# $K$-THEORY OF CLUSTER $C^*$-ALGEBRAS

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**Abstract.** It is proved that the $K_0$-group of a cluster $C^*$-algebra is isomorphic to the corresponding cluster algebra. As a corollary, one gets a shorter proof of the positivity conjecture for cluster algebras. As an example, we consider a cluster $C^*$-algebra $\mathbb{A}(1, 1)$ coming from triangulation of an annulus with one marked point on each boundary component.

**1. Introduction.** Cluster algebras are a class of commutative rings introduced by Fomin & Zelevinsky [5] having deep roots in hyperbolic geometry and Teichmüller theory [17]. Namely, the *cluster algebra* $\mathcal{A}(\mathbf{x}, B)$ of rank $n$ is a subring of the field of rational functions in $n$ variables depending on a *cluster* of variables $\mathbf{x} = (x_1, \ldots, x_n)$ and a skew-symmetric matrix $B = (b_{ij}) \in M_n(\mathbf{Z})$; the pair $(\mathbf{x}, B)$ is called a *seed*. A new cluster $\mathbf{x}' = (x_1, \ldots, x_k', \ldots, x_n)$ and a new skew-symmetric matrix $B' = (b_{ij}')$ is obtained from $(\mathbf{x}, B)$ by the *exchange relations*:

$$x_k x_k' = \prod_{i=1}^n x_i^{\max(b_{ik}, 0)} + \prod_{i=1}^n x_i^{\max(-b_{ik}, 0)},$$

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + (|b_{ik}|b_{kj} + b_{ik}|b_{kj}|)/2 & \text{otherwise.} \end{cases} \tag{1}$$

The seed $(\mathbf{x}', B')$ is said to be a *mutation* of $(\mathbf{x}, B)$ in direction $k$, where $1 \le k \le n$; the algebra $\mathcal{A}(\mathbf{x}, B)$ is generated by cluster variables $\{x_i\}_{i=1}^\infty$ obtained from the initial seed $(\mathbf{x}, B)$ by the iteration of mutations in all possible directions $k$.

The *Laurent phenomenon* proved by Fomin & Zelevinsky [5] says that $\mathcal{A}(\mathbf{x}, B) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$, where $\mathbf{Z}[\mathbf{x}^{\pm 1}]$ is the ring of the Laurent polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$

depending on an initial seed $(\mathbf{x}, B)$; in other words, each generator $x_i$ of algebra $\mathcal{A}(\mathbf{x}, B)$ can be written as a Laurent polynomial in $n$ variables with integer coefficients. The famous *Positivity Conjecture* says that coefficients of the Laurent polynomials corresponding to variables $x_i$ are always non-negative integers, see [5]. A general form of the Positivity Conjecture was proved by Lee & Schiffler [9] using a clever combinatorial formula for the variables $x_i$.

*Cluster $C^*$-algebras* $\mathbb{A}(\mathbf{x}, B)$ are a class of non-commutative rings introduced in [13]. The $\mathbb{A}(\mathbf{x}, B)$ is an *AF-algebra* given by the Bratteli diagram [1]; such a diagram is obtained from a mutation tree of the initial seed $(\mathbf{x}, B)$ modulo an equivalence relation between the seeds lying at the same level, see Section 2.2. (We refer the reader to Figures 1 and 2 for an immediate example of such algebras.)
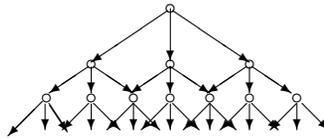


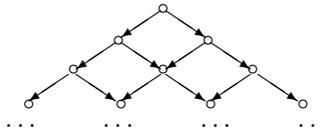Fig. 1. The Bratteli diagram of Markov's cluster $C^*$-algebra.



Fig. 2. Bratteli diagram of algebra $\mathbb{A}(1, 1)$.

The aim of our note is the $K$-theory of the $AF$-algebra $\mathbb{A}(\mathbf{x}, B)$. Namely, the *ordered* abelian group is a pair $(G, G^+)$ consisting of an abelian group $G$ and a semigroup $G^+ \subset G$ of positive elements of $G$; the order $\leq$ on $G$ is defined by the positive cone $G^+$, i.e. $a \leq b$ if and only if $b - a \in G^+$. An *order-isomorphism* $\cong$ between $(G, G^+)$ and $(H, H^+)$ is an isomorphism $\varphi : G \to H$ such that $\varphi(G^+) = H^+$. Denote by $K_0(\mathbb{A}(\mathbf{x}, B))$ the $K_0$-group of the $AF$-algebra $\mathbb{A}(\mathbf{x}, B)$ and by $K_0^+(\mathbb{A}(\mathbf{x}, B)) \subset K_0(\mathbb{A}(\mathbf{x}, B))$ its Grothendieck semigroup [2, Chapter 8]. The pair $(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B)))$ is an invariant of *Morita equivalence* of the $AF$-algebra $\mathbb{A}(\mathbf{x}, B)$ [3]. In view of the Laurent phenomenon, let $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$ be an additive group of the cluster algebra $\mathcal{A}(\mathbf{x}, B)$; let $\mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$ be a semigroup inside the $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$ consisting of the Laurent polynomials with *positive* coefficients. The pair $(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B))$ is an abelian group with order. The order $a > b$ is defined between two elements $a, b \in \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$ if and only if $a - b \in \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$. Our main result can be formulated as follows.

THEOREM 1. $\left(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B))\right) \cong (\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B))$.

An application of Theorem 1 is as follows. Recall that the *dimension group* is a triple $(G, G^+, \Gamma)$ consisting of an abelian group $G$, a semigroup of positive elements $G^+ \subset G$ and a scale $\Gamma \subseteq G^+$, i.e. a generating, hereditary and directed subset of $G^+$ [2, Chapter 7]. For instance, the $\Gamma \cong G^+$ is a scale called *stable*; thus the pair $(G, G^+)$

is a special case of the dimension group. An order-isomorphism $\cong$ between dimension groups $(G, G^+, \Gamma)$ and $(H, H^+, \Gamma')$ is an isomorphism $\varphi : G \to H$ such that $\varphi(G^+) = H^+$ and $\varphi(\Gamma) = \Gamma'$. Denote by $\Gamma \subset K_0^+(\mathbb{A}(\mathbf{x}, B))$ the set of the Murray–von Neumann equivalence classes of projections in the algebra $\mathbb{A}(\mathbf{x}, B)$. It is known that the triple $(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B)), \Gamma)$ is an invariant of the *isomorphism* class of the $AF$-algebra $\mathbb{A}(\mathbf{x}, B)$ [3]. It is not hard to observe that the set $X = \{x_i\}_{i=1}^\infty$ of all variables $x_i$ in the cluster algebra $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$ is a scale, since it is a generating, hereditary and directed subset of $\mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$. Notice that choosing a different initial seed $(\mathbf{x}, B)$ for the Laurent expansion of variables $x_i$ yields a new scale $X'$ such that $(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B), X) \cong (\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B), X')$. But $X \subseteq \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$ for any dimension group; therefore Theorem 1 implies a new proof of the Positivity Conjecture for the cluster algebras.

COROLLARY 1. *The coefficients of the Laurent polynomials corresponding to the cluster variables $x_i$ are non-negative integers.*

The article is organized as follows. The preliminary facts are introduced in Section 2. Theorem 1 and Corollary 1 are proved in Section 3. In Section 4 we consider an example of the cluster $C^*$-algebra $\mathbb{A}(1, 1)$ coming from triangulation of an annulus with one marked point on each boundary component.

**2. Preliminaries.** This section is a brief review of the $AF$-algebras, cluster $C^*$-algebras and Mundici dimension groups. For a general review of $C^*$-algebras we refer the reader to [12]. The $AF$-algebras were introduced in [1]. The general $K$-theory of $C^*$-algebras is covered in [15] and $K$-theory of the $AF$-algebras in [2]. Cluster $C^*$-algebras were the subject of [13]. Mundici dimension groups were introduced by Mundici [10].

**2.1. $AF$-algebras and dimension groups.** A $C^*$-*algebra* is an algebra $A$ over $\mathbf{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $\|ab\| \leq \|a\| \, \|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $A$ can be thought of as a noncommutative topological space.

An $AF$-*algebra* (Approximately Finite $C^*$-algebra) is defined to be the norm closure of a dimension-increasing sequence of finite-dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbf{C}$. Here the index $n = (n_1, \dots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \dots,$$

where $M_i$ are the finite-dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \cdots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \cdots \oplus M_{i'_k}$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \to M_{i'}$ the homomorphism. (To keep it simple, one can assume that $i' = i + 1$.) One has two sets of vertices $V_{i_1}, \dots, V_{i_k}$ and $V_{i'_1}, \dots, V_{i'_k}$ joined by $b_{rs}$ edges whenever the summand $M_{i'_r}$ contains $b_{rs}$ copies of the summand $M_{i_s}$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called the *Bratteli diagram* of the $AF$-algebra.

The matrix $B = (b_{rs})$ is known as a *partial multiplicity* matrix; an infinite sequence of $B_i$ defines a unique *AF*-algebra.

For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$ are *equivalent* if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p+q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $A$. The functor $A \to K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+ \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a *dimension group*; we denote an order-isomorphism class of the latter by $(G, G^+)$.

If $\mathbb{A}$ is an *AF*-algebra, then its dimension group $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), u)$ is a complete isomorphism invariant of algebra $\mathbb{A}$ [3]. The order-isomorphism class $(K_0(\mathbb{A}), K_0^+(\mathbb{A}))$ is an invariant of *Morita equivalence* of the algebra $\mathbb{A}$, i.e. an isomorphism class in the category of finitely generated projective modules over $\mathbb{A}$.

The *scale* $\Gamma$ is a subset of $K_0^+(\mathbb{A})$ which is generating, hereditary and directed, i.e. (i) for each $a \in K_0^+(\mathbb{A})$ there exist $a_1, \ldots, a_r \in \Gamma(\mathbb{A})$, such that $a = a_1 + \cdots + a_r$; (ii) if $0 \le a \le b \in \Gamma$, then $a \in \Gamma$; (iii) given $a, b \in \Gamma$ there exists $c \in \Gamma$ such that $a, b \le c$. If $u$ is an order unit, then the set $\Gamma := \{a \in K_0^+(\mathbb{A}) \,|\, 0 \le a \le u\}$ is a scale; thus the dimension group of algebra $\mathbb{A}$ can be written in the form $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), \Gamma)$.

**2.2. Cluster $C^*$-algebras.** Let $T_n$ be an oriented tree whose vertices correspond to the seeds $(\mathbf{x}, B)$ and outgoing edges correspond to mutations in direction $1 \le k \le n$. Notice that the tree $T_n$ of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ has a grading by levels, i.e. the minimal distance from the root of $T_n$. We shall say that a pair of clusters $\mathbf{x}$ and $\mathbf{x}'$ with exchange matrices $B$ and $B'$ are *$\ell$-equivalent*, if:

(i)  $\mathbf{x}$ and $\mathbf{x}'$ lie at the same level;
(ii)  $\mathbf{x}$ and $\mathbf{x}'$ coincide modulo a cyclic permutation of variables $x_i$;
(iii)  $B = B'$.

It is not hard to see that $\ell$ is an equivalence relation on the set of vertices of graph $T_n$.

DEFINITION 1. By a cluster $C^*$-algebra $\mathbb{A}(\mathbf{x}, B)$ one understands an *AF*-algebra given by the Bratteli diagram $\mathfrak{B}(\mathbf{x}, B)$ of the form

$$\mathfrak{B}(\mathbf{x}, B) := T_n \bmod \ell.$$

REMARK 1. Notice that the graph $\mathfrak{B}(\mathbf{x}, B)$ is no longer a tree; the cycles of $\mathfrak{B}(\mathbf{x}, B)$ appear after gluing together vertices lying at the same level of the tree according to the relation $\ell$. The $\mathfrak{B}(\mathbf{x}, B)$ is not a regular graph, since the valency of vertices can vary. However, the $\mathfrak{B}(\mathbf{x}, B)$ is always a Bratteli diagram, since it is obtained from a regular tree by an addition of extra edges and subsequent contraction of the respective edges. Notice also that the $\mathfrak{B}(\mathbf{x}, B)$ is a finite graph if and only if $\mathcal{A}(\mathbf{x}, B)$ is a finite cluster algebra.

EXAMPLE 1. Let $\mathbf{x} = (x_1, x_2)$ and

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is associated to an ideal triangulation of an annulus with one marked point on each boundary component, see [4, Example 4.4]. The exchange relations (1) in this case can be written as $x_{i-1}x_{i+1} = 1 + x_i^2$ and $B' = -B$. It is easy to verify using Definition 1 that the Bratteli diagram $T_2$ mod $\ell$ of the corresponding cluster $C^*$-algebra $\mathbb{A}(\mathbf{x}, B)$ is given by Figure 2. We refer the reader to Section 4 for an extended discussion of the properties of such an algebra.

EXAMPLE 2 ([13]). Let $\mathbf{x} = (x_1, x_2, x_3)$ and

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is called *Markov's*; it is associated to an ideal triangulation of the hyperbolic torus with a cusp, see e.g. [17]. The Bratteli diagram $T_3$ mod $\ell$ of the cluster $C^*$-algebra $\mathbb{A}(\mathbf{x}, B)$ is shown in Figure 1. (The corresponding mutation tree $T_3$ and the equivalence classes of relation $\ell$ are given in full detail in [13, Figure 4].) The algebra $\mathbb{A}(\mathbf{x}, B)$ has a non-trivial primitive spectrum being isomorphic to an $AF$-algebra $\mathfrak{M}_1$ introduced by Mundici [10]; for a general theory we refer the reader to the monograph [11].

**2.3. Mundici dimension groups.** A broad class of dimension groups has been introduced by Mundici [10]. We shall use such groups as the main technical tool in proof of Theorem 1. We refer the reader to [10] and [11] for a detailed account.

A *lattice-ordered* (or $\ell$-group) is a structure $(G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and $x + (y \vee z) = (x + y) \vee (x + z)$ for all $x, y, z \in G$. An *order unit* in a partially ordered group $G$ is an element $u \geq 0$ such that for each $x \in G$ there is an integer $n \geq 0$ with $x < nu$. A *unital $\ell$-group* is an $\ell$-group with distinguished order unit.

The function $f : [0, 1]^n \to \mathbb{R}$ is called a *McNaughton function* over $[0, 1]^n$ iff $f$ is continuous and there are a finite number of linear functions:

$$\begin{cases} \alpha_1 = b_1 + a_{11}x_1 + \ldots + a_{1n}x_n \\ \alpha_2 = b_2 + a_{21}x_1 + \ldots + a_{2n}x_n \\ \vdots \\ \alpha_m = b_m + a_{m1}x_1 + \ldots + a_{mn}x_n, \end{cases} \tag{2}$$

where all $a_{ij}$ and $b_i$ are integers such that for every $(x_1, \ldots, x_n) \in [0, 1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x_1, \ldots, x_n) = \alpha_i(x_1, \ldots, x_n)$, see [10] and [11]. In other words, the McNaughton function is a piecewise linear function with integer coefficients. It is easy to see that the set of all McNaughton functions over $[0, 1]$ is an $\ell$-group with the pointwise operations $+, -, \vee, \wedge$ of $\mathbb{R}$ and with the constant function 1 as the distinguished order unit. The *Mundici dimension group* $\mathcal{M}_n$ is an $\ell$-group defined by the McNaughton functions over $[0, 1]^n$.

THEOREM 2 ([10], [11]).  $(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B)), u) \cong (\mathcal{M}_n, 1)$, *where* $\mathcal{M}_n$ *is defined by a subset of all McNaughton functions over* $[0, 1]^n$.

REMARK 2.  Theorem 2 for $n = 1$ was proved in [10]. In particular, the Markov cluster $C^*$-algebra $\mathbb{A}(\mathbf{x}, B)$ in Figure 1 has the dimension group $\mathcal{M}_1$. By an extension of the argument of [Mundici 2011] [11], one can prove Theorem 2 for $n \geq 1$.

## 3. Proofs

*Proof of Theorem 1.*  We shall split the proof into a series of lemmas.

LEMMA 1.  *The ordered abelian group* $(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B))$ *is a dimension group with the stable scale* $\Gamma \cong \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$.

*Proof.*  Recall that an ordered abelian group $(G, G^+)$ satisfies the *Riesz interpolation property*, if given $\{a_i, b_j \in G \mid a_i \leq b_j \text{ for } i, j = 1, 2\}$ there exists $c \in G$ such that

$$a_i \leq c \leq b_j. \tag{3}$$

Let us show that the ordered group $(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B))$ satisfies the Riesz interpolation property. Indeed, if $a_i = \sum A_i \mathbf{x}^i$ and $b_j = \sum B_i \mathbf{x}^i$ are the Laurent polynomials of $a_i, b_j \in \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$, then one can choose $c = \sum C_i \mathbf{x}^i$ such that $C_i = A_i$ if $A_i \neq 0$ and $0 < C_i < B_i$ if $A_i = 0$. Clearly, the condition (3) is satisfied.

By the Effros–Handelman–Shen Theorem, a countable ordered abelian group is a dimension group if and only if it satisfies the Riesz interpolation property, see [2, Theorem 3.1]. Thus $(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B))$ is a dimension group with the stable scale. Lemma 1 is proved. ∎

LEMMA 2.  *The exists a canonical isomorphism* $\varphi$ *between the abelian groups* $K_0(\mathbb{A}(\mathbf{x}, B))$ *and* $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$.

*Proof.*  The idea is to construct an isomorphism $\varphi : \mathcal{M}_n \to \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$, where $\mathcal{M}_n$ is the Mundici dimension group. The rest of the proof will follow from Theorem 2.

We assume without loss of generality that the linear functions $\alpha_i$ in the set (2) constitute the Schauder-type basis for $[0, 1]^n$.[1] We shall assign to each $\alpha_i = b_i + a_{i1}x_1 + \ldots + a_{in}x_n$ a Laurent monomial, which is a generator of the group $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$. Roughly speaking, this can be done by an "exponentiation" of the variables $x_i$.

Indeed, consider a map $\varphi$ acting by the formula:

$$b_i + a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \mapsto b_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_n^{a_{in}}, \quad 1 \leq i \leq m, \tag{4}$$

where $a_{ij} \in \mathbf{Z}$ and $b_i \in \mathbf{Z}$.

The map $\varphi$ sends the McNaughton function $f : [0, 1]^n \to \mathbb{R}$ to a Laurent polynomial according to the formula

$$f \mapsto \sum_{i=1}^m b_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_n^{a_{in}} \in \mathbf{Z}[\mathbf{x}^{\pm 1}].$$

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[1] Such a choice of $\alpha_i$ provides injectivity of our construction. I am grateful to the referee for pointing out this fact to me.

It is verified directly that pointwise addition of the McNaughton functions maps to an addition of the Laurent polynomials. One can see that our construction provides an injective mapping into the ring of Laurent polynomials. Working backwards our construction, it can be proved that the mapping is also surjective. Thus one gets an isomorphism of the abelian groups:

$$\varphi : \mathcal{M}_n \to \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B).$$

On the other hand, it follows from Theorem 2 that $\mathcal{M}_n \cong K_0(\mathbb{A}(\mathbf{x}, B))$, where the set of the McNaughton functions over $[0,1]^n$ is defined by the algebra $\mathbb{A}(\mathbf{x}, B)$. Thus one obtains an isomorphism of the abelian groups:

$$\varphi : K_0(\mathbb{A}(\mathbf{x}, B)) \to \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B).$$

Lemma 2 is proved. ∎

REMARK 3. Using the McNaughton functions over $[0,1]^n$, one can see that for the finite cluster algebras the group $K_0(\mathbb{A}(\mathbf{x}, B))$ is isomorphic to a direct sum of finitely many copies of $\mathbf{Z}$. This fact implies that the corresponding tracial simplex is spanned by $n$ extremal traces.

REMARK 4. Lemma 2 implies that the group $K_0(\mathbb{A}(\mathbf{x}, B))$ has the natural structure of a commutative ring, since $K_0(\mathbb{A}(\mathbf{x}, B)) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$. It is an interesting question to find an interpretation of the product in terms of the $K$-theory.

LEMMA 3. *The isomorphism $\varphi$ is order-preserving, i.e.*

$$\big(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B))\big) \cong \big(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)\big).$$

*Proof.* In view of Theorem 2, it is sufficient to show that

$$(\mathcal{M}_n, 1) \cong \big(\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B), \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)\big)$$

are isomorphic dimension groups.

The semi-group $\mathcal{M}_n^+$ of positive elements of the Mundici dimension group $(\mathcal{M}_n, 1)$ consists of all piecewise linear functions with $b_i > 0$. Likewise, the semigroup $\mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$ consists of the Laurent polynomials with $b_i > 0$.

On the other hand, formula (4) says that $\varphi$ sends the coefficient $b_i$ into the coefficient $b_i$ of the Laurent monomial. Thus one gets the equality

$$\varphi(\mathcal{M}_n^+) = \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B). \tag{5}$$

In other words, the isomorphism $\varphi : \mathcal{M}_m \to \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$ preserves the semi-group of positive elements of the respective dimension groups.

Lemma 3 follows from (5) and Theorem 2. ∎

Theorem 1 follows from Lemma 3. ∎

REMARK 5 ([13]). Theorem 1 implies that the category of cluster algebras can be embedded into the category of dimension groups $(G, G^+)$ with the stable scale. The following (partial) characterization of cluster algebras in terms of the dimension groups is true: The cluster algebras correspond to the dimension groups with a non-trivial spectrum $\mathrm{Prim}(G, G^+) \cong \{\mathbf{R}^n \,|\, n \geq 1\}$, where $\mathrm{Prim}(G, G^+)$ is the space of primitive ideals of $(G, G^+)$ endowed with the Jacobson topology.

*Proof of Corollary 1.* Let $\psi$ be an inverse of the map $\varphi$ constructed in Lemma 2. We shall fix an isomorphism class of the $AF$-algebra $\mathbb{A}(\mathbf{x}, B)$ and consider the corresponding dimension group $\left(K_0(\mathbb{A}(\mathbf{x}, B)), K_0^+(\mathbb{A}(\mathbf{x}, B)), \Gamma\right)$. In view of Theorem 1, we have

$$\begin{cases} \mathcal{A}_{\mathrm{add}}(\mathbf{x}, B) = \psi(K_0(\mathbb{A}(\mathbf{x}, B))) \\ \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B) = \psi(K_0^+(\mathbb{A}(\mathbf{x}, B))). \end{cases}$$

Since $\Gamma \subseteq K_0^+(\mathbb{A}(\mathbf{x}, B))$, one gets a scale $\psi(\Gamma) \subseteq \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$ in the cluster algebra $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$.

On the other hand, it is verified directly that the set $X = \{x_i\}_{i=1}^{\infty}$ of all cluster variables $x_i$ is a scale, since it is a generating, hereditary and directed subset of $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$. But given isomorphism class of algebra $\mathbb{A}(\mathbf{x}, B)$ can define only one scale on the cluster algebra $\mathcal{A}_{\mathrm{add}}(\mathbf{x}, B)$; thus $X \cong \psi(\Gamma)$. It remains to recall that $\psi(\Gamma) \subseteq \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$ and therefore $X \subseteq \mathcal{A}_{\mathrm{add}}^+(\mathbf{x}, B)$. In other words, the coefficients of the Laurent polynomials corresponding to the cluster variables $x_i$ are non-negative integers. Corollary 1 is proved. ∎

**4. Applications.** We shall consider some applications of Theorem 1. They include a new shorter proof of the Jones Index Theorem, calculation of the dimension group of the GICAR algebra and solution of the Jones Problem about a relation between the Hecke groups and the Jones Index Theorem. In what follows, we restrict to the cluster $C^*$-algebra $\mathbb{A}(1, 1)$ associated to a triangulation of an annulus with one marked point on each boundary component keeping the original notation of [4, Example 4.4].

**4.1. Cluster $C^*$-algebra $\mathbb{A}(1, 1)$.** Let $\mathfrak{A} = \{z = x + iy \in \mathbf{C} \,|\, r \leq |z| \leq R\}$ be an annulus in the complex plane such that $r < R$. Recall that the Riemann surfaces $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic if and only if $R/r = R'/r'$; the real number $t = R/r$ is called a *modulus* of $\mathfrak{A}$. By $T_{\mathfrak{A}} = \{t \in \mathbf{R} \,|\, t > 1\}$ we understand the Teichmüller space of $\mathfrak{A}$, i.e. the space of all Riemann surfaces $\mathfrak{A}$ endowed with a natural topology. The cluster algebra $\mathcal{A}(\mathbf{x}, B_T)$ of rank two given by the matrix

$$B_T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

is the coordinate ring of $T_{\mathfrak{A}}$ [4, Example 4.4]; the $\mathcal{A}(\mathbf{x}, B_T)$ is related to the Penner coordinates on the space $T_{\mathfrak{A}}$ corresponding to an ideal triangulation $T$ of $\mathfrak{A}$ with one marked point on each boundary component of $\mathfrak{A}$ [17, Section 3].

By $\mathbb{A}(1, 1) := \mathbb{A}(\mathbf{x}, B_T)$ we shall understand a cluster $C^*$-algebra given by matrix $B_T$; the reader is encouraged to verify using formulas (1) that the Bratteli diagram of $\mathbb{A}(1, 1)$ has the form of a Pascal triangle shown in Figure 2. (The $\mathbb{A}(1, 1)$ is the so-called *GICAR algebra* [2, p. 13(e)]; such an algebra has a rich set of ideals [1, Section 5.5].)

By $\{\sigma_t : \mathbb{A}(1, 1) \to \mathbb{A}(1, 1) \,|\, t \in \mathbf{R}\}$ we denote a group of modular automorphisms constructed in [13, Section 4]; the $\sigma_t$ is generated by the geodesic flow $T^t$ on the space $T_{\mathfrak{A}}$.

The $\mathbb{A}(1, 1)$ embeds into an UHF-algebra:

$$M_{2^\infty} := \bigotimes_{i=1}^{\infty} M_2(\mathbf{C}).$$

The $M_{2^\infty}$ is known as a *CAR algebra*; unlike the $\mathbb{A}(1,1)$, it is a simple *AF*-algebra with the Bratteli diagram shown in [2, p. 13(c1)]. The Powers product

$$\left\{ \bigotimes_{i=1}^{\infty} \exp\left( \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right) \mid 0 < \lambda < 1 \right\}$$

defines a group of modular automorphisms $\{\sigma^t : M_{2^\infty} \to M_{2^\infty} \mid t \in \mathbf{R}\}$; it is not hard to observe that $\sigma^t \equiv \sigma_t$ on $\mathbb{A}(1,1)$.

Recall that if $e_{ij}$ are the matrix units in $M_2(\mathbf{C})$, one can define a projection $e \in M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ by the formula

$$e = \frac{1}{1+t}(e_{11} \otimes e_{11} + t e_{22} \otimes e_{22} + \sqrt{t}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12})),$$

where $t \in \mathbf{R}$ is a parameter. If $\theta$ is the shift automorphism of the $UHF$-algebra $M_{2^\infty}$, then projections $e_i := \theta^i(e) \in M_{2^i}$ satisfy the following relations:

$$\begin{cases} e_i e_j = e_j e_i, & \text{if } |i-j| \geq 2 \\ e_i e_{i \pm 1} e_i = t(1+t)^{-2} e_i, \end{cases} \tag{6}$$

and the Powers state $\varphi_t : M_{2^\infty} \to \mathbf{C}$ satisfies the Jones equality:

$$\varphi_t(w e_{n+1}) = \frac{t}{(1+t)^2} \varphi_t(w), \quad \forall w \in M_{2^{n+1}},$$

see [8, Section 5.6] for the details. The $e_i$ generate the algebra $M_{2^\infty}$ and taking new generators $s_i$ such that $\sigma^t(s_i) = t e_i - (1 - e_i)$ one gets a representation of the braid group $B_n = \{s_1, \ldots, s_n \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \; s_i s_j = s_j s_i \text{ if } |i-j| \geq 2\}$ in the algebra $M_{2^n}$.

**4.2. Jones Index Theorem.** As an application of Theorem 1, one gets a short proof of the Jones Index Theorem in terms of the cluster algebras.

COROLLARY 2. *Relations (6) define a $C^*$-algebra if and only if the values of index $\frac{(1+t)^2}{t}$ belong to the set*

$$[4, \infty) \cup \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3 \right\}. \tag{7}$$

*Proof.* To find admissible values of parameter $t$, we shall use a simple analysis of the cluster algebra $\mathcal{A}(1,1) \cong K_0(\mathbb{A}(1,1))$. Recall that algebra $\mathcal{A}(1,1)$ has a unique canonical basis $\mathcal{B}$ consisting of the positive elements of $\mathcal{A}(1,1)$, i.e. the Laurent polynomials with positive integer coefficients; the elements of $\mathcal{B}$ generate the whole algebra $\mathcal{A}(1,1)$. Sherman and Zelevinsky [16, Theorem 2.8] gave an explicit construction of $\mathcal{B}$. Namely,

$$\mathcal{B} = \left\{ x_i^p x_{i+1}^q \mid p, q \geq 0 \right\} \cup \left\{ T_n(x_1 x_4 - x_2 x_3) \mid n \geq 3 \right\}, \tag{8}$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind. Since

$$T_0 = 1 \quad \text{and} \quad T_n\left[ \frac{1}{2}(t + t^{-1}) \right] = \frac{1}{2}(t^n + t^{-n}), \tag{9}$$

we shall look for a modulus $t$ such that $\frac{1}{2}(t + t^{-1}) = x_1 x_4 - x_2 x_3$. This is always possible since the Penner coordinates on the Teichmüller space $T_{\mathfrak{A}}$ are given by the cluster $(x_1, x_2)$, where each $x_i$ is a function of modulus $t$ [17, Section 3].

(i) Since $t > 1$, it is easy to see by direct substitution that the values of index belong to the interval

$$(4, \infty). \tag{10}$$

(ii) To get discrete values, we shall assume that $\mathcal{A}(1,1)$ is a finite cluster algebra, i.e. the number of $x_i$ is finite. It is immediate that $|\mathcal{B}| < \infty$ and from the second series in (8) one obtains

$$T_n(x_1 x_4 - x_2 x_3) = T_0 = 1$$

for some integer $n \geq 1$. But $x_1 x_4 - x_2 x_3 = \frac{1}{2}(t + t^{-1})$ and using formula (9) for the Chebyshev polynomials, one gets the equation

$$t^n + t^{-n} = 2 \tag{11}$$

for (possibly complex) values of modulus $t$. Since (11) is equivalent to the equation $t^{2n} - 2t^n + 1 = (t^n - 1)^2 = 0$, one gets the $n$-th root of unity:

$$t \in \{ e^{2\pi i/n} \,|\, n \geq 1 \}.$$

However, the index

$$\frac{(1+t)^2}{t} = t^{-1} + 2 + t = 2 \left[ \cos\left(\frac{2\pi}{n}\right) + 1 \right] = 4\cos^2\left(\frac{\pi}{n}\right) \tag{12}$$

is a real number. Thus relations (6) define a $C^*$-algebra. (We must exclude the case $n = 2$ corresponding to $t = -1$, because otherwise one gets a division by zero in (6).)

Bringing together (10) and (12) one gets the conclusion of Corollary 2. $\blacksquare$

REMARK 6. The finite cluster algebras corresponding to the discrete moduli come from a triangulation of the $n$-gons or the $n$-gons with one puncture, see [4, Table 1]; such algebras are classified by their Coxeter–Dynkin diagrams of type $A_{n-3}$ and $D_n$, respectively. As explained, the $\mathbb{A}(1,1)$ is a finite-dimensional $C^*$-algebra having the Bratteli diagram similar to one shown in [8, pp. 37–38].

**4.3. Dimension group of the GICAR algebra.** We shall use Theorem 1 to calculate a dimension group of the algebra $\mathbb{A}(1,1)$.

COROLLARY 3. $(K_0(\mathbb{A}(1,1)), K_0^+(\mathbb{A}(1,1)), u) \cong (\mathbf{Z}[x], P^+(0,1), u)$, *where* $P^+(0,1)$ *is the semigroup of all positive-definite polynomials on the interval* $(0,1)$.

*Proof.* It is known that the Chebyshev polynomials of the first kind $T_n(x)$ lie in a basis $\mathcal{B}$ of the cluster algebra $\mathcal{A}(1,1)$ [16, Theorem 2.8]. For each $0 \leq k \leq n$, we shall introduce a new basis $\mathcal{B}'$ in $\mathcal{A}(1,1)$ comprising the elements

$$T_1^k(x)(T_0(x) - T_1(x))^{n-k} = x^k (1-x)^{n-k}.$$

On the other hand, the Bratteli diagram in Figure 2 says that the group $K_0(\mathbb{A}(1,1))$ is generated by the (equivalence classes of) projections $[e_k^n]$ subject to the relations

$$[e_k^n] = [e_k^{n+1}] + [e_{k+1}^{n+1}]. \tag{13}$$

Take a representation $\rho$ of $K_0(\mathbb{A}(1,1))$ in the cluster algebra $\mathcal{A}(1,1)$ given by the formula

$$\rho([e_k^n]) = x^k(1-x)^{n-k}, \quad 0 \leq k \leq n.$$

The reader can verify that relations (13) are satisfied. It is easy to see that $x^k(1-x)^{n-k}$ are generators of the polynomial ring $\mathbf{Z}[x]$ and the rest of the proof repeats the argument in [14, Appendix]. Corollary 3 follows. ■

REMARK 7. Corollary 3 was first proved by Renault [14, Appendix]; the GICAR algebra involved in the original proof is isomorphic to the cluster $C^*$-algebra $\mathbb{A}(1,1)$, see Figure 2.

**4.4. Jones Problem.** The following problem can be found in [8, p. 24]:

*"Consider the subgroup $G_\lambda$ of $SL_2(\mathbf{R})$ generated by $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For what values of $\lambda > 0$ is it discrete? Answer: $\lambda = 2\cos(\frac{\pi}{n})$, $n = 3, 4, \ldots$ or $\lambda \geq 2$. (...) We have been unable to find any direct connection between this result and Theorem 3.1 (Jones Index Theorem). It is a tantalizing situation."*

We refer the reader to [6] for a detailed solution of the Jones Problem. To give an outline, consider the Schottky uniformization of the annulus $\mathfrak{A}$ given by the formula

$$\mathfrak{A} \cong \mathbf{C}P^1/A^{\mathbf{Z}},$$

where $\mathbf{C}P^1 := \mathbf{C} \cup \{\infty\}$ is the Riemann sphere and $A \in SL_2(\mathbf{C})/\pm I$ is a matrix acting on the $\mathbf{C}P^1$ by the Möbius transformations. It follows from the results of Section 4.2 that the index of subfactors in a von Neumann algebra coincides with the square of trace of matrix $A$, i.e.

$$\mathrm{tr}^2(A) \in [4, \infty) \cup \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3 \right\}.$$

The idea is to prove that the $\mathfrak{A}$ is a ramified double cover of the orbifold $\mathbb{H}/G_\lambda$, where $\mathbb{H} := \{x + iy \in \mathbf{C} \mid y > 0\}$ is the Lobachevsky half-plane and the group $G_\lambda$ acts on $\mathbb{H}$ by the linear fractional transformations. Since such a cover takes the square root of the modulus parameter $\mathrm{tr}^2(A)$ of $\mathfrak{A}$, we conclude that $\lambda = \mathrm{tr}(A)$. In other words, the Jones Index Theorem (7) is equivalent to the following result.

COROLLARY 4 ([7, Satz 1, 2 & 6]). *The $G_\lambda$ is a discrete subgroup of $SL_2(\mathbf{R})$ if and only if*

$$\lambda \in [2, \infty) \cup \left\{ 2\cos\left(\frac{\pi}{n}\right) \mid n \geq 3 \right\}.$$

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