrm{red}_{\mathcal{W}}(\omega_k(i)) = 0 \quad \text{for } 1 \le i \le t,\ k_i + 1 \le k \le u_i \tag{5.4}$$

and

$$\mathrm{red}_{\mathcal{W}}(\omega_k(i)) = \mathrm{red}_{\mathcal{W}}(T_k(i)) \quad \text{for } 1 \le i \le t,\ 1 \le k \le k_i. \tag{5.5}$$

Pick such a prime $\mathcal{W}$ that does not divide $\mathcal{U}$. Since by assumption $\mathrm{red}_{\mathcal{W}}(P) \in \mathrm{red}_{\mathcal{W}}(\Lambda)$, we can choose $Q \in \Lambda$ such that $\mathrm{red}_{\mathcal{W}}(P) = \mathrm{red}_{\mathcal{W}}(Q)$. Now we apply the reduction map $\mathrm{red}_{\mathcal{W}}$ to the equation

$$M_2(P-Q)=\sum_{i=1}^{t}\sum_{k=1}^{k_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)M_0\boldsymbol{\omega}_k(i)$$
$$+\sum_{i=1}^{t}\sum_{k=k_i+1}^{u_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)\boldsymbol{\omega}_k(i).$$

Thus we obtain $0=\sum_{i=1}^{t}\sum_{k=1}^{k_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)M_0\,\mathrm{red}_{\mathcal{W}}(\boldsymbol{T}_k(i))$. Since $\mathrm{red}_{\mathcal{W}}$ is injective on a torsion prime to the characteristic of a Drinfeld module (cf. Proposition 4.1), we have $0=\sum_{i=1}^{t}\sum_{k=1}^{k_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)M_0\boldsymbol{T}_k(i)$. Therefore the element $v^{m_0}\sum_{i=1}^{t}\sum_{k=1}^{k_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)\boldsymbol{\eta}_k(i)$ maps to zero in the $A_{\mathcal{U}}/v^{n'}A_{\mathcal{U}}$-module $\mathcal{L}_{\mathcal{U}}/v^{n'}\mathcal{L}_{\mathcal{U}}$. This yields

$$\sum_{i=1}^{t}\sum_{k=1}^{k_i}(\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1)\boldsymbol{\eta}_k(i)\in v^{n'-m_0}\mathcal{L}_{\mathcal{U}}.$$

But since $\eta_k(i)$, $1\le k\le d_i$, constitute a basis of $T_{\mathcal{U}}(\phi_i)$ over $A_{\mathcal{U}}$, we obtain

$$\boldsymbol{\alpha}_k(i)_1-\boldsymbol{\beta}_k(i)_1\in v^{n'-m_0}K(i)_{1,\mathcal{U}}, \tag{5.6}$$

which contradicts (5.3). ∎

We also have the following theorem:

Theorem 5.5. *Let ϕ be a Drinfeld module of rank d defined over $\mathcal{O}_K$ with* $\mathrm{End}(\phi)=A$. *Then the numerical bound in Theorem* 5.1 *is the best possible. That is, for the* $\mathbf{t}$*-module ϕ^{d+1} the local-to-global principle of Theorem* 5.1 *does not hold.*

Proof. Our proof is modelled on the counterexample to the local-to-global principle for abelian varieties constructed by P. Jossen and A. Perucca [JP10]. Let $e=d+1$ and $P_1,\dots,P_e\in\phi(\mathcal{O}_K)$ be points linearly independent over A. Let

$$P:=\begin{bmatrix}P_1\\ \vdots\\ P_e\end{bmatrix},\qquad \Lambda:=\{MP\mid M\in\mathrm{Mat}_e(A),\ \mathrm{tr}\,M=0\}.$$

Set $\kappa=\mathcal{O}_K/\mathcal{P}$. Notice that $P\notin\Lambda$ since $P_1,\dots,P_e$ are A-linearly independent. Let $\mathcal{W}\in\mathcal{O}_K$ be a prime of good reduction for ϕ. We will find $M\in\mathrm{Mat}_e(A)$ such that $\overline{P}=M\overline{P}$ where $\overline{P}=[\overline{P}_1,\dots,\overline{P}_e]^T$ is the reduction mod $\mathcal{W}$ of P. This will show that $\mathrm{red}_{\mathcal{W}}P\in\mathrm{red}_{\mathcal{W}}\Lambda$. Since the Mordell–Weil group $\phi(\kappa)[\mathcal{P}]$ is finite, there exist polynomials $\alpha_1,\dots,\alpha_e\in A$ of minimal degrees such that

$$\begin{aligned}
\alpha_1\overline{P}_1 + m_{1,2}\overline{P}_2 + \cdots + m_{1,e}\overline{P}_e &= 0,\\
m_{2,1}\overline{P}_1 + \alpha_2\overline{P}_2 + \cdots + m_{2,e}\overline{P}_e &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
m_{e,1}\overline{P}_1 + m_{e,2}\overline{P}_2 + \cdots + \alpha_e\overline{P}_e &= 0.
\end{aligned}$$

We will show that $D = \gcd(\alpha_1, \ldots, \alpha_e) = 1$. It is enough to show that for any prime ideal $\mathcal{P} \lhd A$ the polynomial D is not divisible by $\mathcal{P}$. Assume the contrary and let $\tilde{\mathcal{P}}$ be a prime that divides D. This means, by our choice of $\alpha_1, \ldots, \alpha_e$, that $\tilde{\mathcal{P}}$ divides the coefficients of any linear combination of points $\bar{P}_1, \ldots, \bar{P}_e \in \phi^{\tilde{\mathcal{P}}}$. By Proposition 2.1 the group $\phi^{\tilde{\mathcal{P}}}(\kappa)[\mathcal{P}]$ is isomorphic to $(A/\tilde{\mathcal{P}})^{d-\bar{h}}$. Therefore the group $Y = (\bar{P}_1, \ldots, \bar{P}_e) \cap \phi^{\tilde{\mathcal{P}}}(\kappa)[\mathcal{P}]$ is generated by fewer than e elements. We may assume without loss of generality that $Y = (\bar{P}_2, \ldots, \bar{P}_e) \cap \phi^{\tilde{\mathcal{P}}}(\kappa)[\mathcal{P}]$. Let

$$\alpha_1\bar{P}_1 + x_2\bar{P}_2 + \cdots + x_e\bar{P}_e = 0 \tag{5.7}$$

be a linear relation. Then since the left-hand side of (5.7) is in Y, we obtain a contradiction with the minimality of α_1. Thus $D = 1$. Hence there exist $a_1, \ldots, a_e \in A$ such that

$$e = a_1\alpha_1 + \cdots + a_e\alpha_e.$$

Put $m_{i,i} = 1 - a_i\alpha_i$. Then $m_{1,1} + \cdots + m_{e,e} = 0$ and

$$\begin{bmatrix} m_{1,1} & \ldots & m_{1,e}\\ \cdots & \cdots & \cdots\\ m_{e,1} & \ldots & m_{e,e}\end{bmatrix} \begin{bmatrix}\bar{P}_1\\ \cdots\\ \bar{P}_e\end{bmatrix} = \begin{bmatrix}\bar{P}_1\\ \cdots\\ \bar{P}_e\end{bmatrix}.$$

Therefore $\bar{P} \in \bar{\Lambda}$. ∎

Essentially the same proof—with $\mathbb{Z}$ substituted for A—works for abelian varieties with $\operatorname{End}\mathbb{A} = \mathbb{Z}$ and we obtain the following:

Theorem 5.6. *Let $\mathbb{A}/F$ be an abelian variety defined over a number field F with* $\operatorname{End}\mathbb{A} = \mathbb{Z}$. *Let $d = \dim_{\operatorname{End}_{F'}(\mathbb{A}_i)^0} H_1(\mathbb{A}(\mathbb{C}); \mathbb{Q}) = 2g$, where $g =$* $\dim\mathbb{A}$. *Assume* $\operatorname{rank} A(F) > d$. *Then the numerical bound in Theorem* 1.3 *is best possible, that is, for $\mathbb{A}^{d+1}$ the local-to-global principle of Theorem* 1.3 *does not hold.*

6. Some other consequences of the reduction theorem. In [B17] S. Barańczuk introduced a dynamical version of the local-to-global principle 1.1. He considers the case of abelian groups (modules over $\mathbb{Z}$) satisfying the following two axioms:

Assumptions 6.1. *Let B be an abelian group such that there are homomorphisms* $\mathrm{red}_v : B \to B_v$ *for an infinite family of primes $v \in F$, whose targets B_v are finite abelian groups.*

(1) *Let l be a prime number, and $(k_1, \dots, k_m)$ be a sequence of nonnegative integers. If $P_1, \dots, P_m \in B$ are points linearly independent over $\mathbb{Z}$, then there is a family of primes v in F such that $l^{k_i} \parallel \mathrm{ord}_v P_i$ if $k_i > 0$ and $l \nmid \mathrm{ord}_v P_i$ if $k_i = 0$.*
(2) *For almost all v the map $B_{\mathrm{tors}} \to B_v$ is injective.*

Here $\mathrm{ord}_v P$ is the order of a reduced point P mod v. In [B17] and [Ba17] finitely generated abelian groups are considered. In our case we modify Assumption 6.1(2) in order to deal appropriately with infinite torsion in Drinfeld modules. Instead of Assumption 6.1 we assume the following:

Assumptions 6.2. *Let B be an A-module such that there are homomorphisms* $\mathrm{red}_{\mathcal{U}} : B \to B_{\mathcal{U}}$ *for an infinite family $\mathcal{U} \in \mathrm{Spec}(\mathcal{O}_K)$ such that $B_{\mathcal{U}}$ is a torsion A-module. For $P \in B$ let* $\mathrm{ord}_{\mathcal{U}} P$ *be the polynomial of minimal degree in $\mathcal{O}_K$ that annihilates* $\mathrm{red}_{\mathcal{U}} P$.

(1) *For $\mathcal{U}$ and a sequence $(k_1, \dots, k_m)$ of nonnegative integers, the following holds true: If $P_1, \dots, P_m \in B$ are points linearly independent over A, then there is a family of primes $\mathcal{W}$ in $\mathcal{O}_K$ such that $\mathcal{U}^{k_i} \parallel \mathrm{ord}_{\mathcal{W}} P_i$ if $k_i > 0$ and $\mathcal{U} \nmid \mathrm{ord}_{\mathcal{W}} P_i$ if $k_i = 0$.*
(2) *For any $\mathcal{U}$ there are infinitely many primes $\mathcal{W}$ in $\mathcal{O}_K$ such that the reduction map is an injection on $\mathcal{U}$-torsion, i.e.* $\mathrm{red}_{\mathcal{W}} : B[\mathcal{U}] \hookrightarrow \mathcal{B}_{\mathcal{W}}[\mathcal{U}]$.

The main theorem of [B17] can be extended to the case of $\mathbf{t}$-modules considered in this paper in the following way.

Theorem 6.3. *Let $\widehat{\varphi} = \phi_1^{e_1} \times \dots \times \phi_t^{e_t}$ be a $\mathbf{t}$-module where ϕ_i, $1 \le i \le t$, are pairwise nonisogenous Drinfeld modules defined over $\mathcal{O}_K$ such that* $\mathrm{End}\,\phi_i = A$. *Let $\Lambda \subset \phi(\mathcal{O}_K)$ be a finitely generated A-submodule. For $x \in \phi(\mathcal{O}_K)$ and $w(t) \in A$ let $O_{w(t)}(x) = \{w(t)^n(x) \mid n \ge 0\}$ be the orbit of the point x under the iterations of multiplication by $w(t) \in A$. Then the following are equivalent:*

(1) *For almost every $\mathcal{U} \in \mathrm{Spec}(\mathcal{O}_K)$,*
$$O_{w(t)}(\mathrm{red}_{\mathcal{U}}(P)) \cap \mathrm{red}_{\mathcal{U}}(\Lambda) \neq \emptyset.$$
(2) $O_{w(t)}(P) \cap \Lambda \neq \emptyset$.

On the other hand, the main theorem of [Ba17] extended to $\mathbf{t}$-modules reads as follows:

Theorem 6.4. *Let $\widehat{\varphi} = \phi_1^{e_1} \times \dots \times \phi_t^{e_t}$ be a $\mathbf{t}$-module where ϕ_i, $1 \le i \le t$, are pairwise nonisogenous Drinfeld modules defined over $\mathcal{O}_K$ such* *that* $\operatorname{End}\phi_i = A$. *Let* $P, Q \in \phi(\mathcal{O}_K)$ *and* $w_1(t), w_2(t) \in A$. *Suppose that for almost all* $\mathcal{U} \in \operatorname{Spec}(\mathcal{O}_K)$ *there exists a natural number* $n_{\mathcal{U}}$ *such that*

$$\operatorname{red}_{\mathcal{U}}(w_1^{n_{\mathcal{U}}} P - w_2^{n_{\mathcal{U}}} Q) = 0.$$

Then there exists a natural number n *and a torsion point* $T \in \phi(\mathcal{O}_K)$ *of an order that divides some power of* $\gcd(w_1, w_2)$ *such that* $w_1^n P - w_2^n Q = T$.

The proofs of Theorems 6.3 and 6.4 follow the lines of S. Barańczuk's original proofs. In appropriate places one has to replace multiplication by a natural number (viewed as an element of $\operatorname{End} B$) by the A-module action of an element $w(t) \in A$, and Assumption 6.1 by 6.2. Additionally one has to check that Assumption 6.2 is fulfilled. Assumption 6.2(2) is fulfilled by $B = \phi(\mathcal{O}_K)$ (cf. Proposition 4.1(2)), while Assumption 6.2(1) readily follows from Theorem 1.6.

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