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SPECTRA AS UNIVERSAL OBJECTS IN CATEGORIES OF SUPPORTS

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Abstract. Many years ago, André Joyal outlined a method of describing the Zariski spectrum $Spec(R)$ of a commutative ring R in a manner that makes no reference to prime ideals of R. In Joyal's approach, the spectrum is not a topological space, but a distributive lattice that satisfies a certain universal property. Recently, this approach has been shown to be very fruitful in understanding other spectra, such as the spectrum of a tensor triangulated category. In this paper, we take a similar method to describe as universal objects several other 'spectrum like spaces' that arise in commutative algebra and noncommutative algebraic geometry.

1. Introduction. The Zariski spectrum $Spec(R)$ of a commutative ring R is a very familiar object in commutative algebra and algebraic geometry. In a brief 1975 paper, André Joyal [Jo] gave a very different view of $Spec(R)$, by describing it in a manner that makes no reference to prime ideals of the ring. In Joyal's approach, the spectrum is not a topological space but a distributive lattice, or more precisely, a 'frame' (see [J, §II]). The spectrum $Spec(R)$, along with the association $f \mapsto \sqrt{(f)}$ for each $f \in R$ (here $\sqrt{(f)}$ denotes the radical of the principal ideal $(f) \subseteq R$) becomes a universal object; it is initial in the category of supports on R. A support (F, δ) on R consists of a frame F and a map

$$\delta : R \longrightarrow F$$

satisfying the conditions:

(1) $d(1) = 1$, $d(0) = 0$,
(2) $d(ff') = d(f) \wedge d(f')$,
(3) $d(f + f') \leq d(f) \vee d(f')$ for any f, $f' \in R$.

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Therefore, the spectrum $Spec(R)$ is identified with the lattice of radical ideals in the ring R. This description goes well with the larger idea of "point-free topology," where topological spaces are studied by replacing them with the frame of their open sets. For a full view of this subject, we refer the reader to the book of Johnstone [J].

More recently, Kock and Pitsch [KP] showed that a similar frame-theoretic approach may be used to describe the spectrum $Spec(\mathbb{T})$ of a tensor triangulated category $(\mathbb{T}, \otimes, 1)$ constructed by Balmer [Bal]. The spectrum $Spec(\mathbb{T})$ constructed by Paul Balmer in [Bal] brings together a dazzling variety of support theories from various branches of mathematics: from the work of Devinatz, Hopkins and Smith [DHS] in homotopy theory to Thomason's classification of thick subcategories in [T] and to the study of support varieties in modular representation theory by Benson, Carlson and Rickard [BCR].

In this paper, we take the frame-theoretic approach to describe as universal objects several other 'spectrum like spaces' that arise in commutative algebra and noncommutative algebraic geometry. We also build on the lessons learned from our previous work in [Ban1], [Ban2], [Ban3], [Ban4], [Ban5], both from the study of thick tensor ideals in tensor triangulated categories and from the study of spectral spaces. More precisely, we do the following:

(a) The space $SMod(M)$ of submodules of a given module M over a commutative ring R has recently been developed by Finocchiaro, Fontana and Spirito [FFS] in course of their work on a topological version of Hilbert's Nullstellensatz. In fact, the authors in [FFS] show that $SMod(M)$ is a 'spectral space' in the sense of Hochster [Ho]. We construct support data for submodules of M: such a support datum $\Psi = (F, \delta)$ consists of a frame F and a map $\delta : M \longrightarrow F$ satisfying certain conditions. We recover the space $SMod(M)$ constructed in [FFS] as the universal object in the category of support data for submodules of M.

(b) We extend Joyal's notion [Jo] of support to the case of a commutative graded ring R in order to describe the space $Proj(R)$. Indeed, we show that the frame of open subsets of $Proj(R)$ can be obtained as a universal support datum for the graded ring R as well as the frame of homogeneous radical ideals in R.

Thereafter, we consider more generally, the collection of 'homogeneous Ψ-closed ideals' (see Definition 3.7) in place of 'homogeneous radical ideals,' where $\Psi = (F, \delta)$ is a support datum on R. Under certain conditions, we show that the homogeneous Ψ-closed ideals form a frame $hRad^{\Psi}(R)$, which is further isomorphic to a frame obtained from a certain universal factorization of a morphism of supports. When equipped with the 'lower interval topology' (see, for instance, [J, §II.1.8]), the frame $hRad^{\Psi}(R)$ gives a spectral space. In the particular case of the initial support $Proj(R)$, this yields a projective version of the topological Nullstellensatz developed by Finocchiaro, Fontana and Spirito [FFS].

(c) For an abelian category $\mathcal{A}$, its spectrum $\mathfrak{Spec}(\mathcal{A})$ was developed by A. Rosenberg in [R1], [R2], [R3] with a view towards building noncommutative algebraic geometry. When applied to the category of modules over a (not necessarily commutative) ring, this gives a highly versatile theory of 'noncommutative local algebra' (see [R1]). Since this noncommutative local algebra is based on the category of modules instead of the ring itself, there is Morita invariance built into the theory. The spectrum construction of

Rosenberg also allows us to recover a scheme from its category of quasi-coherent sheaves, suggesting that this is a possible way of studying 'noncommutative schemes.'

We take a support datum Ψ on a locally noetherian Grothendieck category $\mathcal{A}$ to consist of a frame F and a certain kind of map

$$\phi : \mathcal{A}_{fg} \longrightarrow F$$

where $\mathcal{A}_{fg} \subseteq \mathcal{A}$ is the full subcategory of finitely generated objects of $\mathcal{A}$. Since $\mathcal{A}$ is locally noetherian, $\mathcal{A}_{fg}$ is in fact an abelian category. Finally, we show that the spectrum $\mathfrak{Spec}(\mathcal{A})$ as constructed by Rosenberg is weakly initial in the category of support data on $\mathcal{A}$.

2. Frames of submodules. We recall (see, for instance, [J, Chapter I]) that a lattice is a partially ordered set $(L, \leq)$ such that every finite subset $S \subseteq L$ has a least upper bound (known as the join $\bigvee_{s\in S} s$) and a greatest lower bound (known as the meet $\bigwedge_{s\in S} s$). In particular, any lattice L has a least element that we denote by 0 and a top element that we will denote by 1. A frame F is a complete lattice such that finite meets distribute over arbitrary joins, i.e.,

$$a \wedge \Big(\bigvee_{s\in S} s\Big) = \Big(\bigvee_{s\in S} (a \wedge s)\Big)$$

for any $a \in F$ and any (not necessarily finite) subset $S \subseteq F$. A morphism of frames is a function that preserves finite meets and arbitrary joins. If X is any topological space, the lattice ΩX of its open sets always forms a frame. For more on the study of topological spaces by means of frames, we refer the reader to [J].

Throughout this section, we let R be a commutative ring and let M be an R-module. Let $SMod(M)$ be the set of all submodules of M. In [FFS], it was shown that $SMod(M)$ is actually a spectral space in the sense of Hochster [Ho], with a subbasis of open sets given by the collection

$$D(m) := \{N \in SMod(M) \mid m \notin N\}, \qquad m \in M.$$

For any finite subset $S \subseteq M$, we set $D(S) := \bigcap_{m\in S} D(m)$. For the sake of convenience, we also set $V(S) := SMod(M) \setminus D(S)$ for any finite subset $S \subseteq M$.

LEMMA 2.1. *For any finite subset $S = \{m_1, \ldots, m_k\} \subseteq M$, the open set $D(S) \subseteq SMod(M)$ is quasi-compact.*

Proof. We consider a covering $D(S) \subseteq \bigcup_{i\in I} D(n_i)$ of $D(S)$ by means of subbasic open sets $\{D(n_i)\}_{i\in I}$ with each $n_i \in M$. Taking complements, we now obtain

$$V(S) = V(m_1) \cup \ldots \cup V(m_k) \supseteq \bigcap_{i\in I} V(n_i).$$

Let N be the submodule generated by the collection $\{n_i\}_{i\in I}$. Clearly, $\{N\} \in V(n_i)$ for each $i \in I$. For the sake of definiteness, suppose that $\{N\} \in V(m_1)$, i.e., $m_1 \in N$. Then, there exists a finite subset $J \subseteq I$ such that m_1 lies in the submodule generated by $\{n_j\}_{j\in J}$. It follows that $D(m_1) \subseteq \bigcup_{j\in J} D(n_j)$. In particular, this gives $D(S) = D(m_1) \cap \ldots \cap D(m_k) \subseteq D(m_1) \subseteq \bigcup_{j\in J} D(n_j)$. By the Alexander subbase theorem (see, for instance, [S, Tag 08ZP]), it follows that $D(S)$ is quasi-compact. ∎

We consider the frame $\Omega SMod(M)$ of open sets of $SMod(M)$. Every element $m \in M$ is associated to an open set $D(m) \in \Omega SMod(M)$. More generally, we now introduce the notion of a support datum for submodules of M.

Definition 2.2. Let R be a commutative ring and let M be an R-module. A *support datum* $\Psi = (F, \delta)$ for submodules of M consists of a frame F and a map

$$\delta : M \longrightarrow F$$

satisfying the following conditions:

(1) $\delta(0) = 0_F$, where 0_F denotes the least element of the frame F,
(2) for any $m_1, m_2 \in M$, we have $\delta(m_1 + m_2) \leq \delta(m_1) \vee \delta(m_2)$,
(3) for any $a \in R$ and $m \in M$, we have $\delta(am) \leq \delta(m)$.

A morphism $f : \Psi \longrightarrow \Psi'$ of support data from $\Psi = (F, \delta)$ to $\Psi' = (F', \delta')$ consists of a morphism $f : F \longrightarrow F'$ of frames such that $f \circ \delta = \delta'$.

We notice that if $N \subseteq M$ is a submodule generated by a set $\{n_i\}_{i \in I}$, conditions (2) and (3) in Definition 2.2 together imply that

$$\delta(n) \leq \bigvee_{i \in I} \delta(n_i) \tag{2.1}$$

for any element $n \in N$.

It is evident that the pair $\Psi_0 = (\Omega SMod(M), D)$ is a support datum for submodules of M. Given an arbitrary support datum $\Psi = (F, \delta)$, we proceed to define $f_\Psi : \Omega SMod(M) \longrightarrow F$ by setting

$$f_\Psi(D(\{m_1, \ldots, m_k\})) = \delta(m_1) \wedge \ldots \wedge \delta(m_k) \tag{2.2}$$

for each finite subset $S = \{m_1, \ldots, m_k\} \subseteq M$. As S varies over all finite subsets of M, the $D(S)$ form a basis for the topology on $SMod(M)$. As such, if $U \in \Omega SMod(M)$ is an open set with $U = \bigcup_{i \in I} D(S_i)$, we set

$$f_\Psi(U) := \bigvee_{i \in I} f_\Psi(D(S_i)). \tag{2.3}$$

Lemma 2.3. *The association $f_\Psi : \Omega SMod(M) \longrightarrow F$ is well-defined.*

Proof. Let $S = \{m_1, \ldots, m_k\} \subseteq M$ be a finite set and let $\{S_i\}_{i \in I}$ be a collection of finite subsets of M such that $D(S) \subseteq \bigcup_{i \in I} D(S_i)$. It suffices to check that

$$\delta(m_1) \wedge \ldots \wedge \delta(m_k) \leq \bigvee_{i \in I} \Big(\bigwedge_{n_{ij} \in S_i} \delta(n_{ij}) \Big).$$

From Lemma 2.1, we know that $D(S)$ is quasi-compact and so we can choose a finite subset $I' \subseteq I$ such that $D(S) \subseteq \bigcup_{i \in I'} D(S_i)$. Taking complements, we obtain:

$$V(S) = V(m_1) \cup \ldots \cup V(m_k) \supseteq \bigcap_{i \in I'} \Big(\bigcup_{n_{ij} \in S_i} V(n_{ij}) \Big) = \bigcup_{\substack{T \subseteq \coprod_{i \in I'} S_i \\ |T \cap S_i| = 1 \ \forall i \in I'}} \Big(\bigcap_{n \in T} V(n) \Big). \tag{2.4}$$

For each $T \subseteq \coprod_{i \in I'} S_i$ such that $T \cap S_i$ is a singleton for all $i \in I'$, we let $N_T \subseteq M$ be the submodule generated by elements $n \in T$. Then, we can choose some integer $1 \leq k_T \leq k$

such that $m_{k_T} \in N_T$. From (2.1), it follows that $\delta(m_{k_T}) \leq \bigvee_{n\in T} \delta(n)$. We now obtain

$$\begin{aligned}\delta(m_1)\wedge\ldots\wedge\delta(m_k) &\leq \bigwedge_{\substack{T\subseteq\coprod_{i\in I'} S_i\\ |T\cap S_i|=1\ \forall i\in I'}} \delta(m_{k_T})\\ &\leq \bigwedge_{\substack{T\subseteq\coprod_{i\in I'} S_i\\ |T\cap S_i|=1\ \forall i\in I'}} \Big(\bigvee_{n\in T}\delta(n)\Big) = \bigvee_{i\in I'}\Big(\bigwedge_{n_{ij}\in S_i}\delta(n_{ij})\Big) \leq \bigvee_{i\in I}\Big(\bigwedge_{n_{ij}\in S_i}\delta(n_{ij})\Big).\end{aligned}$$

This proves the result. ∎

THEOREM 2.4. *Let M be a module over a commutative ring R. Then, the datum $\Psi_0 = (\Omega SMod(M), D)$ is an initial object in the category of support data for submodules of M.*

Proof. Given a support datum $\Psi = (F, \delta)$ for submodules of M, we will show that f_Ψ as defined in (2.2) and (2.3) gives a morphism of support data $f_\Psi : \Psi_0 \longrightarrow \Psi$. From Lemma 2.3, we know that $f_\Psi : \Omega SMod(M) \longrightarrow F$ is well-defined at the level of sets.

Further, we know from (2.3) that if any open $U \in \Omega SMod(M)$ can be expressed as a union $U = \bigcup_{i\in I} D(S_i)$ of some collection of basis elements, we have

$$f_\Psi(U) := \bigvee_{i\in I} f_\Psi(D(S_i)).$$

From this, it is clear that $f_\Psi(\bigcup_{k\in K} U_k) = \bigvee_{k\in K} f_\Psi(U_k)$ for any collection $\{U_k\}_{k\in K}$ of elements from $\Omega SMod(M)$, i.e., f_Ψ preserves arbitrary joins. We now consider elements $U, V \in \Omega SMod(M)$ and their intersection $U \cap V$, i.e., their meet in $SMod(M)$. If we write $U = \bigcup_{i\in I} D(S_i)$ and $V = \bigcup_{j\in J} D(T_j)$ as unions of basis elements, we get:

$$\begin{aligned} f_\Psi(U\cap V) &= f_\Psi\Big(\Big(\bigcup_{i\in I} D(S_i)\Big)\cap\Big(\bigcup_{j\in J} D(T_j)\Big)\Big) = f_\Psi\Big(\bigcup_{(i,j)\in I\times J} D(S_i)\cap D(T_j)\Big)\\ &= f_\Psi\Big(\bigcup_{(i,j)\in I\times J} D(S_i\cup T_j)\Big) = \bigvee_{(i,j)\in I\times J} f_\Psi(D(S_i\cup T_j))\\ &= \bigvee_{(i,j)\in I\times J} \big(f_\Psi(D(S_i))\wedge f_\Psi(D(T_j))\big)\\ &= \Big(\bigvee_{i\in I} f_\Psi(D(S_i))\Big)\wedge\Big(\bigvee_{j\in J} f_\Psi(D(T_j))\Big) = f_\Psi(U)\wedge f_\Psi(V).\end{aligned}$$

It remains to show that $f_\Psi : \Psi_0 \longrightarrow \Psi = (F, \delta)$ is unique. We know from (2.2) and (2.3) that f_Ψ is defined by setting $f_\Psi(D(m)) = \delta(m)$ for each $m \in M$ and extending by means of finite meets and arbitrary joins. This proves the result. ∎

3. Support data and *Proj* of a graded ring. Throughout this section, we let $R = \bigoplus_{i\geq 0} R_i$ be a commutative graded ring. We recall that an ideal $I \subseteq R$ is said to be homogeneous if it is generated by homogeneous elements. In other words, if $f \in I$ can be expressed as a sum $f = f_0 + \ldots + f_n$ with each $f_i \in R_i$, we have $f_i \in I$ for every $0 \leq i \leq n$. Suppose that a homogeneous ideal $\mathfrak{p} \subseteq R$ is such that given $ab \in \mathfrak{p}$ with a, b homogeneous, we have either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Then, $\mathfrak{p}$ is prime (see, for instance, [S, Tag 00JM]).

By definition, $Proj(R)$ consists of all homogeneous prime ideals $\mathfrak{p}$ of R such that $R_+ := \bigoplus_{i>0} R_i \not\subseteq \mathfrak{p}$. Since $Proj(R) \subseteq Spec(R)$, there is an induced subspace topology on $Proj(R)$. We let R_+^h denote the collection of all homogeneous elements in R of degree > 0. We know that a basis for $Proj(R)$ is given by the collection (see [S, Tag 00JP])

$$D_+(f) := \{\mathfrak{p} \in Proj(R) \mid f \notin \mathfrak{p}\}$$

as f varies over all the elements of R_+^h. It is also clear from the definition that $D_+(ff') = D_+(f) \cap D_+(f')$ for any f, $f' \in R_+^h$.

We now extend Joyal's notion (see [Jo]) of a support of a commutative ring to the graded case. We note that the following notion is somewhat implicit in the work of Coquand, Lombardi and Schuster [CLS1], [CLS2]. However, the authors in [CLS1] and [CLS2] stick to the case of a graded ring $R = \bigoplus_{i\geq 0} R_i$ generated as an R_0-algebra by finitely many homogeneous elements in degree 1.

Definition 3.1. Let $R = \bigoplus_{i\geq 0} R_i$ be a commutative graded ring and let R_+^h be the collection of its homogeneous elements of degree > 0. A *support datum* $\Psi = (F, \delta)$ for R consists of a frame F and a map

$$\delta : R_+^h \cup \{0\} \longrightarrow F$$

satisfying the following conditions:

(1) $\delta(0) = 0_F$, where 0_F is the bottom element of the frame F;
(2) $\delta(ff') = \delta(f) \wedge \delta(f')$ for any f, $f' \in R_+^h \cup \{0\}$;
(3) if f, $f' \in R_+^h \cup \{0\}$ are such that $f + f' \in R_+^h \cup \{0\}$, then $\delta(f + f') \leq \delta(f) \vee \delta(f')$;
(4) if 1_F is the top element of the frame F, then $\bigvee_{f\in R_+^h} \delta(f) = 1_F$.

A morphism $f : \Psi = (F, \delta) \longrightarrow (F', \delta')$ of support data consists of a morphism $f : F \longrightarrow F'$ of frames such that $f \circ \delta = \delta'$.

The collection $\Omega Proj(R)$ of open sets of $Proj(R)$ forms a frame. As such, it is evident that the association

$$\delta_0 : R_+^h \longrightarrow \Omega Proj(R) \quad f \mapsto D_+(f)$$

gives a support datum for R which we denote by $\Psi_0 = (\Omega Proj(R), \delta_0)$. For the sake of convenience, we set $\delta_0(0) = D_+(0)$ to be the empty set considered as an open subset of $Proj(R)$, i.e., an element of the frame $\Omega Proj(R)$. We also notice that if $f \in R_+^h$ is nilpotent, then condition (2) gives $\delta(f) = 0_F$.

Lemma 3.2. *Let I be a homogeneous ideal and let $\{f_j\}_{j\in J}$ be a set of homogeneous elements generating I. Suppose that $\Psi = (F, \delta)$ is a support datum for R. Then, any homogeneous $f \in I$ satisfies*

$$\delta(f) \leq \bigvee_{j\in J} \delta(f_j).$$

Proof. Suppose that $\deg(f) = k$. Since f lies in the ideal generated by $\{f_j\}_{j\in J}$, we can find a finite subcollection $f_1, \ldots, f_n$ of elements from $\{f_j\}_{j\in J}$ and some $a_1, \ldots, a_n \in R$ such that

$$f = a_1 f_1 + \ldots + a_n f_n.$$

For each $1 \leq l \leq n$, let $d_l := \deg(f_l)$ and $a_l^{(k-d_l)}$ be the component of a_l in degree $(k-d_l)$. Since f is homogeneous of degree k, we obtain

$$f = \sum_{l=1}^{n} a_l^{(k-d_l)} f_l.$$

From conditions (2) and (3) in Definition 3.1, we now get

$$\delta(f) \leq \bigvee_{l=1}^{n} \big(\delta(a_l^{(k-d_l)}) \wedge \delta(f_l)\big) \leq \bigvee_{l=1}^{n} \delta(f_l) \leq \bigvee_{j\in J} \delta(f_j).$$

This proves the result. ■

We also recall here the following well known facts on graded rings.

Lemma 3.3.

(a) *Let $\mathfrak{p} \subseteq R$ be a prime ideal and $\mathfrak{p}^h$ be the ideal generated by the homogeneous elements of $\mathfrak{p}$. Then, $\mathfrak{p}^h$ is a homogeneous prime ideal of R.*
(b) *Let $I \subseteq R$ be a homogeneous ideal. Then, the radical $\sqrt{I}$ of I equals the intersection of all homogeneous prime ideals containing I.*
(c) *Let $f \in R_+^h$. Then, $D_+(f) = \phi$ if and only if f is nilpotent.*

Proof. The result of (a) is given in [S, Tag 00JT]. For part (b), we notice that if I is a homogeneous ideal and $I \subseteq \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, we have $I \subseteq \mathfrak{p}^h$. From part (a), we see that $\mathfrak{p}^h$ is a homogeneous prime ideal. This proves (b).

For (c), it is clear that if $f \in R_+^h$ is nilpotent, then $D_+(f) = \phi$. Conversely, if $f \in R_+^h$ is not nilpotent, there is a prime ideal $\mathfrak{p} \subseteq R$ such that $f \notin \mathfrak{p}$. Then, $f \notin \mathfrak{p}^h \subseteq \mathfrak{p}$. We cannot have $R_+ \subseteq \mathfrak{p}^h$ because $f \in R_+^h \subseteq R_+$ and $f \notin \mathfrak{p}^h$. We conclude that $\mathfrak{p}^h \in Proj(R)$ and hence $\mathfrak{p}^h \in D_+(f)$, i.e., $D_+(f) \neq \phi$. ■

Proposition 3.4. *Let $\Psi = (F, \delta)$ be a support datum for $R = \bigoplus_{i\geq 0} R_i$. Let $f \in R_+^h \cup \{0\}$ and let $\{f_j\}_{i\in J}$ be a collection of elements from $R_+^h \cup \{0\}$ such that $D_+(f) \subseteq \bigcup_{j\in J} D_+(f_j)$. Then*

$$\delta(f) \leq \bigvee_{j\in J} \delta(f_j).$$

Proof. If some f_j equals 0, we know that $D_+(f_j) = \phi$ and hence we may exclude it from the covering of $D_+(f)$. Therefore, we suppose that every $f_j \in R_+^h$. Let I be the homogeneous ideal generated by the collection $\{f_j\}_{j\in J}$.

If $f = 0$, the result is obvious. Therefore, we assume that $f \in R_+^h$. Let $\mathfrak{p}$ be a homogeneous prime ideal containing I. Since $D_+(f) \subseteq \bigcup_{j\in J} D_+(f_j)$, it follows that $\mathfrak{p} \notin D_+(f)$. Then, either $f \in \mathfrak{p}$ or $R_+ \subseteq \mathfrak{p}$. But if $R_+ \subseteq \mathfrak{p}$, then $f \in R_+^h \subseteq R_+ \subseteq \mathfrak{p}$ and it still follows that $f \in \mathfrak{p}$. From Lemma 3.3, we now see that $f \in \sqrt{I}$, i.e., $f^n \in I$ for some $n \in \mathbb{N}$. Since f is homogeneous, so is f^n. From Lemma 3.2, it now follows that

$$\delta(f^n) \leq \bigvee_{j\in J} \delta(f_j).$$

Finally, from Definition 3.1, it is clear that $\delta(f^n) = \delta(f)$. This proves the result. ■

Theorem 3.5. *Let $\Psi = (F, \delta)$ be a support datum for a commutative graded ring $R = \bigoplus_{i\geq 0} R_i$. Then, there exists a unique morphism $f_\Psi : \Psi_0 = (\Omega Proj(R), \delta_0) \longrightarrow \Psi = (F, \delta)$.*

Proof. For any open set $U \subseteq Proj(R)$, i.e., any $U \in \Omega Proj(R)$, we set

$$f_\Psi(U) := \bigvee_{D_+(f)\subseteq U} \delta(f). \tag{3.1}$$

Then, Proposition 3.4 shows that (3.1) gives a well-defined map $f_\Psi : \Omega Proj(R) \longrightarrow F$ that is compatible with arbitrary joins. Due to Proposition 3.4, it also follows that $f_\Psi(D_+(f)) = \delta(f)$ for any $f \in R_+^h$. For $f, f' \in R_+^h$ we now see that

$$\begin{aligned} f_\Psi(D_+(f) \cap D_+(f')) &= f_\Psi(D_+(ff')) = \delta(ff') \\ &= \delta(f) \wedge \delta(f') = f_\Psi(D_+(f)) \wedge f_\Psi(D_+(f')). \end{aligned} \tag{3.2}$$

Since finite meets distribute over arbitrary joins in the frame F, it follows from (3.2) that $f_\Psi : \Omega Proj(R) \longrightarrow F$ preserves finite meets. The uniqueness of f_Ψ is clear. ∎

For the remainder of this section, we suppose that the homogeneous elements of degree zero in R form a field, i.e., R_0 is a field. This implies in particular that every proper homogeneous ideal $I \subsetneq R$ must be contained in R_+. Indeed, if $x \in I$ and x_0 is the component of x in degree 0, we must have $x_0 \in I$ (since I is homogeneous). But then $x_0 \in R_0$ is a unit unless $x_0 = 0$.

For any homogeneous ideal $I \subseteq R$, we now set

$$V_+(I) := \{\mathfrak{p} \in Proj(R) \mid I \subseteq \mathfrak{p}\}. \tag{3.3}$$

Further, we let $hRad(R)$ denote the collection of all proper and homogeneous radical ideals of R. Since R_0 is a field, R_+ is a radical ideal. We now make $hRad(R)$ into a frame as follows:

(1) For any I_1, $I_2 \in hRad(R)$, their meet is given by the intersection $I_1 \cap I_2$.
(2) For any family of ideals $\{I_k\}_{k\in K}$ in $hRad(R)$, their join is given by taking the radical of their sum, i.e., by taking the ideal $\sqrt{\sum_{k\in K} I_k}$. We notice that since each $I_k \subseteq R_+$, so is their join $\sqrt{\sum_{k\in K} I_k} \subseteq R_+$.

Proposition 3.6. *There is an isomorphism of frames between $hRad(R)$ and $\Omega Proj(R)$ given by the association*

$$I \mapsto Proj(R) \setminus V_+(I). \tag{3.4}$$

Proof. From the definition in (3.3), it is clear that $V_+(I_1 \cap I_2) = V_+(I_1) \cup V_+(I_2)$ for any $I_1, I_2 \in hRad(R)$. Also, if $\{I_k\}_{k\in K}$ is a family of ideals in $hRad(R)$, then

$$V_+\Big(\sqrt{\sum_{k\in K} I_k}\Big) = V_+\Big(\sum_{k\in K} I_k\Big) = \bigcap_{k\in K} V_+(I_k).$$

This shows that the association in (3.4) is a morphism of frames. Additionally, we know that every closed set in $Proj(R)$ is of the form $V_+(I)$ for some homogeneous radical ideal $I \subseteq R$ (see [S, Tag 00JP]). For the case of $I = R$, we see from the definition of $Proj(R)$ that $V_+(R) = V_+(R_+) = \phi$. Since $R_+ \in hRad(R)$, we see that the collection of all $\{V_+(I)\}$ as I varies over all elements of $hRad(R)$ exhausts all the closed sets in $Proj(R)$.

It remains to show that if $V_+(I_1) = V_+(I_2)$ for any $I_1, I_2 \in hRad(R)$, then $I_1 = I_2$. Since the proper homogeneous ideals of R are all contained in R_+, the only homogeneous prime ideals containing some $I \in hRad(R)$ are R_+ and the prime ideals in $V_+(I)$. From Lemma 3.3 it now follows that

$$I_1 = \sqrt{I_1} = R_+ \cap \Big(\bigcap_{\mathfrak{p} \in V_+(I_1)} \mathfrak{p} \Big) = R_+ \cap \Big(\bigcap_{\mathfrak{p} \in V_+(I_2)} \mathfrak{p} \Big) = \sqrt{I_2} = I_2.$$

This proves the result. ∎

The following definition is motivated by [Ban5, Definition 2.3] which referred to the case of thick tensor ideals in tensor triangulated categories.

Definition 3.7. Let $\Psi = (F, \delta)$ be a support datum for R. We will say that a homogeneous ideal $I \subseteq R$ is *Ψ-closed* if it satisfies:

$$\delta(f) \leq \bigvee_{f' \in R_+^h \cap I} \delta(f') \implies f \in I$$

for any homogeneous element $f \in R_+^h \cup \{0\}$. It is clear that any such homogeneous ideal is also radical. As such, the collection of all homogeneous Ψ-closed ideals will be denoted by $hRad^{\Psi}(R)$.

Given a support datum $\Psi = (F, \delta)$, we know from Theorem 3.5 that we have an induced morphism $f_\Psi : \Omega Proj(R) \longrightarrow F$. Combining with Proposition 3.6, we might as well say that we have a morphism of frames $f_\Psi : hRad(R) \longrightarrow F$.

On the other hand, since the category of frames is 'algebraic' (see [J, §I.3]), we know that any morphism of frames may be factored uniquely as a composition of a regular epimorphism followed by a monomorphism (see [J, §II.2.1]). As such, we factor the morphism $f_\Psi : hRad(R) \longrightarrow F$ as $i_\Psi \circ e_\Psi$, where $e_\Psi : hRad(R) \longrightarrow F^\Psi$ is a regular epimorphism and $i_\Psi : F^\Psi \longrightarrow F$ is a monomorphism.

Lemma 3.8. *Let $I \in hRad(R)$ and $\Psi = (F, \delta)$ be a support datum on R. Then, $f_\Psi(I) \in F$ may be expressed as*

$$f_\Psi(I) = \bigvee_{f \in R_+^h \cap I} \delta(f).$$

Proof. Since $I \in hRad(R)$ is radical, it is clear that for any $f \in R_+^h$ we have:

$$f \in I \iff (Proj(R) \setminus D_+(f)) \supseteq V_+(I) \iff D_+(f) \subseteq (Proj(R) \setminus V_+(I)).$$

Combining (3.1) and (3.4), we obtain

$$f_\Psi(I) = \bigvee_{D_+(f) \subseteq (Proj(R) \setminus V_+(I))} \delta(f) = \bigvee_{f \in R_+^h \cap I} \delta(f). \ ∎$$

Theorem 3.9. *Let $R = \bigoplus_{i \geq 0} R_i$ be a commutative graded ring such that R_0 is a field. Let $\Psi = (F, \delta)$ be a support datum on R. Then, the homogeneous Ψ-closed ideals of R form a frame and this frame $hRad^\Psi(R)$ is isomorphic to F^Ψ.*

Proof. We know that the support datum $\Psi = (F, \delta)$ induces a morphism of frames $f_\Psi : hRad(R) \longrightarrow F$. Then, we may treat f_Ψ as a functor between the categories obtained by considering the partially ordered sets underlying $hRad(R)$ and F. In this sense, we know that any morphism of frames admits a right adjoint (see [J, §II.1]) and we let $g_\Psi : F \longrightarrow hRad(R)$ denote the right adjoint of f_Ψ. We should remark here that the adjoint g_Ψ is only a morphism of meet-semilattices and not necessarily a morphism of frames.

We set $j_\Psi := g_\Psi \circ f_\Psi$. Then, the composition $j_\Psi : hRad(R) \longrightarrow hRad(R)$ gives a 'nucleus' on the frame $hRad(R)$ (see [J, §II.2.2]). Further, from [J, §II.2.2–3], it follows that the fixed points of j_Ψ in $hRad(R)$ form a frame that is isomorphic to the frame F^Ψ. In order to prove the result, it therefore suffices to show that the fixed points of j_Ψ are the same as the homogeneous Ψ-closed ideals of R.

For this, we define $g : F \longrightarrow hRad(R)$ as follows: for any $x \in F$ we let $g(x)$ be the ideal in R generated by all the homogeneous elements $f \in R^h_+ \cup \{0\}$ such that $\delta(f) \leq x$. We claim that $g = g_\Psi$, i.e., g is right adjoint to f_Ψ. For this, we must verify that

$$f_\Psi(I) \leq x \iff I \subseteq g(x)$$

for any $I \in hRad(R)$ and any $x \in F$.

First, we suppose that $f_\Psi(I) \leq x$. From Lemma 3.8, it follows that each $f \in R^h_+ \cap I$ satisfies $\delta(f) \leq x$. From the definition of $g(x)$, it is now clear that $I \subseteq g(x)$.

Conversely, suppose that $I \subseteq g(x)$. Then, from Lemma 3.2 and the definition of $g(x)$, it follows that any homogeneous element $f \in g(x)$ satisfies $\delta(f) \leq x$. In particular, any element $f \in I \cap R^h_+$ satisfies $\delta(f) \leq x$. It is now clear from Lemma 3.8 that $f_\Psi(I) \leq x$.

Thus, we know that $g = g_\Psi$. It follows that for any $I \in hRad(R)$, the ideal $j_\Psi(I) = (g_\Psi \circ f_\Psi)(I)$ is generated by all $f \in R^h_+$ such that

$$\delta(f) \leq f_\Psi(I) = \bigvee_{f' \in R^h_+ \cap I} \delta(f'). \tag{3.5}$$

Then, (3.5) shows that the fixed points of j_Ψ are exactly the homogeneous Ψ-closed ideals of R. This proves the result. ■

If we consider the initial support $(\Omega Proj(R), \delta_0)$ for the commutative graded ring R, we notice that for each $f \in R^h_+ \cup \{0\}$, the open subset $\delta_0(f) = D_+(f)$ is quasi-compact as a subspace of $Proj(R)$. This motivates the next definition.

Definition 3.10. We will say that a support datum $\Psi = (F, \delta)$ for a commutative graded ring R *is of finite type* if for each $f \in R^h_+ \cup \{0\}$, the element $\delta(f)$ is a finite element of the frame F. In other words, given any collection $\{x_j\}_{j \in J}$ of elements of F and $f \in R^h_+ \cup \{0\}$ with $\delta(f) \leq \bigvee_{j \in J} x_j$, there exists a finite subset $J' \subseteq J$ such that $\delta(f) \leq \bigvee_{j \in J'} x_j$.

Lemma 3.11. *Let $R = \bigoplus_{i \geq 0} R_i$ be a commutative graded ring such that R_0 is a field. Let $\Psi = (F, \delta)$ be a support datum of finite type on R. Then, every element of the frame $hRad^\Psi(R)$ of homogeneous Ψ-closed ideals is a join of finite elements.*

Proof. From Theorem 3.9, we have an isomorphism of frames from $hRad^{\Psi}(R)$ to F^{Ψ}. By definition, F^{Ψ} is the image of the morphism $f_{\Psi} : hRad(R) \longrightarrow F$. From Lemma 3.8, we know that

$$f_{\Psi}(I) = \bigvee_{f' \in R_+^h \cap I} \delta(f') \tag{3.6}$$

for any $I \in hRad(R)$. By assumption, Ψ is a support datum of finite type, i.e., each $\delta(f')$ appearing on the right hand side of (3.6) is finite. This proves the result. ■

Given a partially ordered set $(A, \leq)$, we recall that a 'lower interval' in A is a set of the form

$$L(x) := \{y \in A \mid y \not\geq x\}$$

for some $x \in A$. The 'lower interval topology' on A is defined by taking $\{L(x)\}_{x \in A}$ to be an open subbasis (see [J, §II.1.8]). The set A equipped with the lower interval topology will be denoted by $LI(A)$.

We also recall that a topological space is said to be 'spectral' if it is homeomorphic to the Zariski spectrum of a commutative ring. A famous result of Hochster [Ho] shows that a space is spectral if and only if it is quasi-compact, has a basis of quasi-compact opens that is closed under intersection and every irreducible closed subset has a unique generic point.

PROPOSITION 3.12. *Let A be a frame and let A^{ω} be the collection of its finite elements. Suppose that every element of A can be expressed as a join of elements in A^{ω}. Then, the space $LI(A)$, i.e., the set A equipped with lower interval topology, forms a spectral space. The collection $\{L(x) \mid x \in A^{\omega}\}$ forms a subbasis of quasi-compact open subspaces of $LI(A)$.*

Proof. The proof of this result is the same as that of [Ban5, Proposition 4.1]. Although we have supposed in the statement of [Ban5, Proposition 4.1] that A is a coherent frame, the only property of A that is used in the proof is that every element in A can be expressed as a join of elements from A^{ω}. ■

LEMMA 3.13. *Let $R = \bigoplus_{i \geq 0} R_i$ be a commutative graded ring such that R_0 is a field. Let $\Psi = (F, \delta)$ be a support datum of finite type on R. Then*:

(a) *The lower interval topology on $hRad^{\Psi}(R)$ is generated by taking the collection*

$$U(f) = \{J \in hRad^{\Psi}(R) \mid f \notin J\} \quad \forall\, f \in R_+^h$$

to be a subbasis of open sets.

(b) *The lower interval topology on F^{Ψ} is generated by taking the collection*

$$W(f) = \{y \in F^{\Psi} \mid y \not\geq \delta(f)\} \quad \forall\, f \in R_+^h$$

to be a subbasis of open sets.

Proof. (a) By definition, the lower interval topology on $hRad^{\Psi}(R)$ is generated by taking sets of the form $\{J \in hRad^{\Psi}(R) \mid J \not\supseteq I\}$, $I \in hRad^{\Psi}(R)$ to be open sets. If we have ideals $J, I \in hRad^{\Psi}(R)$ such that $J \not\supseteq I$, there must be some homogeneous element $f \in I$

such that $f \notin J$. It follows that

$$\{J \in hRad^{\Psi}(R) \mid J \not\supseteq I\} = \bigcup_{f \in R^h_+ \cap I} \{J \in hRad^{\Psi}(R) \mid f \notin J\} = \bigcup_{f \in R^h_+ \cap I} U(f). \tag{3.7}$$

For any homogeneous element $f \in R^h_+$, let $c_\delta(f)$ be the homogeneous Ψ-closed ideal generated by all homogeneous elements $f' \in R^h_+$ satisfying $\delta(f') \leq \delta(f)$. Then, it is clear that any ideal $J \in hRad^{\Psi}(R)$ does not contain f if and only if it does not contain the homogeneous Ψ-closed ideal $c_\delta(f)$. This shows that $U(f) = \{J \in hRad^{\Psi}(R) \mid J \not\supseteq c_\delta(f)\}$. Combining with (3.7), we have the result.

(b) We know that F^{Ψ} is the image of the morphism $f_\Psi : hRad(R) = \Omega Proj(R) \longrightarrow F$. Then, using (3.6), we notice that any subbasis element $\{y \in F^{\Psi} \mid y \not\geq f_\Psi(I)\}$, $I \in hRad(R)$ for $LI(F^{\Psi})$ may be expressed as

$$\{y \in F^{\Psi} \mid y \not\geq f_\Psi(I)\} = \bigcup_{f \in R^h_+ \cap I} \{y \in F^{\Psi} \mid y \not\geq \delta(f)\} = \bigcup_{f \in R^h_+ \cap I} W(f).$$

On the other hand, from (3.6) and Lemma 3.2, it is clear that

$$W(f) = \{y \in F^{\Psi} \mid y \not\geq \delta(f)\} = \{y \in F^{\Psi} \mid y \not\geq f_\Psi(c_\delta(f))\}$$

where $c_\delta(f) \in hRad^{\Psi}(R) \subseteq hRad(R)$ is as in part (a). This proves the result. ■

THEOREM 3.14. *Let $R = \bigoplus_{i \geq 0} R_i$ be a commutative graded ring such that R_0 is a field. Let $\Psi = (F, \delta)$ be a support datum of finite type on R. Then, the following are spectral spaces and there is a homeomorphism between them*:

(1) *The set $hRad^{\Psi}(R)$ with topology generated by taking the collection*

$$U(f) = \{J \in hRad^{\Psi}(R) \mid f \notin J\} \quad \forall f \in R^h_+$$

to be a subbasis of open sets.

(2) *The set F^{Ψ} with topology generated by taking the collection*

$$W(f) = \{y \in F^{\Psi} \mid y \not\geq \delta(f)\} \quad \forall f \in R^h_+$$

to be a subbasis of open sets.

Proof. From Theorem 3.9, we know that there is an isomorphism of frames between $hRad^{\Psi}(R)$ and F^{Ψ}. This gives a homeomorphism between $LI(hRad^{\Psi}(R))$ and $LI(F^{\Psi})$. From the explicit description of the lower interval topologies $LI(hRad^{\Psi}(R))$ and $LI(F^{\Psi})$ in Lemma 3.13, we know that there is a homeomorphism between the spaces described in (1) and (2). Since $\Psi = (F, \delta)$ is a support datum of finite type on R, it follows from Lemma 3.11 that every element in the frame $hRad^{\Psi}(R) \cong F^{\Psi}$ can be described as a join of finite elements. Applying Proposition 3.12, it now follows that the lower interval topologies $LI(hRad^{\Psi}(R))$ and $LI(F^{\Psi})$ give spectral spaces. ■

Applying Theorem 3.14 to the case of the initial support datum $\Psi_0 = (\Omega Proj(R), \delta_0)$, we obtain as a consequence a projective version of the topological Nullstellensatz of Finocchiaro, Fontana and Spirito [FFS, Theorem 4.1].

Corollary 3.15. *Let $R = \bigoplus_{i\geq 0} R_i$ be a commutative graded ring such that R_0 is a field. Then, the following are spectral spaces and there is a homeomorphism between them*:

(1) *The set $hRad(R)$ of homogeneous radical ideals equipped with topology generated by taking the collection $\{J \in hRad(R) \mid f \notin J\}$, $\forall f \in R_+^h$ to be a subbasis of open sets.*
(2) *The collection $\Omega Proj(R)$ of open sets of $Proj(R)$ with topology generated by taking the collection $\{U \in \Omega Proj(R) \mid U \not\supseteq D_+(f)\}$, $\forall f \in R_+^h$ to be a subbasis of open sets.*

4. Support data for a locally noetherian Grothendieck category. In this section, $\mathcal{A}$ will be a Grothendieck abelian category. We begin by recalling the construction of the spectrum $\mathfrak{S}pec(\mathcal{A})$ of the abelian category $\mathcal{A}$ as defined by Rosenberg [R1], [R3].

We start with a relation on the objects of the category: $X \prec Y$ for objects $X, Y \in \mathcal{A}$ if X is a subquotient of a finite direct sum of copies of Y. This gives rise to an equivalence relation: $X \approx Y$ if $X \prec Y$ and $Y \prec X$. For each object X, we consider the full subcategory $\langle X\rangle$ of $\mathcal{A}$ whose objects are given by

$$Ob(\langle X\rangle) := Ob(\mathcal{A}) - \{Y \in Ob(\mathcal{A}) \mid X \prec Y\}.$$

We notice that $X \prec Y$ if and only if $\langle X\rangle \subseteq \langle Y\rangle$. In particular, an object P in $\mathcal{A}$ is said to be spectral if $P \neq 0$ and any nonzero subobject $Q \subseteq P$ satisfies $P \prec Q$. The collection of spectral objects of $\mathcal{A}$ is denoted by $Spec(\mathcal{A})$. Remarkably, when $\mathcal{A} = R{-}Mod$, the module category over a commutative ring R, every spectral object of $R - Mod$ is equivalent to a quotient $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$.

The spectrum $\mathfrak{S}pec(\mathcal{A})$ is defined as follows:

$$\mathfrak{S}pec(\mathcal{A}) := \{\langle P\rangle \mid P \in \mathcal{A} \text{ is spectral}\}.$$

A basis of closed sets for $\mathfrak{S}pec(\mathcal{A})$ is given by the collection (see [R3, §1.6]):

$$Supp(M) := \{\langle P\rangle \mid P \in Spec(\mathcal{A}) \text{ and } P \prec M\} \tag{4.1}$$

where M varies over all the finitely generated objects of $\mathcal{A}$. We observe that the collection in (4.1) is closed under finite unions since $Supp(M_1) \cup Supp(M_2) = Supp(M_1 \oplus M_2)$ for any finitely generated objects M_1, M_2. We recall here that an object $M \in \mathcal{A}$ is said to be finitely generated if the canonical morphism

$$\varinjlim_{\lambda\in\Lambda} Hom(M, N_\lambda) \longrightarrow Hom(M, \varinjlim_{\lambda\in\Lambda} N_\lambda)$$

is an isomorphism for any filtered system $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects in $\mathcal{A}$ connected by monomorphisms. Additionally, we suppose throughout that $\mathcal{A}$ is a 'locally noetherian category,' i.e., it has a small generating family of noetherian objects. In particular, the full subcategory $\mathcal{A}_{fg} \subseteq \mathcal{A}$ consisting of finitely generated objects forms an abelian subcategory.

We know, for instance, that the category of modules over any left noetherian ring is a locally noetherian Grothendieck category (see [V, (3.3)]). Another example is that of the category of quasi-coherent sheaves on a separated noetherian scheme (see [Ha, §II.7] and [S, Tag 077P]). For more on locally noetherian categories, we refer the reader, for instance, to [AR], [St], [V].

Definition 4.1. Let $\mathcal{A}$ be a locally noetherian Grothendieck category. A *support datum* for $\mathcal{A}$ is a pair $\Psi = (F, \phi)$ consisting of a frame F and a mapping

$$\phi : \mathcal{A}_{fg} \longrightarrow F$$

satisfying the following conditions:

(1) If 1_F denotes the top element of the frame F, then $\phi(0) = 1_F$.
(2) Given any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of objects in $\mathcal{A}_{fg}$, we have

$$\phi(M) = \phi(M') \wedge \phi(M'').$$

A morphism $f : (F, \phi) \longrightarrow (F', \phi')$ of support data for $\mathcal{A}$ is a morphism $f : F \longrightarrow F'$ that preserves finite meets and satisfies the condition $f \circ \phi = \phi'$.

It is clear from Definition 4.1 that given a collection $\{M_i\}_{1 \leq i \leq n}$ of objects in $\mathcal{A}_{fg}$, we have

$$\phi\Big(\bigoplus_{i=1}^{n} M_i\Big) = \bigwedge_{i=1}^{n} \phi(M_i). \tag{4.2}$$

Lemma 4.2. *Let $\Psi = (F, \phi)$ be a support datum for $\mathcal{A}$ and let M, N be two finitely generated objects of $\mathcal{A}$ such that $M \prec N$. Then, $\phi(M) \geq \phi(N)$ as elements of the frame F.*

Proof. Since $M \prec N$, we can express M as a subquotient of a direct sum of finitely many copies of N, say $N^{\oplus k}$. From (4.2), it is clear that $\phi(N^{\oplus k}) = \phi(N)$. Since M is a subquotient of $N^{\oplus k}$, it follows from condition (2) in Definition 4.1 that $\phi(M) \geq \phi(N^{\oplus k}) = \phi(N)$. ∎

We now consider the space $\mathfrak{Spec}(\mathcal{A})$. The collection $\Omega\mathfrak{Spec}(\mathcal{A})$ of open sets of $\mathfrak{Spec}(\mathcal{A})$ forms a frame. We also know (see [R1, §5.2.2]) that for any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ we have $Supp(M) = Supp(M') \cup Supp(M'')$. As such, it is evident that the association

$$\phi_0 : \mathcal{A}_{fg} \longrightarrow \Omega\mathfrak{Spec}(\mathcal{A}) \qquad M \mapsto Supp^c(M) := \mathfrak{Spec}(\mathcal{A}) \setminus Supp(M)$$

is a support datum for $\mathcal{A}$ in the sense of Definition 4.1. Our aim is to show that $\Psi_0 = (\Omega\mathfrak{Spec}(\mathcal{A}), \phi_0)$ is a weakly initial object in the category of support data on $\mathcal{A}$.

Lemma 4.3. *As P varies over all the spectral objects of $\mathcal{A}$, the collection $Supp(P)$ forms a subbasis of closed sets for the topological space $\mathfrak{Spec}(\mathcal{A})$.*

Proof. Let $0 \neq N \in \mathcal{A}_{fg}$ be finitely generated. Since $\mathcal{A}$ is locally noetherian, we know (see [R3, §1.6.4.1]) that there is a finite filtration

$$N = N_k \supseteq N_{k-1} \supseteq \ldots \supseteq N_0 = 0$$

such that each successive quotient $N_i/N_{i-1} \in Spec(\mathcal{A})$. It follows that $Supp(N) = \bigcup_{i=1}^{k} Supp(N_i/N_{i-1})$. As such, every element of the basis $\{Supp(N)\}_{N \in \mathcal{A}_{fg}}$ of closed sets for $\mathfrak{Spec}(\mathcal{A})$ may be expressed as a finite union of elements in $\{Supp(P)\}_{P \in Spec(\mathcal{A})}$. This proves the result. ∎

It follows that the collection $\{Supp^c(P) = \mathfrak{Spec}(\mathcal{A}) \setminus Supp(P)\}_{P \in Spec(\mathcal{A})}$ forms a subbasis of open sets for $\mathfrak{Spec}(\mathcal{A})$. For the sake of convenience, we denote this subbasis by $\mathcal{S}$. The collection of finite intersections of elements of $\mathcal{S}$ will be denoted by $\mathcal{B}$. Since

$\langle P\rangle \in Supp(P)$ for any $P \in Spec(\mathcal{A})$, we notice that the open set $\mathfrak{S}pec(\mathcal{A})$ itself does not lie in $\mathcal{B}$. Then, $\mathcal{B}^+ := \mathcal{B}\cup\{\mathfrak{S}pec(\mathcal{A})\}$ becomes a basis of open sets for the space $\mathfrak{S}pec(\mathcal{A})$.

Let $\Psi = (F, \phi)$ be a support datum for $\mathcal{A}$ in the sense of Definition 4.1. For any element $B \in \mathcal{B}$, we set

$$f_\Psi(B) := \bigwedge_{i=1}^{m} \phi(P_i) \tag{4.3}$$

where the set $B \in \mathcal{B}$ can be expressed as an intersection $B = \bigcap_{i=1}^m Supp^c(P_i)$ of subbasis elements.

Lemma 4.4. *The association in* (4.3) *gives a well-defined function $f_\Psi : \mathcal{B} \longrightarrow F$. Additionally, for any B_1, $B_2 \in \mathcal{B}$, we have $f_\Psi(B_1 \cap B_2) = f_\Psi(B_1) \wedge f_\Psi(B_2)$.*

Proof. Suppose that some $B \in \mathcal{B}$ can be expressed as $B = \bigcap_{i=1}^m Supp^c(P_i)$ and also as $B = \bigcap_{j=1}^n Supp^c(Q_j)$ where each P_i, $Q_j \in Spec(\mathcal{A})$. Taking complements, we obtain

$$\bigcup_{i=1}^{m} Supp(P_i) = \bigcup_{j=1}^{n} Supp(Q_j).$$

We now pick some P_i. Since P_i is spectral, it is clear from (4.1) that $\langle P_i\rangle \in Supp(P_i)$. Hence, there is some integer $1 \le q(i) \le n$ such that $\langle P_i\rangle \in Supp(Q_{q(i)})$. Thus, we can choose some $Q' \in Spec(\mathcal{A})$ such that $\langle P_i\rangle = \langle Q'\rangle$ and $Q' \prec Q_{q(i)}$. From Lemma 4.2, we now have $\phi(P_i) = \phi(Q') \ge \phi(Q_{q(i)})$. It follows that

$$\bigwedge_{i=1}^{m} \phi(P_i) \ge \bigwedge_{i=1}^{m} \phi(Q_{q(i)}) \ge \bigwedge_{j=1}^{n} \phi(Q_j).$$

Similarly, we can verify that $\bigwedge_{j=1}^n \phi(Q_j) \ge \bigwedge_{i=1}^m \phi(P_i)$ and it follows that $f_\Psi : \mathcal{B} \longrightarrow F$ is well-defined. The last statement is immediate from the definition in (4.3). ∎

Lemma 4.5. *Let $\Psi = (F, \phi)$ be a support datum for $\mathcal{A}$. We extend $f_\Psi : \mathcal{B} \longrightarrow F$ to $\mathcal{B}^+$ by setting $f_\Psi(\mathfrak{S}pec(\mathcal{A})) := 1_F$. Then, the association*

$$U \mapsto \bigvee_{B\in\mathcal{B}^+, B\subseteq U} f_\Psi(B) \tag{4.4}$$

gives a well-defined map $f_\Psi : \Omega\mathfrak{S}pec(\mathcal{A}) \longrightarrow F$.

Proof. We need to verify that if $U \in \mathcal{B}$, then $f_\Psi(U)$ as defined in (4.3) agrees with $\bigvee_{B\in\mathcal{B}^+, B\subseteq U} f_\Psi(B) = \bigvee_{B\in\mathcal{B}, B\subseteq U} f_\Psi(B)$. Since $U \in \mathcal{B}$, it is immediate that

$$f_\Psi(U) \le \bigvee_{B\in\mathcal{B}, B\subseteq U} f_\Psi(B).$$

We now consider some $B \in \mathcal{B}$ with $B \subseteq U$. We suppose that $U = \bigcap_{i=1}^m Supp^c(P_i)$ and $B = \bigcap_{j=1}^n Supp^c(Q_j)$ where each P_i, $Q_j \in Spec(\mathcal{A})$. Since $B \subseteq U$, we obtain

$$\bigcup_{i=1}^{m} Supp(P_i) \subseteq \bigcup_{j=1}^{n} Supp(Q_j).$$

Proceeding as in the proof of Lemma 4.4, we obtain $f_\Psi(B) = \bigwedge_{j=1}^n \phi(Q_j) \le \bigwedge_{i=1}^m \phi(P_i) = f_\Psi(U)$. It follows that $\bigvee_{B\in\mathcal{B}, B\subseteq U} f_\Psi(B) \le f_\Psi(U)$ and this proves the result. ∎

Theorem 4.6. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and let $\Psi = (F, \phi)$ be a support datum for $\mathcal{A}$. Then, there exists a morphism $f_\Psi : \Psi_0 = (\Omega\mathfrak{S}pec(\mathcal{A}), \phi_0) \longrightarrow \Psi = (F, \phi)$. In other words, $\Psi_0 = (\Omega\mathfrak{S}pec(\mathcal{A}), \phi_0)$ is a weakly initial object in the category of support data on $\mathcal{A}$.*

Proof. From Lemma 4.5, we already know that $f_\Psi : \Omega\mathfrak{S}pec(\mathcal{A}) \longrightarrow F$ is a well-defined map. We need to show that f_Ψ preserves finite meets. In other words, we have to show that for any two opens $U_1, U_2 \subseteq \mathfrak{S}pec(\mathcal{A})$, we have $f_\Psi(U_1 \cap U_2) = f_\Psi(U_1) \wedge f_\Psi(U_2)$. Since $U_1 \cap U_2 \subseteq U_1, U_2$, it is clear from (4.4) that $f_\Psi(U_1 \cap U_2) \leq f_\Psi(U_1), f_\Psi(U_2)$ and hence $f_\Psi(U_1 \cap U_2) \leq f_\Psi(U_1) \wedge f_\Psi(U_2)$.

If either U_1 or $U_2 = \mathfrak{S}pec(\mathcal{A})$, it is already clear that $f_\Psi(U_1 \cap U_2) = f_\Psi(U_1) \wedge f_\Psi(U_2)$. We assume therefore that U_1, $U_2 \subsetneq \mathfrak{S}pec(\mathcal{A})$. We now set

$$\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \subseteq U_1\}, \qquad \mathcal{B}_2 = \{B \in \mathcal{B} \mid B \subseteq U_2\}.$$

By definition, we now have:

$$\begin{aligned} f_\Psi(U_1) \wedge f_\Psi(U_2) &= \Big(\bigvee_{B_1 \in \mathcal{B}_1} f_\Psi(B_1)\Big) \wedge \Big(\bigvee_{B_2 \in \mathcal{B}_2} f_\Psi(B_2)\Big) \\ &= \bigvee_{(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2} \big(f_\Psi(B_1) \wedge f_\Psi(B_2)\big) = \bigvee_{(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2} \big(f_\Psi(B_1 \cap B_2)\big) \end{aligned}$$

where the last equality follows from Lemma 4.4. We notice that for any $(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, $B_1 \cap B_2$ is a basis element contained in $U_1 \cap U_2$. It now follows that

$$f_\Psi(U_1) \wedge f_\Psi(U_2) = \bigvee_{(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2} \big(f_\Psi(B_1 \cap B_2)\big) \leq \bigvee_{B \in \mathcal{B}, B \subseteq U_1 \cap U_2} f_\Psi(B) = f_\Psi(U_1 \cap U_2).$$

Finally, given some $0 \neq N \in \mathcal{A}_{fg}$, as in the proof of Lemma 4.3, we have a finite filtration

$$N = N_k \supseteq N_{k-1} \supseteq \ldots \supseteq N_0 = 0$$

such that each successive quotient $N_i/N_{i-1} \in Spec(\mathcal{A})$. It is clear from condition (2) of Definition 4.1 that $\phi(N) = \bigwedge_{i=1}^{k} \phi(N_i/N_{i-1})$. On the other hand, we have $\phi_0(N) = Supp^c(N) = \bigcap_{i=1}^{k} Supp^c(N_i/N_{i-1})$ and it follows from the definition in (4.3) that

$$f_\Psi(\phi_0(N)) = \bigwedge_{i=1}^{k} \phi(N_i/N_{i-1}) = \phi(N).$$

Besides this, we already have $f_\Psi(\phi_0(0)) = f_\Psi(\mathfrak{S}pec(\mathcal{A})) = 1_F = \phi(0)$. This proves the result. ∎

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