QUANTUM DYNAMICS
BANACH CENTER PUBLICATIONS, VOLUME 120
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2020

# MORITA EQUIVALENCE FOR $k$-ALGEBRAS

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**Abstract.** We review Morita equivalence for finite type $k$-algebras $A$ and also a weakening of Morita equivalence which we call *stratified equivalence*. The spectrum of $A$ is the set of equivalence classes of irreducible $A$-modules. For any finite type $k$-algebra $A$, the spectrum of $A$ is in bijection with the set of primitive ideals of $A$. The stratified equivalence relation

2010 *Mathematics Subject Classification*: 14A22, 16G30, 20C08, 33D80.
*Key words and phrases*: finite type algebras, affine varieties, Morita equivalence, stratified equivalence.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc120-16  

preserves the spectrum of $A$ and also preserves the periodic cyclic homology of $A$. However, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras associated to affine Weyl groups.

**1. Introduction.** The subject of Morita equivalence begins with the classical paper of Morita [Mor]. Morita equivalence of rings is well-established and widely used [Lam, §18–19].

Now let $k$ be the coordinate algebra of a complex affine variety. Equivalently, $k$ is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. Morita equivalence for $k$-algebras, the starting-point of this article, is a more recent development, which introduces several nuances related to the $k$-module structure of the algebras.

A $k$-algebra is an algebra $A$ over the complex numbers $\mathbb{C}$ which is a $k$-module (with an evident compatibility between the algebra structure of $A$ and the $k$-module structure of $A$). $A$ is not required to have a unit. $A$ is not required to be commutative. A $k$-algebra $A$ is of finite type if as a $k$-module $A$ is finitely generated. The spectrum of $A$ is, by definition, the set of equivalence classes of irreducible $A$-modules. For any finite type $k$-algebra $A$, the spectrum of $A$ is in bijection with the set of primitive ideals of $A$.

The set of primitive ideals of $A$ will be denoted $\mathrm{Prim}(A)$. When $A$ is unital and commutative, let $\mathrm{maxSpec}(A)$ denote the set of maximal ideals in $A$, and let $\mathrm{Spec}(A)$ denote the set of prime ideals in $A$. In particular, let $A = k$. Then we have

$$\mathrm{Prim}(A) \simeq \mathrm{maxSpec}(A) \subset \mathrm{Spec}(A).$$

The inclusion is strict, for the prime spectrum $\mathrm{Spec}(A)$ contains a generic point, namely the zero ideal $\mathbf{0}$ in $A$. The set $\mathrm{Prim}(A)$ consists of the $\mathbb{C}$-rational points of the affine variety underlying $k$.

So $\mathrm{Prim}(A)$ is a generalisation, for $k$-algebras $A$ of finite type, of the maximal ideal spectrum of a unital commutative ring, rather than the prime spectrum of a unital commutative ring. For a finite type $k$-algebra $A$, the zero ideal $\mathbf{0}$ is a primitive ideal if and only if $k = \mathbb{C}$ and $A = M_n(\mathbb{C})$ the algebra of all $n$ by $n$ matrices with entries in $\mathbb{C}$. Quite generally, any irreducible representation of a finite type $k$-algebra $A$ is a surjection of $A$ onto $M_n(\mathbb{C})$ for some $n$. This implies that any primitive ideal is maximal.

In some situations, Morita equivalence can be too strong and we are led to introduce a weakening of this concept, which we call *stratified equivalence*.

The stratified equivalence relation preserves the spectrum of $A$ and also preserves the periodic cyclic homology of $A$. In addition, the stratified equivalence relation permits a tearing apart of *strata* in the primitive ideal space which is not allowed by Morita equivalence.

Denote by $\Gamma$ a finite group. In §10, we show that stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.

A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras. Let $(X, Y, R, R^\vee)$ be a root datum in the standard sense [L]. This root datum delivers the following items:

— a finite Weyl group $W_0$
— an extended affine Weyl group $W_0X := W_0 \ltimes X$
— for each $q \in \mathbb{C}^\times$, an affine Hecke algebra $\mathcal{H}_q$
— a complex torus $T := \mathrm{Hom}(X, \mathbb{C}^\times)$
— a complex variety $X := T/W_0$
— a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$.

Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^\times$, $\mathcal{H}_q$ is a unital finite type $k$-algebra. If $q = 1$, then $\mathcal{H}_1$ is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X].$$

THEOREM 1.1. *Except for $q$ in a finite set of roots of unity, none of which is 1, $\mathcal{H}_q$ and $\mathcal{H}_1$ are stratified equivalent as $k$-algebras. If $q \neq 1$ then $\mathcal{H}_q$ and $\mathcal{H}_1$ are not Morita equivalent as $k$-algebras.*

A finite type $k$-algebra can be viewed as a noncommutative complex affine variety. So the setting of this article can be viewed as noncommutative algebraic geometry.

We conclude this article with a detailed account of the affine Hecke algebra $\mathcal{H}_q$ attached to the Lie group $\mathrm{SL}_2(\mathbb{C})$.

For an application of stratified equivalence to the representation theory of $p$-adic groups, see [ABPS].

## 2. $k$-algebras.

If $X$ is an affine algebraic variety over the complex numbers $\mathbb{C}$, then $\mathcal{O}(X)$ will denote the coordinate algebra of $X$. Set $k = \mathcal{O}(X)$. Equivalently, $k$ is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

$$\begin{pmatrix}\text{unital commutative finitely generated} \\ \text{nilpotent-free } \mathbb{C}\text{-algebras}\end{pmatrix} \sim \begin{pmatrix}\text{affine complex} \\ \text{algebraic varieties}\end{pmatrix}^{op}$$

$$\mathcal{O}(X) \mapsto X.$$

Here “op” denotes the opposite category.

DEFINITION 2.1. A *$k$-algebra* is a $\mathbb{C}$-algebra $A$ such that $A$ is a unital (left) $k$-module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda\omega)a \quad \forall(\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1a_2) = (\omega a_1)a_2 = a_1(\omega a_2) \quad \forall(\omega, a_1, a_2) \in k \times A \times A.$$

REMARK 2.2. $A$ is not required to have a unit.

The centre of $A$ will be denoted by $Z(A)$:

$$Z(A) := \{c \in A \mid ca = ac \; \forall\, a \in A\}.$$

Remark 2.3. Let $A$ be a unital $k$-algebra. Denote the unit of $A$ by $1_A$. The map $\omega \mapsto \omega.1_A$ is then a unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A),$$

i.e. unital $k$-algebra = unital $\mathbb{C}$-algebra $A$ with a given unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A).$$

If $A$ does not have a unit, a $k$-structure is equivalent to a unital morphism of $\mathbb{C}$-algebras

$$k \to Z(\mathcal{M}A)$$

where $\mathcal{M}A$ is the multiplier algebra of $A$.

Definition 2.4. Let $A, B$ be two $k$-algebras. A *morphism of $k$-algebras* is a morphism of $\mathbb{C}$-algebras

$$f\colon A \to B$$

which is also a morphism of (left) $k$-modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$

Definition 2.5. Let $A$ be a $k$-algebra. A *representation* of $A$ [or a (left) *$A$-module*] is a $\mathbb{C}$-vector space $V$ with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V), \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that

1. $k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital;
2. $(\omega a)v = \omega(av) = a(\omega v) \quad \forall (\omega, a, v) \in k \times A \times V$.

*From now on in this article, $A$ will denote a $k$-algebra.*

A representation of $A$, namely $A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$ with $k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$, will often be denoted by $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$, it being understood that the action of $k$ on $V$ is part of the given structure.

Definition 2.6. A representation $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is *non-degenerate* if $AV = V$, i.e. for any $v \in V$, there are $v_1, v_2, \ldots, v_r$ in $V$ and $a_1, a_2, \ldots, a_r$ in $A$ with

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_r v_r.$$

Definition 2.7. A representation $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is *irreducible* if $AV \neq \{0\}$ and there is no sub-$\mathbb{C}$-vector space $W$ of $V$ with:

$$\{0\} \neq W, \quad W \neq V$$

and

$$\omega w \in W \quad \forall (\omega, w) \in k \times W \quad \text{and} \quad aw \in W \quad \forall (a, w) \in A \times W.$$

Definition 2.8. Two representations of the $k$-algebra $A$

$$\varphi_1\colon A \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_1) \quad \text{and} \quad \varphi_2\colon A \to \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of $\mathbb{C}$-vector spaces

$$T\colon V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V \quad \text{and} \quad T(\omega v) = \omega T(v) \quad \forall (\omega, v) \in k \times V.$$

The *spectrum* of $A$, also denoted by $\mathrm{Irr}(A)$, is the set of equivalence classes of irreducible representations of $A$.

$$\mathrm{Irr}(A) := \{\text{Irreducible representations of A}\}/\sim.$$

It can happen that the spectrum is empty. For example, let $A$ comprise all 2 by 2 matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

with $x \in \mathbb{C}$. Then $A$ is a commutative non-unital algebra of finite type (with $k = \mathbb{C}$) such that $\mathrm{Irr}(A) = \emptyset$.

**3. On the action of $k$.** For a $k$-algebra $A$, $A_{\mathbb{C}}$ denotes the underlying $\mathbb{C}$-algebra of $A$. Then $A_{\mathbb{C}}$ is obtained from $A$ by forgetting the action of $k$ on $A$. For $A_{\mathbb{C}}$ there are the usual definitions: A *representation* of $A_{\mathbb{C}}$ [or a (left) $A_{\mathbb{C}}$-*module*] is a $\mathbb{C}$-vector space $V$ with a given morphism of $\mathbb{C}$-algebras

$$A_{\mathbb{C}} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V).$$

An $A_{\mathbb{C}}$-module $V$ is *irreducible* if $A_{\mathbb{C}}V \neq \{0\}$ and there is no sub-$\mathbb{C}$-vector space $W$ of $V$ with:

$$\{0\} \neq W, \qquad W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A_{\mathbb{C}} \times W.$$

Two representations of $A_{\mathbb{C}}$

$$\varphi_1 \colon A \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_1) \quad \text{and} \quad \varphi_2 \colon A \to \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of $\mathbb{C}$-vector spaces

$$T \colon V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V.$$

$\mathrm{Irr}(A_{\mathbb{C}}) := \{\text{Irreducible representations of } A_{\mathbb{C}}\}/\sim.$

An $A_{\mathbb{C}}$-module $V$ for which the following two properties are valid is *strictly non-degenerate*:

— $A_{\mathbb{C}}V = V$;
— if $v \in V$ has $av = 0 \quad \forall\, a \in A_{\mathbb{C}}$, then $v = 0$.

Lemma 3.1. *Any irreducible $A_{\mathbb{C}}$-module is strictly non-degenerate.*

*Proof.* Let $V$ be an irreducible $A_{\mathbb{C}}$-module. First, consider $A_{\mathbb{C}}V \subset V$. $A_{\mathbb{C}}V$ is preserved by the action of $A_{\mathbb{C}}$ on $V$. We cannot have $A_{\mathbb{C}}V = \{0\}$ since this would contradict the irreducibility of $V$. Therefore $A_{\mathbb{C}}V = V$.

Next, set

$$W = \{v \in V \mid av = 0 \quad \forall\, a \in A_{\mathbb{C}}\}.$$

$W$ is preserved by the action of $A_{\mathbb{C}}$ on $V$. $W$ cannot be equal to $V$ since this would imply $A_{\mathbb{C}}V = \{0\}$. Hence $W = \{0\}$. ∎

Lemma 3.2. *Let $A$ be a $k$-algebra, and let $V$ be a strictly non-degenerate $A_{\mathbb{C}}$-module. Then there is a unique unital morphism of $\mathbb{C}$ algebras*

$$k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

*which makes $V$ an $A$-module.*

*Proof.* Given $v \in V$, choose $v_1, v_2, \dots, v_r \in V$ and $a_1, a_2, \dots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_r v_r.$$

For $\omega \in k$, define $\omega v$ by

$$\omega v = (\omega a_1)v_1 + (\omega a_2)v_2 + \ldots + (\omega a_r)v_r.$$

The second condition in the definition of strictly non-degenerate implies that $\omega v$ is well-defined. ■

Lemma 3.2 will be referred to as the "$k$-action for free lemma".

If $V$ is an $A$-module, $V_{\mathbb{C}}$ will denote the underlying $A_{\mathbb{C}}$-module. $V_{\mathbb{C}}$ is obtained from $V$ by forgetting the action of $k$ on $V$.

Lemma 3.3. *If $V$ is any irreducible $A$-module, then $V_{\mathbb{C}}$ is an irreducible $A_{\mathbb{C}}$-module.*

*Proof.* Suppose that $V_{\mathbb{C}}$ is not an irreducible $A_{\mathbb{C}}$-module. Then there is a sub-$\mathbb{C}$-vector space $W$ of $V$ with:

$$0 \neq W, \qquad W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A \times W.$$

Consider $AW \subset W$. $AW$ is preserved by both the $A$-action on $V$ and the $k$-action on $V$. Thus if $AW \neq \{0\}$, then $V$ is not an irreducible $A$-module. Hence $AW = \{0\}$. Consider $kW \supset W$. $kW$ is preserved by the $k$-action on $V$ and is also preserved by the $A$-action on $V$ because $A$ annihilates $kW$. Since $A$ annihilates $kW$, we cannot have $kW = V$. Therefore $\{0\} \neq kW$, $kW \neq V$, which contradicts the irreducibility of the $A$-module $V$. ■

A corollary of Lemma 3.2 is

Corollary 3.4. *For any $k$-algebra $A$, the map*

$$\mathrm{Irr}(A) \to \mathrm{Irr}(A_{\mathbb{C}}), \qquad V \mapsto V_{\mathbb{C}}$$

*is a bijection.*

*Proof.* Surjectivity follows from Lemmas 3.1 and 3.2. For injectivity, let $V, W$ be two irreducible $A$-modules such that $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are equivalent $A_{\mathbb{C}}$-modules. Let $T \colon V \to W$ be an isomorphism of $\mathbb{C}$-vector spaces with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V.$$

Given $v \in V$ and $\omega \in k$, choose $v_1, v_2, \dots, v_r \in V$ and $a_1, a_2, \dots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_r v_r.$$

Then

$$\begin{aligned}T(\omega v) &= T((\omega a_1)v_1 + (\omega a_2)v_2 + \ldots + (\omega a_r)v_r)\\ &= (\omega a_1)Tv_1 + (\omega a_2)Tv_2 + \ldots + (\omega a_r)Tv_r\\ &= \omega(a_1Tv_1 + a_2Tv_2 + \ldots + a_rTv_r) = \omega(Tv).\end{aligned}$$

Hence $T\colon V \to W$ intertwines the $k$-actions on $V, W$ and thus $V, W$ are equivalent $A$-modules. ■

**4. Finite type $k$-algebras.** An ideal $I$ in a $k$-algebra $A$ is *primitive* if $I$ is the null-space of an irreducible representation of $A$, i.e. there is an irreducible representation of $A$, $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ with

$$I = \{a \in A \mid \varphi(a) = 0\}.$$

$\mathrm{Prim}(A)$ denotes the set of all primitive ideals in $A$. The evident map

$$\mathrm{Irr}(A) \to \mathrm{Prim}(A)$$

sends an irreducible representation to its null-space. On $\mathrm{Prim}(A)$ there is the Jacobson topology. If $S$ is any subset of $\mathrm{Prim}(A)$, $S \subset \mathrm{Prim}(A)$, then the closure $\overline{S}$ of $S$ is

$$\overline{S} := \Big\{I \in \mathrm{Prim}(A) \mid I \supset \bigcap_{L \in S} L\Big\}.$$

Definition 4.1. A $k$-algebra $A$ is of *finite type* if, as a $k$-module, $A$ is finitely generated.

For any finite type $k$-algebra $A$, the following three statements are valid:

— If $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ is any irreducible representation of $A$, then $V$ is a finite dimensional $\mathbb{C}$-vector space and $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ is surjective.
— The evident map $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection.
— Any primitive ideal in $A$ is a maximal ideal.

Since $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection, the Jacobson topology on $\mathrm{Prim}(A)$ can be transferred to $\mathrm{Irr}(A)$ and thus $\mathrm{Irr}(A)$ is topologized. Equivalently, $\mathrm{Irr}(A)$ is topologized by requiring that $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ be a homeomorphism.

For a finite type $k$-algebra $A$ ($k = \mathcal{O}(X)$), the central character is a map

$$\mathrm{Irr}(A) \longrightarrow X$$

defined as follows. Let $\varphi$, given by the morphisms

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V), \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V),$$

be an irreducible representation of $A$. $I_V$ denotes the identity operator of $V$.

For $\omega \in k = \mathcal{O}(X)$, define $T_\omega\colon V \to V$ by

$$T_\omega(v) = \omega v$$

for all $v \in V$. Then $T_\omega$ is an intertwining operator for $A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$. By Lemma 3.3 and Schur's Lemma, $T_\omega$ is a scalar multiple of $I_V$,

$$T_\omega = \lambda_\omega I_V \qquad \lambda_\omega \in \mathbb{C}.$$

The map

$$\omega \mapsto \lambda_\omega$$

is a unital morphism of $\mathbb{C}$-algebras $\mathcal{O}(X) \to \mathbb{C}$ and thus is given by evaluation at a unique ($\mathbb{C}$-rational) point $p_\varphi$ of $X$:

$$\lambda_\omega = \omega(p_\varphi) \quad \forall\, \omega \in \mathcal{O}(X).$$

The *central character* $\mathrm{Irr}(A) \longrightarrow X$ is

$$\varphi \mapsto p_\varphi .$$

REMARK. Corollary 3.4 states that $\mathrm{Irr}(A)$ depends only on the underlying $\mathbb{C}$-algebra $A_\mathbb{C}$. The central character $\mathrm{Irr}(A) \to X$, however, does depend on the structure of $A$ as a $k$-module. A change in the action of $k$ on $A_\mathbb{C}$ will change the central character.

The central character $\mathrm{Irr}(A) \to X$ is continuous where $\mathrm{Irr}(A)$ is topologized as above and $X$ has the Zariski topology. For a proof of this assertion see [KNS, Lemma 1]. From a somewhat heuristic non-commutative geometry point of view, $A_\mathbb{C}$ is a non-commutative complex affine variety, and a given action of $k$ on $A_\mathbb{C}$, making $A_\mathbb{C}$ into a finite type $k$-algebra $A$, determines a morphism of algebraic varieties $A_\mathbb{C} \to X$.

## 5. Morita equivalence for $k$-algebras

DEFINITION 5.1. Let $B$ be a $k$-algebra. A *right $B$-module* is a $\mathbb{C}$-vector space $V$ with given morphisms of $\mathbb{C}$-algebras

$$B^{op} \longrightarrow \mathrm{Hom}_\mathbb{C}(V, V), \qquad k \longrightarrow \mathrm{Hom}_\mathbb{C}(V, V)$$

such that:

1. $k \to \mathrm{Hom}_\mathbb{C}(V, V)$ is unital;
2. $v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v, \omega, b) \in V \times k \times B$.

$B^{op}$ is the opposite algebra of $B$. $V$ is non-degenerate if $VB = V$.

Note that a right $B$-module is the same as a left $B^{op}$-module.

With $k$ fixed, let $A, B$ be two $k$-algebras. An $A - B$ bimodule, denoted by ${}_AV_B$, is a $\mathbb{C}$-vector space $V$ such that:

1. $V$ is a left $A$-module,
2. $V$ is a right $B$-module,
3. $a(vb) = (av)b \quad \forall (a, v, b) \in A \times V \times B$,
4. $\omega v = v\omega \quad \forall (\omega, v) \in k \times V$.

An $A - B$ bimodule ${}_AV_B$ is non-degenerate if $AV = V = VB$. $I_V$ is the identity map of $V$. $A$ is an $A - A$ bimodule in the evident way.

DEFINITION 5.2. A $k$-algebra $A$ has *local units* if given any finite set $\{a_1, a_2, \ldots, a_r\}$ of elements of $A$, there is an idempotent $Q \in A$ ($Q^2 = Q$) with

$$Qa_j = a_jQ = a_j, \qquad j = 1, 2, \ldots, r.$$

DEFINITION 5.3. Let $A$, $B$ be two $k$-algebras with local units. A *Morita equivalence* (between $A$ and $B$) is given by a pair of non-degenerate bimodules

$$ {}_A V_B \qquad {}_B W_A $$

together with isomorphisms of bimodules

$$ \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B $$

such that there is commutativity in the diagrams:

$$ \begin{array}{ccc} V \otimes_B W \otimes_A V & \xrightarrow{I_V \otimes \beta} & V \otimes_B B \\ \downarrow{\scriptstyle \alpha \otimes I_V} & & \downarrow{\scriptstyle \cong} \\ A \otimes_A V & \xrightarrow[\cong]{} & V \end{array} $$

$$ \begin{array}{ccc} W \otimes_A V \otimes_B W & \xrightarrow{I_W \otimes \alpha} & W \otimes_A A \\ \downarrow{\scriptstyle \beta \otimes I_W} & & \downarrow{\scriptstyle \cong} \\ B \otimes_B V & \xrightarrow[\cong]{} & W. \end{array} $$

Let $A, B$ be two $k$-algebras with local units, and suppose given a Morita equivalence

$$ {}_A V_B \qquad {}_B W_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B. $$

The *linking algebra* is

$$ L({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}, $$

i.e. $L({}_A V_B, {}_B W_A)$ consists of all $2 \times 2$ matrices having $(1,1)$ entry in $A$, $(2,2)$ entry in $B$, $(2,1)$ entry in $W$, and $(1,2)$ entry in $V$. Addition and multiplication are matrix addition and matrix multiplication. Note that $\alpha$ and $\beta$ are used in the matrix multiplication.

$L({}_A V_B, {}_B W_A)$ is a $k$-algebra. With $\omega \in k$, the action of $k$ on $L({}_A V_B, {}_B W_A)$ is given by

$$ \omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix}. $$

A Morita equivalence between $A$ and $B$ determines an equivalence of categories between the category of non-degenerate left $A$-modules and the category of non-degenerate left $B$-modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between $A$ and $B$) of Hochschild homology, cyclic homology, and periodic cyclic homology.

EXAMPLE. For $n$ a positive integer, let $\mathrm{M}_n(A)$ be the $k$-algebra of all $n \times n$ matrices with entries in $A$. If $A$ has local units, $A$ and $\mathrm{M}_n(A)$ are Morita equivalent as follows. For $m, n$ positive integers, denote by $\mathrm{M}_{m,n}(A)$ the set of all $m \times n$ (i.e. $m$ rows and $n$ columns) matrices with entries in $A$. Matrix multiplication then gives a map

$$ \mathrm{M}_{m,n}(A) \times \mathrm{M}_{n,r}(A) \longrightarrow \mathrm{M}_{m,r}(A). $$

With this notation, $\mathrm{M}_{n,n}(A) = \mathrm{M}_n(A)$ and $\mathrm{M}_{1,1}(A) = \mathrm{M}_1(A) = A$. Hence matrix multiplication gives maps

$$\mathrm{M}_{1,n}(A) \times \mathrm{M}_n(A) \longrightarrow \mathrm{M}_{1,n}(A), \qquad \mathrm{M}_n(A) \times \mathrm{M}_{n,1} \longrightarrow \mathrm{M}_{n,1}(A).$$

Thus $\mathrm{M}_{1,n}(A)$ is a right $\mathrm{M}_n(A)$-module and $\mathrm{M}_{n,1}(A)$ is a left $\mathrm{M}_n(A)$-module. Similarly, $\mathrm{M}_{1,n}(A)$ is a left $A$-module and $\mathrm{M}_{n,1}(A)$ is a right $A$-module.

With $A = A$ and $B = \mathrm{M}_n(A)$, the bimodules of the Morita equivalence are $V = \mathrm{M}_{1,n}(A)$ and $W = \mathrm{M}_{n,1}(A)$. Note that the required isomorphisms of bimodules

$$\alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$

are obtained by observing that the matrix multiplication maps

$$\mathrm{M}_{1,n}(A) \times \mathrm{M}_{n,1}(A) \to A, \qquad \mathrm{M}_{n,1}(A) \times \mathrm{M}_{1,n}(A) \to \mathrm{M}_n(A)$$

factor through the quotients $\mathrm{M}_{1,n}(A) \otimes_{\mathrm{M}_n(A)} \mathrm{M}_{n,1}(A)$, $\mathrm{M}_{n,1}(A) \otimes_A \mathrm{M}_{1,n}(A)$ and so give bimodule isomorphisms

$$\alpha\colon \mathrm{M}_{1,n}(A) \otimes_{\mathrm{M}_n(A)} \mathrm{M}_{n,1}(A) \to A$$
$$\beta\colon \mathrm{M}_{n,1}(A) \otimes_A \mathrm{M}_{1,n}(A) \to \mathrm{M}_n(A).$$

If $A$ has local units, then $\alpha$ and $\beta$ are isomorphisms. Therefore $A$ and $\mathrm{M}_n(A)$ are Morita equivalent.

If $A$ does not have local units, then $\alpha$ and $\beta$ can fail to be isomorphisms, and there is no way to prove that $A$ and $\mathrm{M}_n(A)$ are Morita equivalent. In examples, this already happens with $n = 1$, and there is then no way to prove (when $A$ does not have local units) that $A$ is Morita equivalent to $A$. For more details on this issue, see below, where the proof is given that in the new equivalence relation $A$ and $\mathrm{M}_n(A)$ are equivalent even when $A$ does not have local units.

A finite type $k$-algebra $A$ has local units iff $A$ is unital.

**6. The exceptional set $\mathfrak{E}_A(X)$.** A Morita equivalence between two finite type $k$-algebras $A, B$ preserves the central character, i.e. there is commutativity in the diagram

$$\begin{array}{ccc} \mathrm{Irr}(A) & \longrightarrow & \mathrm{Irr}(B) \\ {\scriptstyle cc}\big\downarrow & & \big\downarrow{\scriptstyle cc} \\ X & \xrightarrow[I_X]{} & X \end{array}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and $I_X$ is the identity map of $X$.

By the *exceptional set* $\mathfrak{E}(A)$ we shall mean the set of all $x \in X$ such that the fibre over $x$ has cardinality greater than 1:

$$\mathfrak{E}(A) := \{x \in X \mid \sharp cc^{-1}(x) > 1\}.$$

If $A$ and $B$ are Morita equivalent as $k$-algebras, then we will have $\mathfrak{E}(A) = \mathfrak{E}(B)$. Contrapositively, we have the following useful

COROLLARY 6.1. *If $\mathfrak{E}(A) \neq \mathfrak{E}(B)$ then $A$ and $B$ are not Morita equivalent as $k$-algebras.*

**7. Spectrum preserving morphisms.** Let $A$, $B$ two finite type $k$-algebras, and let $f\colon A \to B$ be a morphism of $k$-algebras.

Definition 7.1. $f$ is *spectrum preserving* if

1. Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $I \supset f^{-1}(J)$.
2. The resulting map $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is a bijection.

For $I$ and $J$ as above, let $V$ be the simple $B$-module with annihilator $J$. Then the simple $A$-module with annihilator $I$ can be realized as a quotient of $V$ and also as a subspace of $V$.

Example 7.2. Let $A, B$ be two unital finite type $k$-algebras, and suppose given a Morita equivalence

$$_AV_B \quad _BW_A \quad \alpha\colon V \otimes_B W \to A \quad \beta\colon W \otimes_A V \to B.$$

With the linking algebra $L(_AV_B, {}_BW_A)$ as above, the inclusions

$$A \hookrightarrow L(_AV_B, {}_VW_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

are spectrum preserving morphisms of finite type $k$-algebras. The bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

so obtained is the bijection determined by the given Morita equivalence.

If $f\colon A \to B$ is a spectrum preserving morphism of finite type $k$-algebras, then the resulting bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

is a homeomorphism. For a proof of this assertion see [BN, Theorem 3]. Consequently, if $A$, $B$ are two unital finite type $k$-algebras, and

$$_AV_B \quad _BW_A \quad \alpha\colon V \otimes_B W \to A \quad \beta\colon W \otimes_A V \to B$$

is a Morita equivalence, then the resulting bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

is a homeomorphism.

Definition 7.3. An ideal $I$ in a $k$-algebra $A$ is a *$k$-ideal* if $\omega a \in I \quad \forall (\omega, a) \in k \times I$.

Note that any primitive ideal in a $k$-algebra $A$ is a $k$-ideal.

Given $A, B$ two finite type $k$-algebras, let $f\colon A \to B$ be a morphism of $k$-algebras.

Definition 7.4. $f$ is *spectrum preserving with respect to filtrations* if there are $k$-ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_{r-1} \subset I_r = A \quad \text{in } A$$

and $k$-ideals

$$0 = J_0 \subset J_1 \subset \ldots \subset J_{r-1} \subset J_r = B \quad \text{in } B$$

with $f(I_j) \subset J_j$ $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to J_j/J_{j-1}$ $(j = 1, 2, \ldots, r)$ is spectrum preserving.

The primitive ideal spaces of the subquotients $I_j/I_{j-1}$ and $J_j/J_{j-1}$ are the strata for stratifications of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$. Each stratum of $\mathrm{Prim}(A)$ is mapped homeomorphically onto the corresponding stratum of $\mathrm{Prim}(B)$. However, the map

$$\mathrm{Prim}(A) \to \mathrm{Prim}(B)$$

might not be a homeomorphism.

**8. Algebraic variation of $k$-structure.** Let $A$ be a unital $\mathbb{C}$-algebra, and let

$$\Psi\colon k \to Z(A[t, t^{-1}])$$

be a unital morphism of $\mathbb{C}$-algebras. Here $t$ is an indeterminate, so $A[t, t^{-1}]$ is the algebra of Laurent polynomials with coefficients in $A$. As above $Z$ denotes "center". For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $ev(\zeta)$ denotes the "evaluation at $\zeta$" map:

$$ev(\zeta)\colon A[t, t^{-1}] \to A$$
$$\sum a_j t^j \mapsto \sum a_j \zeta^j.$$

Consider the composition

$$k \xrightarrow{\Psi} Z(A[t, t^{-1}]) \xrightarrow{ev(\zeta)} Z(A).$$

Denote the unital $k$-algebra so obtained by $A_\zeta$. For all $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, the underlying $\mathbb{C}$-algebra of $A_\zeta$ is $A$.

$$(A_\zeta)_{\mathbb{C}} = A \quad \forall \zeta \in \mathbb{C}^\times.$$

Such a family $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, of unital $k$-algebras, will be referred to as *an algebraic variation of $k$-structure with parameter space* $\mathbb{C}^\times$.

**9. Stratified equivalence.** With $k$ fixed, consider the collection of all finite type $k$-algebras. On this collection, *stratified equivalence* is, by definition, the equivalence relation generated by the two elementary steps:

*Elementary Step* 1. If there is a morphism of $k$-algebras $f : A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.

*Elementary Step* 2. If there is $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of $k$-structure with parameter space $\mathbb{C}^\times$, such that each $A_\zeta$ is a unital finite type $k$-algebra, then for any $\zeta, \eta \in \mathbb{C}^\times$, $A_\zeta \sim A_\eta$.

Thus, two finite type $k$-algebras $A, B$ are equivalent if and only if there is a finite sequence $A_0, A_1, A_2, \ldots, A_r$ of finite type $k$-algebras with $A_0 = A$, $A_r = B$, and for each $j = 0, 1, \ldots, r-1$ one of the following three possibilities is valid:

— a morphism of $k$-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
— a morphism of $k$-algebras $A_j \leftarrow A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
— $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of $k$-structure with parameter space $\mathbb{C}^\times$, is given such that each $A_\zeta$ is a unital finite type $k$-algebra, and $\eta, \tau \in \mathbb{C}^\times$ have been chosen with $A_j = A_\eta, A_{j+1} = A_\tau$.

To give a stratified equivalence relating $A$ and $B$, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology [BN] are determined:

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad \mathrm{HP}_*(A) \cong \mathrm{HP}_*(B).$$

PROPOSITION 9.1. *If two unital finite type $k$-algebras $A, B$ are Morita equivalent (as $k$-algebras) then they are stratified equivalent.*

$$A \underset{\text{Morita}}{\sim} B \Longrightarrow A \sim B.$$

*Proof.* Let $A$, $B$ two unital finite type $k$-algebras, and suppose given a Morita equivalence

$$_AV_B \qquad _BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B.$$

The linking algebra is

$$L(_AV_B, {_BW_A}) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}.$$

The inclusions

$$A \hookrightarrow L(_AV_B, {_VW_A}) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

are spectrum preserving morphisms of finite type $k$-algebras. Hence $A$ and $B$ are stratified equivalent. ∎

According to the above, a Morita equivalence of $A$ and $B$ gives a homeomorphism

$$\mathrm{Prim}(A) \simeq \mathrm{Prim}(B).$$

However, the bijection

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B)$$

obtained from a stratified equivalence might not be a homeomorphism, as in the following example.

EXAMPLE 9.2. Let $Y$ be a sub-variety of $X$. We will write $\mathcal{I}_Y$ for the ideal in $\mathcal{O}(X)$ determined by $Y$, so that

$$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall\, p \in Y\}.$$

Let $A$ be the algebra of all $2 \times 2$ matrices whose diagonal entries are in $\mathcal{O}(X)$ and whose off-diagonal entries are in $\mathcal{I}_Y$. Addition and multiplication in $A$ are matrix addition and matrix multiplication. As a $k$-module, $A$ is the direct sum of $\mathcal{O}(X) \oplus \mathcal{O}(X)$ with $\mathcal{I}_Y \oplus \mathcal{I}_Y$.

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}.$$

Set $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$, so that $B$ is the coordinate algebra of the disjoint union $X \sqcup Y$. We have $\mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y$. As a $k = \mathcal{O}(X)$-module, $B$ is the direct sum $\mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y)$.

THEOREM 9.3. *The algebras $A$ and $B$ are stratified equivalent but not Morita equivalent:*

$$A \sim B \qquad A \underset{\text{Morita}}{\nsim} B.$$

*Proof.* Let $\mathrm{M}_2(\mathcal{O}(X))$ denote the algebra of all $2 \times 2$ matrices with entries in $\mathcal{O}(X)$. Consider the algebra morphisms

$$\begin{array}{ccc} A \longrightarrow \mathrm{M}_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y) \longleftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y) \\ T \mapsto (T, t_{22}|Y) \qquad\qquad (\omega, \theta) \mapsto (T_\omega, \theta) \end{array}$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \qquad T_\omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}.$$

The filtration of $A$ is given by

$$\{0\} \subset \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{I}_Y \end{pmatrix} \subset A$$

and the filtration of $\mathrm{M}_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y)$ is given by

$$\{0\} \subset \mathrm{M}_2(\mathcal{O}(X) \oplus \{0\}) \subset \mathrm{M}_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y).$$

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that

$$A \sim B.$$

Note that

$$\begin{aligned} &\mathrm{Prim}(A) = X \text{ with each point of } Y \text{ replaced by two points} \\ \text{and} \quad &\mathrm{Prim}(B) = \mathrm{Prim}(\mathcal{O}(X) \oplus \mathcal{O}(Y)) = X \sqcup Y. \end{aligned}$$

The spaces $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are not homeomorphic, and therefore $A$ is not Morita equivalent to $B$. ∎

Unlike Morita equivalence, stratified equivalence works well for finite type $k$-algebras whether or not the algebras are unital, e.g. $A$ and $\mathrm{M}_n(A)$ are stratified equivalent even when $A$ is not unital. See Proposition 9.5 below.

For any $k$-algebra $A$ there is the evident isomorphism of $k$-algebras $\mathrm{M}_n(A) \cong A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. Hence, using this isomorphism, if $W$ is a representation of $A$ and $U$ is a representation of $\mathrm{M}_n(\mathbb{C})$, then $W \otimes_{\mathbb{C}} U$ is a representation of $\mathrm{M}_n(A)$.

As above, $\mathrm{M}_{n,1}(\mathbb{C})$ denotes the $n \times 1$ matrices with entries in $\mathbb{C}$. Matrix multiplication gives the usual action of $\mathrm{M}_n(\mathbb{C})$ on $\mathrm{M}_{n,1}(\mathbb{C})$.

$$\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_{n,1}(\mathbb{C}) \longrightarrow \mathrm{M}_{n,1}(\mathbb{C}).$$

This is the unique irreducible representation of $\mathrm{M}_n(\mathbb{C})$. For any $k$-algebra $A$, if $W$ is a representation of $A$, then $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is a representation of $\mathrm{M}_n(A)$.

LEMMA 9.4. *Let $A$ be a finite type $k$-algebra and let $n$ be a positive integer. Then*:

(i) *If $W$ is an irreducible representation of $A$, $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is an irreducible representation of $\mathrm{M}_n(A)$.*
(ii) *The resulting map $\mathrm{Irr}(A) \to \mathrm{Irr}(\mathrm{M}_n(A))$ is a bijection.*

*Proof.* For (i), suppose given an irreducible representation $W$ of $A$. Let $J$ be the primitive ideal in $A$ which is the null space of $W$. Then the null space of $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is $J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. Consider the quotient algebra $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})/J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C}) = (A/J) \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. This is isomorphic to $\mathrm{M}_{rn}(\mathbb{C})$ where $A/J \cong \mathrm{M}_r(\mathbb{C})$, and so $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is irreducible. ∎

PROPOSITION 9.5. *Let $A$ be a finite type $k$-algebra and let $n$ be a positive integer, then $A$ and $\mathrm{M}_n(A)$ are stratified equivalent.*

*Proof.* Let $f\colon A \to \mathrm{M}_n(A)$ be the morphism of $k$-algebras which maps $a \in A$ to the diagonal matrix

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix}$$

It will suffice to prove that $f\colon A \to \mathrm{M}_n(A)$ is spectrum preserving.

Let $J$ be an ideal in $A$. Denote by $J^{\diamondsuit}$ the ideal in $\mathrm{M}_n(A)$ consisting of all $[a_{ij}] \in \mathrm{M}_n(A)$ such that each $a_{ij}$ is in $J$. Equivalently, $\mathrm{M}_n(A)$ is $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ and $J^{\diamondsuit} = J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. It will suffice to prove

1. If $J$ is a primitive ideal in $A$, then $J^{\diamondsuit}$ is a primitive ideal in $\mathrm{M}_n(A)$.
2. If $L$ is any primitive ideal in $\mathrm{M}_n(A)$, then there is a primitive ideal $J$ in $A$ with $L = J^{\diamondsuit}$.

For 1., $J$ primitive $\Longrightarrow J^{\diamondsuit}$ primitive, because the quotient algebra $\mathrm{M}_n(A)/J^{\diamondsuit}$ is

$$(A/J) \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$$

which is (isomorphic to) $\mathrm{M}_{rn}(\mathbb{C})$ where $A/J \cong \mathrm{M}_r(\mathbb{C})$.

For 2., since $\mathbb{C}$ is commutative, the action of $\mathbb{C}$ on $A$ can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of $\mathrm{M}_n(\mathbb{C})$ on $\mathrm{M}_n(A)$

$$\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(A) \to \mathrm{M}_n(A), \qquad \mathrm{M}_n(A) \times \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(A).$$

for which the associativity rule

$$(\alpha\theta)\beta = \alpha(\theta\beta) \quad \alpha, \beta \in \mathrm{M}_n(A) \quad \theta \in \mathrm{M}_n(\mathbb{C})$$

is valid.

If $V$ is any representation of $\mathrm{M}_n(A)$, the associativity rule

$$(\alpha\theta)(\beta v) = \alpha[(\theta\beta)v] \qquad \alpha, \beta \in \mathrm{M}_n(A) \quad \theta \in \mathrm{M}_n(\mathbb{C}) \quad v \in V$$

is valid.

Now let $V$ be an irreducible representation of $\mathrm{M}_n(A)$, with $L$ as its null-space. Define a (left) action

$$\mathrm{M}_n(\mathbb{C}) \times V \to V$$

of $\mathrm{M}_n(\mathbb{C})$ on $V$ by proceeding as in the proof of Lemma 3.2 (the "$k$-action for free" lemma), i.e. given $v \in V$, choose $v_1, v_2, \dots, v_r \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathrm{M}_n(A)$ with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r.$$

For $\theta \in \mathrm{M}_n(\mathbb{C})$, define $\theta v$ by

$$\theta v = (\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \ldots + (\theta\alpha_r)v_r.$$

The strict non-degeneracy, Lemmas 3.1 and 3.3, of $V$ implies that $\theta v$ is well-defined as follows. Suppose that $u_1, u_2, \ldots, u_s \in V$ and $\beta_1, \beta_2, \ldots, \beta_r \in \mathrm{M}_n(A)$ are chosen with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_s u_s.$$

If $\alpha$ is any element of $\mathrm{M}_n(A)$, then

$$\begin{aligned}\alpha[(\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \ldots + (\theta\alpha_r)v_r - (\theta\beta_1)u_1 - (\theta\beta_2)u_2 - \ldots - (\theta\beta_s)u_s] \\ = (\alpha\theta)[\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r - \beta_1 u_1 - \beta_2 u_2 - \ldots - \beta_s u_s] = (\alpha\theta)[v - v] = 0.\end{aligned}$$

Use $f\colon A \to \mathrm{M}_n(A)$ to make $V$ into an $A$-module

$$av := f(a)v, \qquad a \in A, \quad v \in V.$$

The actions of $A$ and $\mathrm{M}_n(\mathbb{C})$ on $V$ commute. Thus for each $\theta \in \mathrm{M}_n(\mathbb{C})$, $\theta V$ is a sub-$A$-module of $V$, where $\theta V$ is the image of $v \mapsto \theta v$. Denote by $E_{ij}$ the matrix in $\mathrm{M}_n(\mathbb{C})$ which has 1 for its $(i,j)$ entry and zero for all its other entries. Then, as an $A$-module, $V$ is the direct sum

$$V = E_{11}V \oplus E_{22}V \oplus \ldots \oplus E_{nn}V.$$

Moreover, the action of $E_{ij}$ on $V$ maps $E_{jj}V$ isomorphically (as an $A$-module) onto $E_{ii}V$. Hence as an $\mathrm{M}_n(A) = A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ module, $V$ is isomorphic to $(E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ — where $\mathbb{C}^n$ is the standard representation of $\mathrm{M}_n(\mathbb{C})$, i.e. is the unique irreducible representation of $\mathrm{M}_n(\mathbb{C})$,

$$V \cong (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n.$$

$E_{11}V$ is an irreducible $A$-module since if not $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ would not be an irreducible $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$-module.

If $J$ is the null space (in $A$) of $E_{11}V$, then $J^{\diamondsuit} = J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ is the null space of $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ and this completes the proof. ∎

## 10. Permanence properties

Theorem 10.1. *Denote by $\Gamma$ a finite group. Stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.*

Proposition 10.2. *Given $k_1$-algebras $A \sim A'$ and $k_2$-algebras $B \sim B'$ then*

$$A \otimes B \sim A' \otimes B'$$

*as $k_1 \otimes k_2$-algebras, all tensor products over $\mathbb{C}$.*

*Proof.* We first prove that $A \otimes B \sim A' \otimes B$. Let $\{I_j\}$ be the filtration for $A$, and $\{I'_j\}$ be the filtration for $A'$. Then $\{I_j \otimes B\}$ (resp. $\{I'_j \otimes B\}$) will serve as filtrations for $A \otimes B$ (resp. $A' \otimes B$), in the sense that the maps

$$I_j/I_{j-1} \otimes B \to I'_j/I'_{j-1} \otimes B$$

are spectrum-preserving. A similar argument will prove that $A' \otimes B \sim A' \otimes B'$. Then

$$A \otimes B \sim A' \otimes B \sim A' \otimes B'$$

as required.

Let $\Psi$ be a variation of $k_1$-structure for $A$:

$$k_1 \to Z(A[t, t^{-1}])$$

and let

$$\Psi_B : k_2 \to Z(B)$$

be the $k_2$-structure for $B$. Then

$$\Psi \otimes \Psi_B : k_1 \otimes k_2 \to Z(A[t, t^{-1}]) \otimes Z(B) \simeq Z(A \otimes B)[t, t^{-1}]$$

will be a variation of $k_1 \otimes k_2$-structure for $A \otimes B$. ∎

LEMMA 10.3. *Assume that $A$ and $B$ both have $\Gamma$ acting (as automorphisms of $k$-algebras), and we are given a $\Gamma$-equivariant morphism $\phi : A \to B$ which is spectrum preserving. Then $\phi \rtimes \Gamma : A \rtimes \Gamma \to B \rtimes \Gamma$ is a spectrum preserving morphism of $k$-algebras.*

*Proof.* According to Lemma A.1 in [ABPS], $\mathrm{Irr}(A \rtimes \Gamma), \mathrm{Irr}(B \rtimes \Gamma)$ are in bijection with the (twisted) extended quotients:

$$(\mathrm{Irr}(A)//\Gamma)_\natural \simeq \mathrm{Irr}(A \rtimes \Gamma) \qquad (\mathrm{Irr}(B)//\Gamma)_\natural \simeq \mathrm{Irr}(B \rtimes \Gamma).$$

Since $\phi : A \to B$ is spectrum preserving, the 2-cocycles $\natural$ of $\Gamma$ associated to $\mathrm{Irr}(A)$ and $\mathrm{Irr}(B)$ can be identified. Hence $\phi$ determines a bijection of the (twisted) extended quotients

$$(\mathrm{Irr}(A)//\Gamma)_\natural \longleftarrow (\mathrm{Irr}(B)//\Gamma)_\natural.$$

By applying Lemma A.1 of [ABPS], this implies that $\phi \rtimes \Gamma$ is spectrum preserving. ∎

PROPOSITION 10.4. *If $A$ and $B$ both have $\Gamma$ acting (as automorphisms of $k$-algebras), and we are given a $\Gamma$-equivariant stratified equivalence $A \sim B$, then*

$$A \rtimes \Gamma \sim B \rtimes \Gamma.$$

*Proof.* Suppose now that we are given a $\Gamma$-equivariant morphism of $k$-algebras $\phi : A \to B$ which is, in a $\Gamma$-equivariant way, spectrum preserving with respect to filtrations, i.e. the filtrations of $A$ and $B$ are preserved by the action of $\Gamma$. This gives a map of $k$-algebras $\phi \rtimes \Gamma : A \rtimes \Gamma$ to $B \rtimes \Gamma$. We claim that this map is spectrum-preserving with respect to filtrations.

Using the filtrations, consider the map $I_j/I_{j-1} \to J_j/J_{j-1}$. This is a $\Gamma$-equivariant spectrum preserving map. Hence by Lemma 10.3, the map

$$(I_j/I_{j-1}) \rtimes \Gamma \to (J_j/J_{j-1}) \rtimes \Gamma$$

is spectrum preserving. So the filtrations of $A \rtimes \Gamma$ and $B \rtimes \Gamma$ given by $I_j \rtimes \Gamma$ and $J_j \rtimes \Gamma$ are the required filtrations.

Suppose given a $\Gamma$-equivariant variation of $k$-structure for $A$, i.e. a $\Gamma$-equivariant unital map of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A[t, t^{-1}])$$

where the Γ-action on $k$ is trivial. Due to the triviality of this action, $k$ is mapped to

$$Z(A[t,t^{-1}])^{\Gamma} = (Z(A))^{\Gamma}[t,t^{-1}].$$

By the standard inclusion

$$(Z(A))^{\Gamma} \subset Z(A \rtimes \Gamma)$$

$k$ is mapped to $Z(A \rtimes \Gamma)[t,t^{-1}]$:

$$k \longrightarrow Z(A \rtimes \Gamma)[t,t^{-1}]$$

which is the required variation of $k$-structure for the crossed product $A \rtimes \Gamma$. ∎

**11. Affine Hecke algebras.** Let $(X, Y, R, R^{\vee})$ be a root datum in the standard sense [L, p. 73]. This root datum delivers the following items:

— a finite Weyl group $W_0$,
— an extended affine Weyl group $W_0X := W_0 \ltimes X$,
— for each $q \in \mathbb{C}^{\times}$, an affine Hecke algebra $\mathcal{H}_q$,
— a complex torus $T := \mathrm{Hom}(X, \mathbb{C}^{\times})$,
— a complex variety $X := T/W_0$,
— a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$.

The detailed construction of $\mathcal{H}_q$ is described in the article of Lusztig [L, p. 74]. Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^{\times}$, $\mathcal{H}_q$ is a unital finite type $k$-algebra. If $q = 1$, then $\mathcal{H}_1$ is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X].$$

### 11.1. Stratified equivalence for affine Hecke algebras

THEOREM 11.1. *Except for $q$ in a finite set of roots of unity, none of which is* 1*, $\mathcal{H}_q$ is stratified equivalent to $\mathcal{H}_1$*:

$$\mathcal{H}_q \sim \mathcal{H}_1.$$

*Proof.* Let $J$ be Lusztig's asymptotic algebra [L, §8]. As a $\mathbb{C}$-vector space, $J$ has a basis $\{t_v \mid v \in W_0X\}$, and there is a canonical structure of associative $\mathbb{C}$-algebra on $J$. The algebra $\mathcal{H}_q$ is viewed as a $k$-algebra via the canonical isomorphism

$$\mathcal{O}(T/W_0) \cong Z(\mathcal{H}_q).$$

Lusztig's map $\phi_q$ maps $Z(\mathcal{H}_q)$ to $Z(J)$ by Proposition 1 in [ABP] and thus determines a unique $k$-structure for $J$ such that the map $\phi_q$ is a morphism of $k$-algebras. $J$ with this $k$-structure will be denoted by $J_q$. Let

$$a : W_0X \to \mathbb{N} \cup \{0\}$$

be the weight function defined by Lusztig.

The algebra $J$ has a unit element of the form $1 = \sum_{d \in \mathcal{D}}$ where $\mathcal{D}$ is a certain set of involutions in $W_a$. For any $q \in \mathbb{C}^{\times}$, the $\mathbb{C}$-linear map $\psi_q : \mathcal{H}_q \to J$ defined by

$$\phi_q(C_w) = \sum h_{w,d,v} t_v \tag{1}$$

is a $\mathbb{C}$-linear homomorphism preserving 1. The summation is over the set

$$\{d \in \mathcal{D},\ v \in W_0X \mid a(v) = a(d)\}.$$

The elements $C_w$ are defined in [L, p. 82] and form a $\mathbb{C}$-basis of $\mathcal{H}_q$. The coefficients $h_{w,d,v}$ originate in the multiplication rule for the $C_w$, see [L, p. 82].

The map $\phi_q$ is injective. Thus all algebras $\mathcal{H}_q$ with $q \in \mathbb{C}^\times$ appear as subalgebras of a single $\mathbb{C}$-algebra $J$.

Define

$$\begin{aligned} J_k &= \text{ideal generated by } \{t_v \in J \mid a(v) \le k\} \\ I_k &= \phi_q^{-1}(J_k). \end{aligned}$$

We have the filtrations

$$\begin{aligned} &0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = \mathcal{H}_q \\ &0 \subset J_1 \subset J_2 \subset \ldots \subset J_r = J, \end{aligned} \tag{2}$$

and

$$\mathrm{Irr}(J) = \bigsqcup \mathrm{Irr}(J_k/J_{k-1}), \qquad \mathrm{Irr}(\mathcal{H}_q) = \bigsqcup \mathrm{Irr}(I_k/I_{k-1}).$$

Let $M$ be a simple $\mathcal{H}_q$-module (resp. simple $J$-module). The *weight* $a_M$ of $M$ is defined by Lusztig in [L, p. 82]. Let $\mathrm{Irr}(\mathcal{H}_q)_k$ (resp. $\mathrm{Irr}(J)_k$) denote the set of simple $\mathcal{H}_q$-modules (resp. simple $J$-modules) of weight $k$. It follows from Lusztig's definition that

$$\mathrm{Irr}(J_k/J_{k-1}) = \mathrm{Irr}(J)_k, \qquad \mathrm{Irr}(I_k/I_{k-1}) = \mathrm{Irr}(\mathcal{H}_q)_k.$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_q & \xrightarrow{\phi_q} & J \\ \uparrow & & \uparrow \\ I_k & \xrightarrow{\phi_q|I_k} & J_k \\ \downarrow & & \downarrow \\ I_k/I_{k-1} & \xrightarrow[(\phi_q)_k]{} & J_k/J_{k-1}. \end{array}$$

The middle horizontal map is the restriction of $\phi_q$ to $I_k$. This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D},\ v \in W_0X \mid a(v) = a(d) \le k\}.$$

The bottom horizontal map is middle horizontal map after descent to the quotient $I_k/I_{k-1}$. This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D},\ v \in W_0X \mid a(v) = a(d) = k\}.$$

We now apply one of Lusztig's main theorems, Theorem 8.1 in [L]: Assume that $q \in \mathbb{C}^\times$ is either 1 or is not a root of 1. Let $M'$ be a simple $J$-module of weight $a_{M'}$. The pre-image $\phi_q^{-1}(M')$ will contain a unique subquotient $M$ of maximal weight $a_M = a_{M'}$. Then the map $M' \mapsto M$ is a bijection from the set of simple $J$-modules (up to isomorphism) to the set of simple $\mathcal{H}_q$-modules (up to isomorphism).

We reconcile our construction with Lusztig's bijection by observing that our map

$$(\phi_q)_k : I_k/I_{k-1} \to J_k/J_{k-1}$$

determines a map

$$\mathrm{Irr}(J)_k \to \mathrm{Irr}(\mathcal{H}_q)_k. \tag{3}$$

This map is weight-preserving, by construction, and coincides with the Lusztig bijection. The map (3) is spectrum-preserving by Lusztig's theorem, and so the original map $\phi_q$ is spectrum preserving with respect to the filtrations (2).

$\mathcal{H}_q$ is then stratified equivalent to $\mathcal{H}_1$ by the three elementary steps

$$\mathcal{H}_q \rightsquigarrow J_q \rightsquigarrow J_1 \rightsquigarrow \mathcal{H}_1.$$

The second elementary step (i.e. passing from $J_q$ to $J_1$) is an algebraic variation of $k$-structure with parameter space $\mathbb{C}^\times$. The first elementary step uses Lusztig's map $\phi_q$, and the third elementary step uses Lusztig's map $\phi_1$. Hence (provided $q$ is not in the exceptional set of roots of unity — none of which is 1) $\mathcal{H}_q$ is stratified equivalent to

$$\mathcal{H}_1 = \mathbb{C}[W_0 X] = \mathcal{O}(T) \rtimes W_0. \ \blacksquare$$

REMARK 11.2. In Theorem 11.1, the condition on $q$ can be replaced by the following more precise condition:

$$\sum_{w \in W_0} q^{\ell(w)} \neq 0$$

where $\ell : W_0 X \to \mathbb{N}$ is the length function, see [Xi2, Theorem 3.2].

If $q \neq 1$ then $\mathcal{H}_q$ and $\mathcal{H}_1$ are not Morita equivalent as $k$-algebras, for the exceptional sets are not equal:

$$q \neq 1 \implies \mathfrak{E}(\mathcal{H}_q) \neq \mathfrak{E}(\mathcal{H}_1).$$

**11.2. The affine Hecke algebras of $\mathrm{SL}_2(\mathbb{C})$.** If the root datum

$$(X, Y, R, R^\vee)$$

arises from a connected complex reductive Lie group $G$, then we will write $\mathcal{H}_q(G) = \mathcal{H}_q$.

An interesting situation arises for $\mathcal{H}_q(\mathrm{SL}_2(\mathbb{C}))$. Let $s, t$ be the simple reflections in the (extended) affine Weyl group $W_0 X$. When $q + 1 \neq 0$ there is an isomorphism of $\mathbb{C}$-algebras between $\mathcal{H}_q$ and $\mathcal{H}_1$:

$$\mathcal{H}_q \simeq \mathcal{H}_1$$
$$T_s \mapsto \frac{q+1}{2} s + \frac{q-1}{2}, \qquad T_t \mapsto \frac{q+1}{2} t + \frac{q-1}{2}.$$

On the other hand, when $q \neq 1$, $\mathcal{H}_q$ and $\mathcal{H}_1$ cannot be isomorphic as $k$-algebras, for the exceptional sets are different. Let $T$ denote the standard maximal torus in $\mathrm{SL}_2(\mathbb{C})$:

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \;\middle|\; z \neq 0 \right\}.$$

We have $W_0 = \mathbb{Z}/2\mathbb{Z}$, $k = \mathcal{O}(T/W_0)$ the algebra of $\mathbb{Z}/2\mathbb{Z}$-invariant Laurent polynomials in one variable. Note that the non-trivial element of $W$ acts by

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}.$$

The quotient variety $T/W_0$ comprises unordered pairs $\{z_1, z_2\}$ of nonzero complex numbers which satisfy the equation $z_1 z_2 = 1$.

The exceptional sets are

$$\mathfrak{E}(\mathcal{H}_1) = \{-1,-1\} \sqcup \{1,1\}$$
$$\mathfrak{E}(\mathcal{H}_q) = \{-1,-1\} \sqcup \{q, 1/q\}.$$

As $q$ with $q \neq 1$ is deformed to 1, the point $\{-1,-1\}$ stays fixed, and the point $\{q, 1/q\}$ moves to the point $\{1,1\}$.

**11.3. The affine Hecke algebras of** $\mathrm{SL}_3(\mathbb{C})$**.** The case of $\mathrm{SL}_3(\mathbb{C})$ is considered in §11.7 of Xi [Xi] where it is proved that $\mathcal{H}_q$ and $\mathcal{H}_1$ are not isomorphic as $\mathbb{C}$-algebras whenever $q \neq 1$. As $k$-algebras, $\mathcal{H}_q$ is not Morita equivalent to $\mathcal{H}_1$ whenever $q \neq 1$ because the exceptional set $\mathfrak{E}(\mathcal{H}_q)$ with $q \neq 1$ is not the same as $\mathfrak{E}(\mathcal{H}_1)$.

In contrast to this, except for a finite set of roots of unity (none of which is 1), according to Theorem 11.1, $\mathcal{H}_q$ is stratified equivalent to $\mathcal{H}_1$.

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