TRANSLATIONS IN QUANTUM GROUPS

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Abstract. Let H be the Hopf C^*-algebra of continuous functions on a (locally) compact quantum group of either reduced or full type. We show that endomorphisms of H that respect its right regular comodule structure are translations by elements of the largest classical subgroup of G.

Furthermore, we show that for compact G such an endomorphism is automatically an automorphism regardless of the quantum group norm on the C^*-algebra H.

Introduction. The starting point for the present note is the observation that for a compact group G the map sending $g \in G$ to the translation

$$C(G) \ni f \mapsto f(g-) \in C(G)$$

induces an isomorphism between the group G and the monoid of C^*-algebra endomorphisms $\alpha : C(G) \to C(G)$ that respect the right comodule structure of $C(G)$ in the sense that

$$
\begin{array}{ccc}
 & \xrightarrow{\ \Delta\ } C(G) \otimes C(G) \xrightarrow{\ \alpha \otimes \mathrm{id}\ } & \\
C(G) & & C(G) \otimes C(G) \\
 & \xrightarrow{\ \alpha\ } C(G) \xrightarrow[\ \Delta\ ]{} &
\end{array}
$$

commutes.

The function algebra $H = C(G)$ of a compact *quantum* group is similarly equipped with a comultiplication, and one can ask for a similar description of all *translations* of the quantum group: those endomorphisms $\alpha : H \to H$ for which the analogous diagram is commutative.

More generally, the same problem can be posed in the context of *locally* compact quantum groups in the sense of [9, 10].

To make this more precise, let G be a locally compact quantum group, $H = C_0(G)$ the reduced C^*-algebra of functions on G vanishing at infinity and $H^u = C_0^u(G)$ its universal analogue. We write G_{cl} for the largest classical quantum subgroup of G: this is nothing but the group of characters $H^u \to \mathbb{C}$ equipped with the convolution product. An element $\chi \in G_{\mathrm{cl}}$ induces an endomorphism α of H^u in the category of non-unital C^*-algebras (i.e. a non-degenerate morphism $H^u \to M(H^u)$) defined by

$$H^u \xrightarrow[\alpha]{\Delta} M(H^u \otimes H^u) \xrightarrow{\chi \otimes \mathrm{id}} M(H^u). \tag{1}$$

Note that in fact α takes values in H^u, and furthermore it respects the right H^u coactions on its domain and codomain as discussed before. α is, in other words, a *translation* of H^u. One can now ask

- whether this correspondence is an isomorphism between G_{cl} and the monoid of translations of H^u;
- whether every such α descends to a translation of the reduced version H;
- if so, then do we once more obtain an isomorphism between G_{cl} and the monoid of translations of H?

That the classical answers survive in the quantum setting is one of the main result of the paper (an aggregate of Theorems 2.9, 2.10, 2.12 and 2.13):

THEOREM 0.1. *Let G be a locally compact quantum group and $C_0(G)$ and $C_0^u(G)$ its reduced and respectively full C^*-algebras of functions vanishing at infinity. The correspondence $\chi \mapsto \alpha$ described in (1) gives isomorphisms between G_{cl} and the monoids of translations of both $C_0^u(G)$ and $C_0(G)$.*

Von Neumann algebra versions of this result appear in [5] (for Kac algebras) and [6]. More specifically, [6, Theorems 3.7, 3.9 and 3.11] jointly amount to the fact that von Neumann algebra morphisms $L^\infty(G) \to L^\infty(G)$ that preserve the right coaction are in bijection with G_{cl} in the fashion described in Theorem 0.1.

Let us now specialize to the case when G is compact, i.e. $C(G) := C_0(G)$ is unital. In general, one can equip the unique dense Hopf $*$-subalgebra of $C(G)$ with a number of different norms with that of $C(G)$ being minimal and that of $C^u(G)$ maximal (see [1, discussion preceding Theorem 2.1] for justification of the claim that the norm making the Haar state faithful is minimal).

The intermediate norms interpolating between $C(G)$ and $C^u(G)$ are often referred to as *exotic*; see e.g. [11, 18, 14, 2]. Although I do not know whether Theorem 0.1 holds for exotic quantum group norms, it is nevertheless the case that every translation of a quantum group C^*-algebra is bijective (see Corollary 2.4):

THEOREM 0.2. *Every translation of the underlying C^*-algebra of a compact quantum group is an automorphism of said C^*-algebra.*

This answers a question posed by Piotr M. Hajac.

1. Preliminaries

1.1. Locally compact quantum groups. We will work with *locally* compact quantum groups (LCQGs for short) of either reduced or full type, in the sense of [9] and respectively [7]. For locally compact quantum group morphisms we refer to [12]; see also the preliminary section of [4] for a quick review of the theory.

The category of non-unital C^*-algebras we work with is the usual one, where morphisms $A \to B$ are non-degenerate homomorphisms $A \to M(B)$ to the *multiplier algebra* $M(B)$ of B (this is the space denoted by $\mathrm{Mor}(A, B)$ in [4, §1.1]). Throughout the present subsection $H = C_0(G)$ will be the underlying non-unital C^*-algebra of a reduced LCQG G, and H_u the associated C^*-algebra (whose dual object we denote, as before, by G_u).

[9, Section 1] contains all of the C^* weight theory we will need (much more in fact). For a weight φ on a (possibly non-unital) C^*-algebra A we define

$$M_\varphi^+ := \{x \in A^+ \mid \varphi(x) < \infty\}, \ M_\varphi := \mathrm{span}\, M_\varphi^+,$$
$$N_\varphi := \{x \in A \mid \varphi(x^*x) < \infty\}.$$

We write $C_0(G)$ and $C_0^u(G)$ for the reduced and full function C^*-algebras attached to the LCQG G. As has become customary in the literature, denote by φ the left-invariant Haar weight. It is *proper* (i.e. non-zero, lower semicontinuous and densely defined), faithful on $C_0(G)$, and left invariance means that

$$\varphi((\omega \otimes \mathrm{id})\Delta x) = \omega(1)\varphi(x) \quad \forall x \in M_\varphi^+ \ \forall \omega \in C_0(G)_+^* \tag{2}$$

(see [9, Definition 2.2]). Recall (e.g. [9, §1.2]) that proper weights admit unique extensions to the multiplier algebras of their domains; we take such extensions for granted.

We have a GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda)$ attached to φ as in [9, Definition 1.2]: $\mathcal{H}_\varphi$ is a Hilbert space and

$$\pi_\varphi : C_0^u(G) \to C_0(G) \to B(\mathcal{H}_\varphi)$$
$$\Lambda : N_\varphi \to \mathcal{H}_\varphi$$

satisfying $\Lambda(xy) = \pi_\varphi(x)\Lambda(y)$ for all relevant x and y. We regard $C_0(G)$ as a subalgebra of $B(\mathcal{H}_\varphi)$ via π_φ, and $L^\infty(G)$ is its von Neumann closure therein.

The *multiplicative unitary* $W \in B(\mathcal{H}_\varphi^{\otimes 2})$ is defined by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1)) \quad \forall x, y \in N_\varphi.$$

Then W induces the comultiplication on $L^\infty(G)$ via

$$\Delta(x) = W^*(1 \otimes x)W,$$

and W is contained in $M(C_0(\widehat{G}) \otimes C_0(G))$ for the *Pontryagin dual* $\widehat{G}$ of G: by definition, $C_0(\widehat{G})$ is the norm-closure of

$$\left\{ (\omega \otimes \mathrm{id})W \mid \omega \in B(\mathcal{H}_\varphi)^* \right\}$$

([9, Definition 8.1]). We have

$$(\Delta \otimes \mathrm{id})W = W_{13}W_{23}$$
$$(\mathrm{id} \otimes \Delta)W = W_{13}W_{12},$$

relations which we will use below (note that the right hand comultiplication in the second line of the display above is that of $C_0(\widehat{G})$).

1.2. Compact quantum groups. Although these are technically examples of LCQGs (i.e. those for which $C_0(G)$ is unital), there are several aspects peculiar to the compact case that we outline here very briefly, referring to [20, 8] for further background.

The underlying Hopf C^*-algebra $H = C(G)$ has a unique dense Hopf $*$-algebra we denote by $\mathrm{Pol}(G)$ (the *Peter–Weyl algebra* of G). It is cosemisimple as a coalgebra and hence breaks up as a direct sum of matrix coalgebras, one for each irreducible right H-comodule.

2. Translations. Let $H = C(G)$ be the Hopf C^*-algebra underlying a CQG G.

DEFINITION 2.1. A *translation* of G or H is a C^*-algebra morphism α making the diagram

$$ \begin{array}{ccc} & H \otimes H & \xrightarrow{\alpha \otimes \mathrm{id}} \\ H \nearrow^{\Delta} \searrow_{\alpha} & & H \otimes H \\ & H & \nearrow_{\Delta} \end{array} \tag{3}$$

commutative.

We denote by $\mathcal{T}(H)$ or $\mathcal{T}(G)$ the monoid of translations of G equipped with *opposite* composition.

REMARK 2.2. The convention that we compose translations backwards obviates the need to reverse the multiplication on G_{cl} in Theorem 0.1.

We topologize $\mathcal{T}(H)$ for $H = C(G)$ in the standard way by the topology of pointwise convergence in the norm of H.

Now let $\alpha \in \mathcal{T}(G)$. For each $\gamma \in \mathrm{Irr}(G)$ the matrix coalgebra $H_\gamma \subset H$ is preserved by α. If d_γ is the dimension of γ and $u_\gamma = (u_{ij}) \in M_{d_\gamma}(H)$ is a unitary matrix whose entries form a basis for H_γ, then the action of α on H_γ is implemented by multiplying u_γ by a unitary $d_\gamma \times d_\gamma$ matrix $T_\gamma = T_\gamma^\alpha$.

Since $\varphi \in \mathcal{T}(G)$ is uniquely determined by its action on the dense Hopf $*$-subalgebra $\mathrm{Pol}(G)$, we thus have an *embedding*

$$\mathcal{T}(G) \subseteq U_H := \prod_{\gamma \in \mathrm{Irr}(G)} U_{d_\gamma} \tag{4}$$

sending α to the tuple $(T_\gamma^\alpha)_\gamma$.

Our first observation is

PROPOSITION 2.3. *The embedding* (4) *realizes* $\mathcal{T}(G)$ *as a closed sub-monoid of the right hand side.*

Proof. We have already noted the injectivity of the map (4), and by its very definition it intertwines the composition of translations and the group operation on its codomain U_H. It thus remains to argue that the map is continuous and its image is closed.

Concerning the closure of $\mathcal{T}(G)$ in U_H via (4) consider a net $(\alpha_\eta)_\eta \subset \mathcal{T}(G)$ converging to $\alpha \in U_H$. The latter can be identified with a self-map of

$$\mathrm{Pol}(G) = \bigoplus_\gamma H_\gamma$$

as before, acting as multiplication by its γ-component unitary matrix on H_γ. This is a right comodule endomorphism of $\mathrm{Pol}(G)$. It is also an algebra morphism: multiplicativity is a condition on arbitrary pairs of elements, and hence a closed condition under pointwise convergence.

It remains to argue that α must lift to an endomorphism of the C^*-algebra H assuming that α_η are such endomorphisms. Equivalently, since $\mathrm{Pol}(G)$ is dense in H, it is enough to show that α is contractive on $\mathrm{Pol}(G)$ (with respect to the norm of H).

To see this, fix $x \in \mathrm{Pol}(G)$. x is contained in some finite-dimensional subspace

$$V = \bigoplus_{\gamma \in F} H_\gamma$$

for a finite subset $F \subset \mathrm{Irr}(G)$. V is preserved by α_η as well as α, and hence

$$\alpha_\eta(x) \to \alpha(x)$$

in the unique separated vector space topology on V. In particular this is also the topology inherited from the norm $\| - \|$ on H, and hence

$$\|\alpha(x)\| = \lim_\eta \|\alpha_\eta(x)\| \le \|x\|.$$

because α_η all lift to endomorphisms of the C^*-algebra H and are thus contractive. This finishes the proof. ∎

As an immediate consequence of Proposition 2.3 we have

COROLLARY 2.4. *Every translation of* $H = C(G)$ *is an automorphism of the* C^*-*algebra* H.

Proof. We know from Proposition 2.3 that $\mathcal{T} = \mathcal{T}(H)$ is a closed sub-monoid of a compact group U_H. Being a sub-monoid of a group $\mathcal{T}$ is *cancellative* (i.e. $ab = ab' \Rightarrow b = b'$ and similarly for $ba = b'a$). This means that it is, in fact, a closed sub*group*: see e.g. [13, Lemma B.1] for (a strengthening of) the well known result to the effect that compact cancellative semigroups are groups. ∎

2.1. Classifying translations. A simple observation: recall that for any algebra A, every (left, say) A-module endomorphism of A is automatically of the form

$$A \ni x \mapsto xa$$

for some $a \in A$. In other words, A is its own algebra of A-module endomorphisms.

Dually, for a coalgebra C over a field k, every right C-comodule endomap α of C will be of the form

$$c \mapsto \chi(c_1)c_2,$$

where $c \mapsto c_1 \otimes c_2$ is Sweedler notation for comultiplication and $\chi : C \to k$ is a linear map: one simply sets $\chi = \varepsilon \circ \alpha$.

If furthermore $C = H$ is a Hopf algebra (or even just a bialgebra) and α is also a translation in the sense of Definition 2.1, then $\chi : H \to k$ is easily seen to be an algebra map; it is, after all, just the composition

$$H \xrightarrow[\alpha]{\Delta} H \otimes H \xrightarrow{\chi \otimes \mathrm{id}} H \xrightarrow{\varepsilon} k \qquad (5)$$

of α and the counit ε, both of which are algebra morphisms. Finally, $*$-structures come along for the ride for Hopf $*$-algebras (i.e. if α is a $*$-algebra endomorphism then so is $\chi : H \to \mathbb{C}$, and conversely).

NOTATION 2.5. In order to indicate the dependence of α and χ on each other in (5), we will sometimes denote them by α_χ and χ_α respectively.

Now, if $H = C(G)$ is a Hopf C^*-algebra as we have been assuming and $\alpha \in \mathcal{T}(G)$ is a translation, then the preceding discussion applies to the dense Hopf $*$-subalgebra $\mathrm{Pol}(H)$ (which is automatically preserved by α). This means that the restriction of α to $\mathrm{Pol}(H)$ is of the form α_χ (see Notation 2.5) for a unique character χ of the $*$-algebra $\mathrm{Pol}(H)$, or equivalently, of the full Hopf C^*-algebra H_u with $\mathrm{Pol}(H_u) = \mathrm{Pol}(H)$.

NOTATION 2.6. For a CQG $H = C(G)$ we denote by $\mathrm{pt}(G)$ (for 'points of G') the space of characters $H \to \mathbb{C}$, or equivalently, the space of characters of the abelianization $H \to H_{\mathrm{cl}}$ of H.

To summarize:

PROPOSITION 2.7. *Let $H = C(G)$ for a CQG G. Then, the map*

$$\mathcal{T}(G) \ni \alpha \mapsto \chi_\alpha$$

is an embedding of $\mathcal{T}(G)$ as a closed subgroup of the compact group G_{cl}.

Conversely, for every character $\chi : H \to \mathbb{C}$ we can define the translation α_χ via the left hand half of (5). We thus obtain the following version of Proposition 2.3.

PROPOSITION 2.8. *The map*

$$\mathrm{pt}(G) \ni \chi \mapsto \alpha_\chi$$

is an embedding of $\mathrm{pt}(G)$ as a closed sub-semigroup of the compact group $\mathcal{T}(G)$.

Moreover, since for full CQGs we have $G = G_u$ and hence the sandwich

$$\mathrm{pt}(G) \subseteq \mathcal{T}(G) \subseteq G_{\mathrm{cl}}$$

provided by Propositions 2.7 and 2.8 collapses, we obtain

THEOREM 2.9. *If G is a full compact quantum group, $\mathcal{T}(G)$ is isomorphic to the maximal classical subgroup $\mathrm{pt}(G) \subseteq G$.*

We have an analogous result for *reduced* compact quantum groups:

THEOREM 2.10. *If G is a reduced compact quantum group then the embedding $\mathcal{T}(G) \to G_{\mathrm{cl}}$ from Proposition 2.7 is an isomorphism.*

Proof. Let $H = C(G)$. We have to argue that every element $\chi \in G_{\mathrm{cl}}$, acting as an automorphism of the CQG algebra $\mathrm{Pol}(H)$, lifts to an automorphism of the reduced C^* closure H of $\mathrm{Pol}(H)$. This follows from the fact that H is the GNS closure with respect to the Haar state h on $\mathrm{Pol}(H)$, and h is preserved by χ. ∎

2.2. The locally compact case. We write $H = C_0(G)$ and $H^u = C_0^u(G)$. The present subsection is the reason why we have chosen to work with *left* comodule morphisms α: it allows better agreement with the literature on LCQGs that we reference here.

We can define the group $\mathcal{T}(G_u) = \mathcal{T}(H^u)$ of translations of G_u as in Definition 2.1; it consists of non-degenerate endomorphisms α of H^u that preserve the right regular coaction of H^u, in the sense that the diagram

$$H^u \xrightarrow{\Delta} M(H^u \otimes H^u) \xrightarrow{\alpha \otimes \mathrm{id}} M(H^u \otimes H^u) \qquad (6)$$
$$H^u \xrightarrow{\alpha} M(H^u) \xrightarrow{\Delta}$$

analogous to (3) commutes.

We again equip $\mathcal{T}(G_u)$ with its *standard topology*: as before, this is the pointwise-norm topology when regarded as a space of C^*-algebra morphisms $H^u \to M(H^u)$.

REMARK 2.11. It is easy to see that with its standard topology, $\mathcal{T}(G_u)$ is a locally compact group.

As in the discussion at the beginning of Section 2, we denote by G_{cl} the maximal classical closed subgroup of G_u; it is the spectrum of the abelianization $(H^u)_{\mathrm{cl}}$, i.e. the space of characters of H^u. The version of (5) relevant to universal locally compact quantum groups is

$$H^u \xrightarrow{\Delta} M(H^u \otimes H^u) \xrightarrow{\chi \otimes \mathrm{id}} M(H^u) \xrightarrow{\varepsilon} k, \qquad (7)$$
$$H^u \xrightarrow{\alpha} \qquad \xrightarrow{\chi}$$

and defines back-and-forth mutually inverse maps

$$\mathcal{T}(G_u) \underset{\alpha_\chi \mapsfrom \chi}{\overset{\alpha \mapsto \chi_\alpha}{\rightleftarrows}} G_{\mathrm{cl}}, \qquad (8)$$

where the fact that χ is not the zero map follows from the non-degeneracy of α. We thus arrive at

THEOREM 2.12. *The rightward map in* (8) *defines an isomorphism*

$$\mathcal{T}(G_u) \cong G_{\mathrm{cl}}.$$

Proof. We already have mutually inverse bijections preserving the group structures, so the only claim still to be verified is that these bijections are both continuous.

It is immediate from the definition of the pointwise-norm topology that for a fixed morphism f of (non-unital) C^*-algebras the map $g \mapsto g \circ f$ is continuous. Since both $\alpha \mapsto \chi_\alpha$ and $\chi \mapsto \alpha_\chi$ are of this form, this identifies $\mathcal{T}(G_u)$ topologically with the space of characters $H^u \to \mathbb{C}$ equipped with the pointwise-norm topology.

Since characters $H^u \to \mathbb{C}$ coincide with characters of the abelianization $C_0(G_{\mathrm{cl}})$ of H^u again topologized via the pointwise-norm topology, the claim amounts to the fact that the locally compact topology of G_{cl} can be recovered as the pointwise-norm topology on

$$\text{characters } C_0(G_{\mathrm{cl}}) \to \mathbb{C}.$$

This, however, is nothing but the Gelfand–Naimark theorem. ∎

We can now define $\mathcal{T}(H) = \mathcal{T}(G)$ as in the universal case. The analogue, in this case, of Corollary 2.4 is Theorem 2.13 below. It is analogous to [6, Theorems 3.7, 3.9 and 3.11], being a C^* (as opposed to von Neumann) variant of that material. In fact, we will apply the results of [6] directly once we transport the problem to GNS von Neumann algebras.

THEOREM 2.13. *Every translation of $H^u = C_0^u(G)$ descends to one of $H = C_0(G)$, and this correspondence induces an isomorphism*

$$\mathcal{T}(G) \cong G_{\mathrm{cl}}.$$

Proof. We already know from Theorem 2.12 that translations of H^u are actual left translations by elements in the classical locally compact group G_{cl}. These preserve the left-invariant Haar weight φ on H^u and hence descend to the quotient H of H^u on which φ is faithful.

It now remains to show that *every* translation of H arises in this fashion, as a left translation by some element in G_{cl}. We fix a translation $\alpha \in \mathcal{T}(H)$ throughout the rest of the proof.

CLAIM. *α lifts to a normal self-map $L^\infty(G)$.*

The goal here is to show that α sends M_φ^+ to $\overline{M}_\varphi^+$ and $\varphi \circ \alpha = \varphi$. To that end, recall that (2) is valid for all positive elements in H (or indeed even $M(H)$). For that reason, applying $\omega \otimes \varphi$ to the image of $x \in M_\varphi^+$ through (6) we obtain

$$\omega(1)\varphi(\alpha(x)) = \varphi\big((\omega \otimes \mathrm{id})\Delta\alpha(x)\big) = \varphi\big((\omega \circ \alpha \otimes \mathrm{id})\Delta x\big)$$

for all positive $x \in M_\varphi^+$ and positive functionals $\omega \in H_+^*$, where the second equality uses the commutativity of (6). In turn, this equals

$$\omega(\alpha(1))\varphi(x) = \omega(1)\varphi(x).$$

This does indeed prove the desired conclusion $\varphi \circ \alpha = \varphi$ and hence the claim because then α lifts to the closure $L^\infty(G)$ of H in the GNS representation of the α-invariant weight φ.

As explained before, once we have lifted α to L^∞ we can conclude by [6, Theorems 3.7, 3.9 and 3.11], with 3.7 and 3.11 applied to G and 3.9 applied to the dual $\widehat{G}$. ∎

As a consequence, we have the analogue of Corollary 2.4 in the locally compact setting, albeit only in the universal and reduced cases:

COROLLARY 2.14. *Every translation of a reduced or universal locally compact quantum group is an isomorphism of the respective C^*-algebra $C_0(G)$ or $C_0^u(G)$.*

Proof. It is immediate from the characterization of translations given in Theorems 2.12 and 2.13: they are all translations by elements of the classical group G_{cl}, and each such map admits an inverse (namely translation by the inverse of said element). ∎

REMARK 2.15. There is a more general approach to locally compact quantum groups, based around the notion of a *manageable multiplicative unitary* (see e.g. [16, 19, 17]).

Theorem 2.13 holds in the more general setting as well (i.e. translations are in bijection with the elements of the maximal classical subgroup) because [3, Proposition 3.2], which in turn [15, Theorem 6] paraphrases, holds in the more general setup.

References

[1] E. Bédos, G. J. Murphy, L. Tuset, *Co-amenability of compact quantum groups*, J. Geom. Phys. 40 (2001), 130–153.

[2] N. P. Brown, E. P. Guentner, *New C^*-completions of discrete groups and related spaces*, Bull. Lond. Math. Soc. 45 (2013), 1181–1193.

[3] M. Daws, *Completely positive multipliers of quantum groups*, Internat. J. Math. 23 (2012), 1250132.

[4] M. Daws, P. Kasprzak, A. Skalski, P. M. Sołtan, *Closed quantum subgroups of locally compact quantum groups*, Adv. Math. 231 (2012), 3473–3501.

[5] J. De Cannière, *On the intrinsic group of a Kac algebra*, Proc. London Math. Soc. (3) 40 (1980), 1–20.

[6] M. Kalantar, M. Neufang, *From quantum groups to groups*, Canad. J. Math. 65 (2013), 1073–1094.

[7] J. Kustermans, *Locally compact quantum groups in the universal setting*, Internat. J. Math. 12 (2001), 289–338.

[8] J. Kustermans, L. Tuset, *A survey of C^*-algebraic quantum groups. I*, Irish Math. Soc. Bull. 43 (1999), 8–63.

[9] J. Kustermans, S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. 33 (2000), 837–934.

[10] J. Kustermans, S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. 92 (2003), 68–92.

[11] D. Kyed, P. M. Sołtan, *Property* (T) *and exotic quantum group norms*, J. Noncommut. Geom. 6 (2012), 773–800.

[12] R. Meyer, S. Roy, S. L. Woronowicz, *Homomorphisms of quantum groups*, Münster J. Math. 5 (2012), 1–24.

[13] D. Robbie, S. Svetlichny, *An answer to A. D. Wallace's question about countably compact cancellative semigroups*, Proc. Amer. Math. Soc. 124 (1996), 325–330.

[14] Z.-J. Ruan, M. Wiersma, *On exotic group C^*-algebras*, J. Funct. Anal. 271 (2016), 437–453.

[15] P. Salmi, A. Skalski, *Idempotent states on locally compact quantum groups. II*, Q. J. Math. 68 (2017), 421–431.

[16] P. M. Sołtan, S. L. Woronowicz, *A remark on manageable multiplicative unitaries*, Lett. Math. Phys. 57 (2001), 239–252.

[17] P. M. Sołtan, S. L. Woronowicz, *From multiplicative unitaries to quantum groups. II*, J. Funct. Anal. 252 (2007), 42–67.

[18] M. Wiersma, *Constructions of exotic group C^*-algebras*, Illinois J. Math. 60 (2016), 655–667.

[19] S. L. Woronowicz, *From multiplicative unitaries to quantum groups*, Internat. J. Math. 7 (1996), 127–149.

[20] S. L. Woronowicz, *Compact quantum groups*, in: Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, 845–884.