PULLBACKS OF GRAPH C*-ALGEBRAS
FROM ADMISSIBLE PUSHOUTS OF GRAPHS

PIOTR M. HAJAC

Instytut Matematyczny Polskiej Akademii Nauk

Śniadeckich 8, 00-656 Warszawa, Poland

and

Department of Mathematics, University of Colorado Boulder

2300 Colorado Avenue, Boulder, CO 80309-0395, USA
ORCID: 0000-0001-9764-5697   E-mail: pmh@impan.pl

SARAH REZNIKOFF

Mathematics Department, Kansas State University

138 Cardwell Hall, Manhattan, KS 66502, USA
ORCID: 0000-0001-8640-5890   E-mail: sarahrez@ksu.edu

MARIUSZ TOBOLSKI

Instytut Matematyczny Polskiej Akademii Nauk

Śniadeckich 8, 00-656 Warszawa, Poland
ORCID: 0000-0001-7738-8515   E-mail: mtobolski@impan.pl

Abstract. We define an admissible decomposition of a graph E into subgraphs F_1 and F_2, and consider the intersection graph $F_1 \cap F_2$ as a subgraph of both F_1 and F_2. We prove that, if the decomposition of the graph E into the subgraphs F_1 and F_2 is admissible, then the graph C*-algebra $C^*(E)$ of E is the pullback C*-algebra of the canonical surjections from $C^*(F_1)$ and $C^*(F_2)$ onto $C^*(F_1 \cap F_2)$.

1. Introduction and preliminaries. Pushouts of graphs have proven to be very useful in the theory of free groups [13]. We hope that our approach to pullbacks of graph algebras through pushouts of underlying graphs will also turn out to be beneficial.

A graph C*-algebra is the universal C*-algebra associated to a directed graph. If one considers a specific class of morphisms of directed graphs (e.g., see [1, Definition 1.6.2]), then the graph C*-algebra construction yields a covariant functor from the category of directed graphs to the category of C*-algebras. On the other hand, Hong and Szymański [8] showed that a pushout diagram in the category of directed graphs can lead to a pullback of C*-algebras. The purpose of this paper is to find conditions on the pushout diagram of graphs that give rise to the pullback diagram of the associated graph C*-algebras. This leads to a notion of an admissible decomposition of a directed graph, which we present in Section 2. The main result is contained in Section 3 and examples are in Section 4.

Our result is closely related to [9, Corollary 3.4], where it is proven, in an appropriate form, for k-graphs without sinks. Herein, we focus our attention on 1-graphs but possibly with sinks. Thus our results are complementary and lead to the following question: Is it possible to get rid of both of these assumptions ("only 1-graphs" and "no sinks") at the same time to prove a more general pushout-to-pullback theorem?

In this paper, by a graph E we will always mean a *directed graph*, i.e. a quadruple (E^0, E^1, s_E, r_E), where E^0 is the set of vertices, E^1 is the set of edges, $s_E : E^1 \to E^0$ is the source map and $r_E : E^1 \to E^0$ is the range map. (The sets E^0 and E^1 are usually assumed to be countable.) A graph E is called *row finite* if each vertex emits only a finite number of edges. Next, E is called *finite* if both E^0 and E^1 are finite. A vertex is called a *sink* in E if $s_E^{-1}(v) = \varnothing$. By a finite *path* μ in E we mean a finite sequence of composable edges $\mu := e_1 e_2 \ldots e_k$ or a vertex. The *length* $|\mu|$ of the finite path μ is the number k of composable edges. We treat vertices as paths of length zero. The set of all finite paths for a graph E is denoted by $\mathrm{Path}(E)$. One extends the source and the range maps to $\mathrm{Path}(E)$ in a natural way. We denote the extended source and range maps by s_{PE} and r_{PE}, respectively.

DEFINITION 1.1. The *graph C*-algebra* $C^*(E)$ of a graph E is the universal C*-algebra generated by mutually orthogonal projections $P := \{P_v \mid v \in E^0\}$ and partial isometries $S := \{S_e \mid e \in E^1\}$ with mutually orthogonal ranges satisfying the *Cuntz–Krieger relations* [4, 6]:

$$S_e^* S_e = P_{r_E(e)} \qquad \text{for all } e \in E^1, \tag{CK1}$$

$$\sum_{e \in s_E^{-1}(v)} S_e S_e^* = P_v \qquad \text{for all } v \in E^0 \text{ that are not sinks,} \tag{CK2}$$

$$S_e S_e^* \leq P_{s_E(e)} \qquad \text{for all } e \in E^1. \tag{CK3}$$

The datum $\{S, P\}$ is called a *Cuntz–Krieger E-family*.

One can show that the above relations imply the standard path-algebraic relations:

$$S_f^* S_e = 0 \quad \text{for} \quad e \neq f, \quad P_{s_E(e)} S_e = S_e = S_e P_{r_E(e)}. \tag{1.1}$$

Any graph C*-algebra $C^*(E)$ can be endowed with a natural circle action (called the *gauge action*)

$$\alpha : U(1) \longrightarrow \mathrm{Aut}(C^*(E)) \tag{1.2}$$

defined by its values on the generators:

$$\alpha_\lambda(P_v) = P_v, \quad \alpha_\lambda(S_e) = \lambda S_e, \quad \text{where} \quad \lambda \in U(1), \quad v \in E^0, \quad e \in E^1. \qquad (1.3)$$

A subset H of E^0 is called *hereditary* iff, for any $v \in H$ such that there is a path starting at v and ending at $w \in E^0$, we have $w \in H$.

REMARK 1.2. One can equivalently define the property of being hereditary by replacing "path" with "edge".

Next, a subset H of E^0 is called *saturated* iff there does *not* exist a vertex $v \notin H$ such that

$$0 < |s_E^{-1}(v)| < \infty \qquad \text{and} \qquad r_E(s_E^{-1}(v)) \subseteq H. \qquad (1.4)$$

Finally, a subset H of E^0 is called *unbroken* iff there does *not* exist a vertex $v \notin H$ (called a *breaking vertex* in [1, Definition 2.4.4]) such that it emits infinitely many edges and

$$0 < \left|\left\{e \in s_E^{-1}(v) \,|\, r_E(s_E^{-1}(v)) \subseteq E^0 \setminus H\right\}\right| < \infty. \qquad (1.5)$$

Unbroken saturated hereditary subsets play a fundamental role in the theory of gauge-invariant ideals of graph C*-algebras. It follows from [2, p. 1163] that, for any hereditary subset H, the algebraic ideal generated by $\{P_v \,|\, v \in H\}$ is of the form

$$I_E(H) := \text{span}\left\{S_x S_y^* \,|\, x, y \in \text{Path}(E), \ r_{PE}(x) = r_{PE}(y) \in H\right\}. \qquad (1.6)$$

Here, for any path $\mu = e_1 \dots e_k$, we adopt the notation $S_\mu := S_{e_1} \dots S_{e_k}$. Furthermore, if μ is a vertex, then $S_\mu := P_\mu$.

By [2, Proposition 3.4], quotients by closed ideals generated by unbroken saturated hereditary subsets can also be realised as graph C*-algebras by constructing a quotient graph. Given a hereditary subset H of E^0, the *quotient graph* E/H is given by

$$(E/H)^0 := E^0 \setminus H \quad \text{and} \quad (E/H)^1 := E^1 \setminus r_E^{-1}(H). \qquad (1.7)$$

Note that the restriction-corestriction of the range map r_E to $(E/H)^1 \to (E/H)^0$ makes sense for any H, but the same restriction-corestriction of the source map s_E exists because H is hereditary. Moreover, if H is also saturated and unbroken, we obtain the *-isomorphism

$$C^*(E)/\overline{I_E(H)} \cong C^*(E/H), \qquad (1.8)$$

where $\overline{I_E(H)}$ is the norm closure of $I_E(H)$.

2. Admissible decompositions of graphs. Given two graphs $E = (E^0, E^1, s_E, r_E)$ and $G = (G^0, G^1, s_G, r_G)$, one can define a *graph morphism* $f : E \to G$ as a pair of mappings $f^0 : E^0 \to G^0$ and $f^1 : E^1 \to G^1$ satisfying

$$s_G \circ f^1 = f^0 \circ s_E \quad \text{and} \quad r_G \circ f^1 = f^0 \circ r_E. \qquad (2.1)$$

We call the thus obtained category the *category of directed graphs*.

A *subgraph* of a graph $E = (E^0, E^1, s_E, r_E)$ is a graph $F = (F^0, F^1, s_F, r_F)$ such that

$$F^0 \subseteq E^0, \quad F^1 \subseteq E^1, \quad \forall \, e \in F^1 \colon s_F(e) = s_E(e) \text{ and } r_F(e) = r_E(e). \qquad (2.2)$$

Next, let F_1 and F_2 be two subgraphs of a graph E. We define their *intersection* and *union* as follows:

$$F_1 \cap F_2 := (F_1^0 \cap F_2^0, F_1^1 \cap F_2^1, s_\cap, r_\cap),$$
$$\forall\, e \in F_1^1 \cap F_2^1:\; s_\cap(e) := s_E(e),\; r_\cap(e) := r_E(e),$$
$$F_1 \cup F_2 := (F_1^0 \cup F_2^0, F_1^1 \cup F_2^1, s_\cup, r_\cup),$$
$$\forall\, e \in F_1^1 \cup F_2^1:\; s_\cup(e) := s_E(e),\; r_\cup(e) := r_E(e). \tag{2.3}$$

Let us recall the pushout construction in the category of sets and directed graphs following [5]. Assume that X, Y and Z are sets and that $f : Z \to X$ and $g : Z \to Y$ are mappings. The *pushout P* of f and g is defined by $P := (X \sqcup Y)/\sim$, where $\sim$ is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$. If the mappings f and g are understood, we simply say that P is a pushout of X and Y over Z. Additionally, we call the diagram

$$\begin{array}{ccc} & P & \\ \nearrow & & \nwarrow \\ X & & Y \\ \nwarrow & & \nearrow \\ f & Z & g \end{array} \tag{2.4}$$

a *pushout diagram* of X and Y over Z. Here the upper arrows are compositions of the natural mappings from X and Y into their disjoint union followed by the quotient map $X \sqcup Y \to P$.

Directed graphs are defined in terms of sets, and therefore the pushout construction can be applied separately to the set of vertices and the set of edges. The only restriction is that all the mappings used need to preserve the source map and the range map. If a graph E has two subgraphs F_1 and F_2 such that $E = F_1 \cup F_2$, then the diagram

$$\begin{array}{ccc} & E & \\ \nearrow & & \nwarrow \\ F_1 & & F_2 \\ \nwarrow & & \nearrow \\ & F_1 \cap F_2 & \end{array} \tag{2.5}$$

is automatically a pushout diagram of F_1 and F_2 over their intersection. Let us illustrate the concept of a pushout diagram of graphs with the following example:

$$\tag{2.6}$$

.

We are now ready to define an admissible decomposition of a graph:

DEFINITION 2.1. An unordered pair $\{F_1, F_2\}$ of subgraphs of a graph E is called an *admissible decomposition* of E iff the following conditions are satisfied:

(1) $E = F_1 \cup F_2$,
(2) if v is a sink in $F_1 \cap F_2$, then v is a sink in F_i, $i = 1, 2$,
(3) $F_1^1 \cap F_2^1 = r_{F_i}^{-1}(F_1^0 \cap F_2^0)$, $i = 1, 2$,
(4) $F_i^0 \setminus F_j^0$ is unbroken in F_i^0 and E^0, $i \neq j$, $i, j = 1, 2$.

Observe that Diagram (2.6) gives an example of an admissible decomposition of a graph. Note also that if $\{F_1, F_2\}$ is an admissible decomposition of E, then E is automatically a pushout of F_1 and F_2 over their intersection by (1) in Definition 2.1. The goal of this paper is to use the other admissibility conditions to obtain a pullback diagram of the associated graph C*-algebras.

Definition 2.1 prompts the following two lemmas.

LEMMA 2.2. *Let $\{F_1, F_2\}$ be an admissible decomposition of a graph E. Then we have $F_1 \cap F_2 = F_i/(F_i^0 \setminus (F_1^0 \cap F_2^0))$ and $F_i = E/(E^0 \setminus F_i^0)$, for $i = 1, 2$.*

Proof. First, note that $F_i^0 \setminus (F_1^0 \cap F_2^0)$ is hereditary in F_i. Indeed, take $e \in F_i^1$. Then $s_{F_i}(e) \notin F_1^0 \cap F_2^0$ implies $e \notin F_1^1 \cap F_2^1$. Hence, by Definition 2.1(3), $r_{F_i}(e) \notin F_1^0 \cap F_2^0$. By Remark 1.2 we are done. Therefore, we can define $F_i/(F_i^0 \setminus (F_1^0 \cap F_2^0))$, which coincides with $F_1 \cap F_2$ due to Definition 2.1(3).

Next, note that

$$E^0 \setminus F_i^0 = (F_i^0 \cup F_j^0) \setminus F_i^0 = F_j^0 \setminus F_i^0 = F_j^0 \setminus (F_i^0 \cap F_j^0), \tag{2.7}$$

where $j \neq i$ and $j = 1, 2$, so we already know that $E^0 \setminus F_i^0$ is hereditary in F_j. To see that it is hereditary in E, we only need to exclude edges starting in $E^0 \setminus F_i^0$ and ending in $E^0 \setminus F_j^0$. They do not exist because $E^1 = F_i^1 \cup F_j^1$, so $E^0 \setminus F_i^0$ is hereditary in E.

It remains to verify that $F_i^1 = r_E^{-1}(F_i^0)$. To this end, taking advantage of the admissibility of $(F_i \cap F_j) \subseteq F_i$, we compute

$$r_E^{-1}(F_i^0) \setminus F_i^1 = r_{F_j}^{-1}(F_i^0) \setminus F_i^1 = r_{F_j}^{-1}(F_i^0 \cap F_j^0) \setminus F_i^1 = (F_i^1 \cap F_j^1) \setminus F_i^1 = \varnothing. \tag{2.8}$$

Therefore, as $F_i^1 \subseteq r_E^{-1}(F_i^0)$, we conclude that $F_i^1 = r_E^{-1}(F_i^0)$, as desired. ∎

LEMMA 2.3. *Let $\{F_1, F_2\}$ be an admissible decomposition of a graph E. Then the subset $F_i^0 \setminus (F_1^0 \cap F_2^0)$ is saturated in F_i^0 and in E^0 for $i = 1, 2$.*

Proof. If $F_i^0 \setminus (F_1^0 \cap F_2^0)$ were not saturated in F_i^0, then there would exist a vertex v in

$$F_i^0 \setminus (F_i^0 \setminus (F_1^0 \cap F_2^0)) = F_1^0 \cap F_2^0 \tag{2.9}$$

such that

$$s_{F_i}^{-1}(v) \neq \varnothing \quad \text{and} \quad r_{F_i}(s_{F_i}^{-1}(v)) \subseteq F_i^0 \setminus (F_1^0 \cap F_2^0). \tag{2.10}$$

Thus we would have a vertex in $F_1 \cap F_2$ that is a sink in $F_1 \cap F_2$ but not in F_i, which contradicts Definition 2.1(2).

Much in the same way, suppose that $F_i^0 \setminus (F_1^0 \cap F_2^0)$ is not saturated in E^0. Then there exists a vertex

$$w \in E^0 \setminus (F_i^0 \setminus (F_1 \cap F_2^0)) = F_j^0, \tag{2.11}$$

where $j \neq i$ and $j = 1, 2$, such that

$$s_E^{-1}(w) \neq \varnothing \quad \text{and} \quad r_E(s_E^{-1}(w)) \subseteq F_i^0 \setminus (F_1^0 \cap F_2^0). \tag{2.12}$$

Hence, there is $e \in s_E^{-1}(w)$ such that $r_E(e) \notin F_j^0$. As $E^1 = F_i^1 \cup F_j^1$, it follows that $e \in F_i^1$, so $w = s_E(e) \in F_i^0$. Consequently, w is a sink in $F_1 \cap F_2$ but not in F_i, which again contradicts Definition 2.1(2). ■

3. Pullbacks of graph C*-algebras. Let $\{F_1, F_2\}$ be an admissible decomposition of a graph E. Then, by Lemma 2.2 and Lemma 2.3, we can take advantage of the formula (1.8) to define the canonical quotient maps:

$$\begin{aligned}
\pi_1 &: C^*(E) \longrightarrow C^*(E)/\overline{I_E(F_2^0 \setminus F_1^0)} \cong C^*(F_1), \\
\pi_2 &: C^*(E) \longrightarrow C^*(E)/\overline{I_E(F_1^0 \setminus F_2^0)} \cong C^*(F_2), \\
\chi_1 &: C^*(F_1) \longrightarrow C^*(F_1)/\overline{I_{F_1}(F_1^0 \setminus (F_1^0 \cap F_2^0))} \cong C^*(F_1 \cap F_2), \\
\chi_2 &: C^*(F_2) \longrightarrow C^*(F_2)/\overline{I_{F_2}(F_2^0 \setminus (F_1^0 \cap F_2^0))} \cong C^*(F_1 \cap F_2).
\end{aligned} \tag{3.1}$$

Note that quotient maps are automatically $U(1)$-equivariant for the gauge action.

This brings us to the main theorem:

THEOREM 3.1. *Let $\{F_1, F_2\}$ be an admissible decomposition of a graph E. Then there exist canonical quotient gauge-equivariant $*$-homomorphisms rendering the diagram*

$$\begin{array}{ccc}
 & C^*(E) & \\
\pi_1 \swarrow & & \searrow \pi_2 \\
C^*(F_1) & & C^*(F_2) \\
\chi_1 \searrow & & \swarrow \chi_2 \\
 & C^*(F_1 \cap F_2) &
\end{array} \tag{3.2}$$

commutative. Moreover, this is a pullback diagram *of $U(1)$-C*-algebras.*

Proof. Note first that all the canonical surjections in the diagram are well defined due to the admissibility conditions of the decomposition of the graph E (see the discussion at the beginning of this section). The commutativity of the diagram is obvious as all maps are canonical surjections. Finally, using [10, Proposition 3.1] and the surjectivity of χ_1 and χ_2, to prove that (3.2) is a pullback diagram, it suffices to show that $\ker \pi_1 \cap \ker \pi_2 = \{0\}$ and that $\ker \chi_2 \subseteq \pi_2(\ker \pi_1)$.

Since $\ker \pi_1$ and $\ker \pi_2$ are closed ideals in a C*-algebra, we know that

$$\ker \pi_1 \cap \ker \pi_2 = \ker \pi_1 \ker \pi_2. \tag{3.3}$$

Next, as $F_1^0 \setminus F_2^0$ and $F_2^0 \setminus F_1^0$ are saturated hereditary subsets of E^0, it follows from (1.8) that

$$\ker \pi_1 = \overline{I_E(F_2^0 \setminus F_1^0)} \qquad \text{and} \qquad \ker \pi_2 = \overline{I_E(F_1^0 \setminus F_2^0)}. \tag{3.4}$$

Furthermore, using the characterization (1.6) of ideals generated by hereditary subsets, we know that an arbitrary element of $\ker \pi_1 \ker \pi_2$ is in the closed linear span of elements

of the form $S_\alpha S_\beta^* S_\gamma S_\delta^*$, where $\alpha, \beta \in \text{Path}(E)$ with

$$r_{PE}(\alpha) = r_{PE}(\beta) \in F_2^0 \setminus F_1^0 \,, \tag{3.5}$$

and $\gamma, \delta \in \text{Path}(E)$ with

$$r_{PE}(\gamma) = r_{PE}(\delta) \in F_1^0 \setminus F_2^0 \,. \tag{3.6}$$

The conclusion $\ker \pi_1 \cap \ker \pi_2 = \{0\}$ follows from the analysis of all possible paths satisfying the above conditions. Indeed, it follows from Definition 1.1 that $S_\beta^* S_\gamma \neq 0$ is possible only if $s_{PE}(\beta) = s_{PE}(\gamma)$. As $E^1 = F_1^1 \cup F_2^1$, $r_{PE}(\beta) \in F_2^0 \setminus F_1^0$ and $r_{PE}(\gamma) \in F_1^0 \setminus F_2^0$, if $\beta = e_1 \dots e_m$ and $\gamma = f_1 \dots f_n$, we infer that

$$r_E(e_{m-1}) = s_E(e_m) \in F_2^0 \quad \text{and} \quad r_E(f_{n-1}) = s_E(f_n) \in F_1^0 \,. \tag{3.7}$$

Hence $r_E(e_{m-1}) \in F_1^0 \cap F_2^0$ or $r_E(e_{m-1}) \in F_2^0 \setminus F_1^0$. Now, we continue by induction using Definition 2.1(3) for the intersection case of the alternative. This brings us to conclusion that $s_{PE}(\beta) \in F_2^0$. Much in the same way, we argue that $s_{PE}(\gamma) \in F_1^0$. It follows that $s_{PE}(\beta) = s_{PE}(\gamma) \in F_1^0 \cap F_2^0$. Furthermore, as $r_{PE}(\beta) \in F_2^0 \setminus F_1^0$ and $r_{PE}(\gamma) \in F_1^0 \setminus F_2^0$, we conclude that $\beta \neq \gamma$, so there exists the smallest index i such that $e_i \neq f_i$. Now, remembering the relation (CK1) and (1.1), we compute

$$\begin{aligned} S_\beta^* S_\gamma &= S_{e_{i+1}\dots e_m}^* S_{e_i}^* S_{e_{i-1}}^* \dots S_{e_1}^* S_{e_1} \dots S_{e_{i-1}} S_{f_i} S_{f_{i+1}\dots f_n} \\ &= S_{e_{i+1}\dots e_m}^* S_{e_i}^* S_{f_i} S_{f_{i+1}\dots f_n} = 0. \end{aligned} \tag{3.8}$$

Finally, if β or γ is a path of length zero, i.e. a vertex, then it is straightforward to conclude that $S_\beta^* S_\gamma = 0$.

Next, again taking advantage of (1.6) and (1.8), we obtain

$$\ker \chi_2 = \overline{I_{F_2}(F_2^0 \setminus F_1^0)} = \overline{\text{span}} \big\{ S_\alpha S_\beta^* \mid \alpha, \beta \in \text{Path}(F_2), \ r_{PF_2}(\alpha) = r_{PF_2}(\beta) \in F_2^0 \setminus F_1^0 \big\}.$$

Any element of $I_{F_2}(F_2^0 \setminus F_1^0)$ is an element of $I_E(F_2^0 \setminus F_1^0)$, and $\pi_2(S_\alpha) = S_\alpha$ for all $\alpha \in \text{Path}(F_2)$. Hence $I_{F_2}(F_2^0 \setminus F_1^0) \subseteq \pi_2(I_E(F_2^0 \setminus F_1^0))$. Finally, from the continuity of π_2, we conclude that $\ker \chi_2 \subseteq \pi_2(\ker \pi_1)$. $\blacksquare$

REMARK 3.2. One can also prove Theorem 3.1 in the setting of Leavitt path algebras [1]. A proof of the Leavitt version of Theorem 3.1 is completely analogous due to [1, Corollary 2.5.11]. Furthermore, in the case of row-finite graphs, one should compare Theorem 3.1 with [12, Theorem 3.3].

**4. Examples.** We end the paper by providing motivating examples from noncommutative topology.

4.1. Even quantum spheres. Not only does the graph at the top of Diagram (2.6) representing the generic Podleś quantum sphere [11] admit a natural admissible decomposition, but also the finite graphs L_{2n} [7, Section 5.1] representing, respectively, the C*-algebras $C(S_q^{2n})$ of all even quantum spheres enjoy natural admissible decompositions $\{F_{2n}^1, F_{2n}^2\}$. Here $C^*(F_{2n}^1) = C^*(F_{2n}^2)$ coincides with the C*-algebra $C(B_q^{2n})$ of the Hong–Szymański quantum $2n$-ball [8, Section 3.1], and $C^*(F_{2n}^1 \cap F_{2n}^2)$ coincides [7, Section 4.1] with the C*-algebra $C(S_q^{2n-1})$ of the boundary Vaksman–Soibelman quantum odd sphere [14]. Thus we recover in terms of graphs the classical fact that an even sphere is a gluing of even balls over the boundary odd sphere.

As Theorem 3.1 applies, we infer that the diagram

$$
\begin{array}{ccc}
 & C(S_q^{2n}) & \\
{\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\
C(B_q^{2n}) & & C(B_q^{2n}) \\
{\scriptstyle\chi_1}\searrow & & \swarrow{\scriptstyle\chi_2} \\
 & C(S_q^{2n-1}) &
\end{array}
\tag{4.1}
$$

is a pullback diagram. This fact was already proved in [8, Proposition 5.1] by direct considerations of generators and relations.

The case $n = 3$ is illustrated by the diagram:

$$(4.2)$$
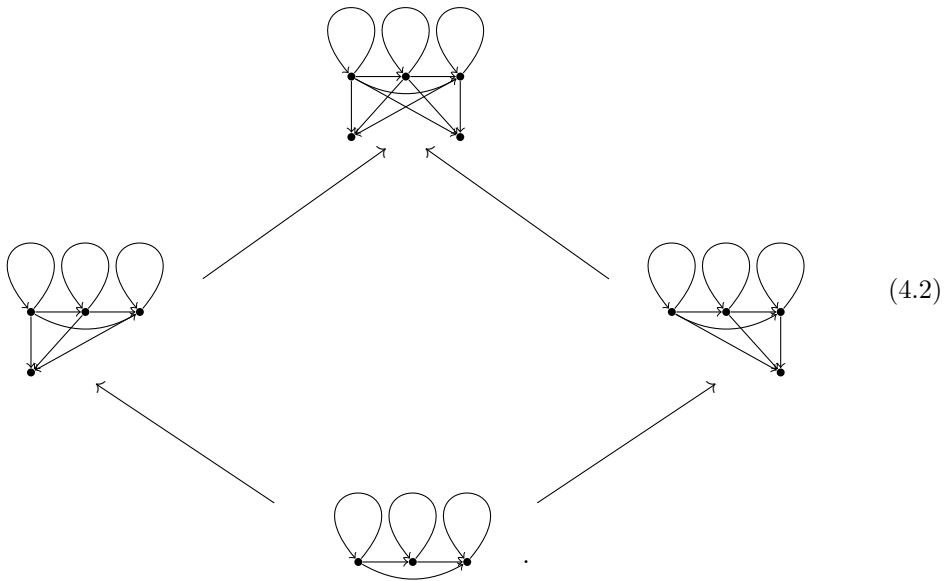
4.2. Quantum lens space $L_q^3(l; 1, l)$. The C*-algebra $C(L_q^3(l; 1, l))$ of the quantum lens space $L_q^3(l; 1, l)$ can be viewed as the graph C*-algebra (e.g., see [3]) of the graph L_l^3:

$$(4.3)$$
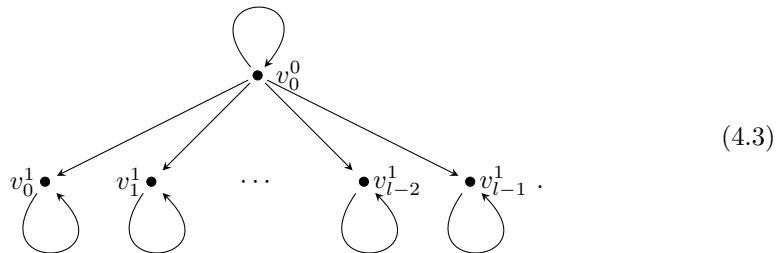
The graph L_l^3 enjoys an admissible decomposition $\{L_k^3, L_{l-k}^3\}$, where $k \in \{1, \ldots, l-1\}$, yielding, by Theorem 3.1, the pullback diagram:

$$C(L_q^3(l; 1, l))$$

$$\pi_1 \qquad\qquad \pi_2$$

$$C(L_q^3(k; 1, k)) \qquad\qquad C(L_q^3(l-k; 1, l-k)) \qquad (4.4)$$

$$\chi_1 \qquad\qquad \chi_2$$

$$C(S^1) \ .$$

Recall that $C^*(L_1^3) \cong C(S_q^3)$, so, for $l = 2$, we obtain the following pullback diagram:

$$C(L_q^3(2; 1, 2))$$

$$\pi_1 \qquad\qquad \pi_2$$

$$C(S_q^3) \qquad\qquad\qquad C(S_q^3) \qquad (4.5)$$

$$\chi_1 \qquad\qquad \chi_2$$

$$C(S^1) \ .$$

Since the above diagram is $U(1)$-equivariant, it induces a pullback diagram for $U(1)$-fixed-point subalgebras:

$$C(\mathbb{W}P_q^1(1, 2))$$

$$\pi_1 \qquad\qquad \pi_2$$

$$C(\mathbb{C}P_q^1) \qquad\qquad C(\mathbb{C}P_q^1) \qquad (4.6)$$

$$\chi_1 \qquad\qquad \chi_2$$

$$\mathbb{C} \ .$$

Here $C(\mathbb{C}P_q^1)$ and $C(\mathbb{W}P_q^1(1, 2))$ denote the quantum complex projective space (see [7, Section 2.3]) and the quantum weighted projective space (see [3, Section 3]), respectively. Interestingly, the C*-algebras in the above diagram can be viewed as graph C*-algebras, and an infinite graph representing $C(\mathbb{W}P_q^1(1, 2))$ is a pushout of infinite graphs representing $C(\mathbb{C}P_q^1)$ over the graph consisting of one vertex and no edges representing $\mathbb{C}$ (see Diagram (4.7) below). Better still, this example is also within the scope of Theorem 3.1.

$$(4.7)$$

Here edges with (∞) denote countably infinitely many edges.

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