STUDY OF MULTIPLE STRUCTURES ON PROJECTIVE SUBVARIETIES

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Abstract. Let $k$ an algebraically closed field, char $k = 0$. We study multiplicity-$r$ structures on varieties for $r \in \mathbb{N}, r \geq 2$. Let $Z$ be a reduced irreducible nonsingular $(N - 2)$-dimensional variety such that $rZ = X \cap F$, where $X$ is a normal $(N - 1)$-fold of degree $n$, $F$ is a smooth $(N - 1)$-fold of degree $m$ in $\mathbb{P}^N$, such that $r \in \mathbb{N}, r \geq 2$, $Z \cap \text{Sing}(X) \neq \emptyset$. There are effective divisors $V$ and $D_1$ on $Z$ such that $O_Z(V - (r - 1)D_1) \simeq \omega_Z(r(-rm - n + (N + 1)r))$, where $\omega_Z$ is the canonical sheaf of $Z$.

Let $Z \subset \mathbb{P}^N$ be a reduced irreducible subvariety of codimension 2. Let $Y$ be an irreducible hypersurface in $\mathbb{P}^N$, $Z \subset Y$. Let $\omega^o_Z$ be the dualizing sheaf of $Z$. Then, there exists a hypersurface $X$ in $\mathbb{P}^N$ such that $Z = Y \cap X$ is a scheme-theoretical complete intersection if and only if

\begin{itemize}
  \item $\omega^o_Z \simeq \omega_{\mathbb{P}^N} \otimes \mathcal{L}^2N_{Z/\mathbb{P}^N}$.
  \item $\deg Y$ divides $\deg Z$.
  \item $\omega^o_Z \simeq O_Z(\deg Y + (\frac{\deg Z}{\deg Y}) - N - 1)$.
\end{itemize}

Introduction. There are two sections in this paper. In Section 1 we study multiplicity-$r$ structures on varieties $r \in \mathbb{N}$.

Let $k$ be an algebraically closed field of characteristic 0. Most of our results could also be extended to char $k = p, p > 0$. Let $Y$ be a nonsingular variety in $\mathbb{P}^N$, with ideal sheaf $I_Y$. A non-reduced structure $\tilde{Y}$ is a multiplicity-$r$ structure on $Y$ if

\begin{itemize}
  \item $I_{\tilde{Y}} \subset I_Y$;
  \item $\tilde{Y}$ is locally a complete intersection;
\end{itemize}

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• \( \tilde{Y} \) has multiplicity \( r \), i.e. for each point \( P \in Y \) and a general hyperplane \( H \) through \( P \), the local intersection multiplicity is

\[
i(P; \tilde{Y}, H) = \dim \frac{O_P}{I(\tilde{Y} \cap H)} = r.
\]

Multiplicity-2 structures on nonsingular varieties appear in several instances; for example, when studying nonsingular curves on a Kummer surface in \( \mathbb{P}^3 \), passing through some of its nodes. To define a multiplicity-2 structure \( \tilde{Y} \) on a codimension 2 nonsingular variety \( Y \), under some conditions, equivalent to defining a subbundle \( L \subset N_Y|_P \), assuming that \( I_Y/I_{\tilde{Y}} \) is locally free. W. Barth gave a construction of the Horrocks–Mumford bundle assuming the existence of a nonsingular irreducible curve with certain properties; among them a double structure on the curve (see [2]). The Horrocks–Mumford bundle is a stable indecomposable rank 2 vector bundle over \( \mathbb{P}^4 \). The study of varieties which are complete intersections with a non-reduced structure on them could be used in the construction of vector bundles in \( \mathbb{P}^n \).

Multiple structures on curves (‘thickenings of curves’) appear in the enumerative geometry of Calabi–Yau \( n \)-folds. In particular, Gromov–Witten theory can be used to define an enumerative geometry of curves in Calabi–Yau 5-folds (see [6]). Also thickenings of rational curves, analytically contractible, on Calabi–Yau 3-folds are used to study the genus zero Gopakumar–Vafa invariants (see [5]).

Let \( Z \) be a reduced irreducible nonsingular \((N - 2)\)-dimensional variety such that \( rZ = X \cap F \), \( r \in \mathbb{N} \), \( r \geq 2 \), where \( F \) is a \((N - 1)\)-fold in \( \mathbb{P}^N \), \( X \) is a normal \((N - 1)\)-fold with \( Z \cap \text{Sing}(X) \neq \emptyset \). We prove that there are effective divisors \( V \) and \( D_1 \) on \( Z \) such that \( O_Z(V - (r - 1)D_1) \simeq \omega_Z^r(-rm - n + (N + 1)r) \), where \( \omega_Z \) is the canonical sheaf of \( Z \).

In Section 2 we give conditions for a reduced irreducible subvariety \( Z \), in \( \mathbb{P}^N \), of codimension 2, to be a scheme-theoretical complete intersection. Our motivation is the extension of a type of Noether–Lefschetz Theorem from surfaces to \( n \)-folds (see [4]).

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1. Let \( Z \) be a reduced irreducible nonsingular \((N - 2)\)-dimensional variety such that \( rZ = X \cap F \), where \( X \) is a normal \((N - 1)\)-fold, \( F \) is a smooth \((N - 1)\)-fold in \( \mathbb{P}^N \), such that \( r \in \mathbb{N} \), \( r \geq 2 \), \( Z \cap \text{Sing}(X) \neq \emptyset \). Let \( m \) be the degree of \( F \) and \( n \) be the degree of \( X \). Let \( I_Z \) be the ideal sheaf of \( Z \) in \( \mathbb{P}^N \).

Let us write \( T = rZ \), \( r \in \mathbb{N} \), \( r \geq 2 \). Consider the Cohen–Macaulay filtration

\[
Z = T_1 \subset T_2 \subset \ldots \subset T_r = rZ, \quad \text{where} \quad T_i = iZ, \ 1 \leq i \leq r.
\]

Let \( J_i = I_Z^i + O_{\mathbb{P}^N}(-F) \), \( 1 \leq i \leq r \). Note that \( I_Z = J_1, I_Z^2 \subset J_2 \subset I_Z \). \( \frac{J_i}{J_{i+1}} \) is a rank 1 locally free \( O_Z \)-module, \( \frac{J_i}{J_{i+1}} \) is a rank 2 locally free \( O_Z \)-module, \( 1 \leq i \leq r \).

We generalize some results of [1].
Let $\mathcal{L} = \frac{I_Z}{J_2}$. We consider the exact sequence

$$0 \to \frac{J_2}{IZ^2} \to \frac{I_Z}{IZ^2} \to \mathcal{L} \to 0 \quad (1)$$

$\mathcal{L}$ is a quotient of $\frac{I_Z}{IZ^2}$, the conormal bundle of $Z$ in $\mathbb{P}^N$. $\frac{I_Z}{IZ^2}$ is a locally free sheaf of rank 2 over $O_Z = O_{\mathbb{P}^N}/I_Z$.

Since $Z$ is contained in the hypersurface $F$ of degree $m$, for $I_F = O_{\mathbb{P}^N}(-m)$, we have

$$0 \to O_{\mathbb{P}^N}(-m) \to I_Z,$$

which, by restriction to $Z$, gives $0 \to O_Z(-m) \to \frac{I_Z}{IZ^2}$, since $\text{Tor}_1(O_{\mathbb{P}^N}(-m), O_Z) = 0$.

We also have

$$0 \to O_{\mathbb{P}^N}(-m) \to J_2 \to I_Z, \quad (2)$$

Restricting (2) to $Z$, we have $O_Z(-m) \to \frac{J_2}{IZ^2} \to \frac{I_Z}{IZ^2}$.

Let $M_1 = \frac{J_2}{IZ^2}$. It is a rank 1 locally free sheaf on $Z$ since so it is $\mathcal{L}$ and (1) is exact.

Let $\alpha : O_Z(-m) \to \frac{I_Z}{IZ^2}$. Let $\delta : \frac{I_Z}{IZ^2} \to \text{Coker} \alpha$.

We also have

$$0 \to O_Z(-m) \to \frac{J_2}{IZ^2} \to \frac{M_1}{O_Z(-m)} \to 0,$$

$$0 \to \frac{M_1}{O_Z(-m)} \to \text{Coker} \alpha \to \mathcal{L} \to 0.$$

Let $\gamma : \text{Coker} \alpha \to \mathcal{L} \to 0$. Let us consider $\gamma \circ \delta$. Hence, there exists an effective divisor $D_1$ on $Z$ such that

$$0 \to O_Z(-m + D_1) \to \frac{I_Z}{IZ^2} \to \mathcal{L} \to 0. \quad (3)$$

Thus, $M_1 = O_Z(D_1 - m)$.

Taking exterior powers in the exact sequence

$$0 \to \frac{I_Z}{IZ^2} \to \Omega_{\mathbb{P}^N} \otimes O_Z \to \omega_Z \to 0,$$

we obtain $\wedge^2(\frac{I_Z}{IZ^2}) \simeq \omega_Z^{-1}(-(N + 1))$.

From the latter and the exact sequence (1),

$$\mathcal{L} = \frac{I_Z}{J_2} \simeq \wedge^2 \left( \frac{I_Z}{IZ^2} \right) \otimes M_1^{-1} \simeq \omega_Z^{-1}(-(N + 1)) \otimes O_Z(m - D_1).$$

We can obtain similar result from (3).

Consider the exact sequence

$$0 \to \frac{I_Z^2}{IZJ_2} \to \frac{J_2}{IZJ_2} \to \frac{J_2}{IZ^2} \to 0,$$

$$\wedge^2 \left( \frac{J_2}{IZJ_2} \right) \simeq \frac{I_Z^2}{IZJ_2} \otimes \frac{J_2}{IZ^2} \simeq \omega_Z^{-2}(-2(N + 1)) \otimes O_Z(m - D_1),$$

since $(\frac{J_2}{IZ})^\otimes \simeq \frac{I_Z^2}{IZJ_2}$ and $(\frac{J_2}{IZ})^\otimes \simeq \omega_Z^{-2}(-2(N + 1)) \otimes O_Z(2m - 2D_1)$.

We also have

$$0 \to O_{\mathbb{P}^N}(-m) \to J_3 \to J_2 \to I_Z, \quad (4)$$
After restricting to $Z$, we have $O_Z(-m) \to \frac{J_3}{I_Z J_3} \to \frac{J_2}{I_Z J_2}$. Let $\mu_1$ be their composition; 
$\mu_1 : O_Z(-m) \to \frac{J_3}{I_Z J_3}$. Consider $\mu_{11} : \frac{J_2}{I_Z J_2} \to \text{Coker } \mu_1$.

Consider the exact sequence

$$0 \to \frac{J_3}{I_Z J_2} \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{J_3} \to 0. \quad (5)$$

Then $M_2 \simeq \frac{J_3}{I_Z J_2}$ is a rank 1 locally free sheaf on $Z$. Let $\mu_{12} : O_Z(-m) \to \frac{J_3}{I_Z J_2}$. The image of $\mu_1$ maps to zero in $\frac{J_2}{J_3}$. We have an exact sequence

$$0 \to \text{Coker } \mu_{12} \to \text{Coker } \mu_1 \to \frac{J_2}{J_3} \to 0.$$ 

Then there exists an effective divisor $D_2$ such that $M_2 = O_Z(D_2 - m)$ since

$$0 \to O_Z(-m) \to M_2 \to \frac{M_2}{O_Z(-m)} \to 0,$$

and

$$0 \to O_Z(-m + D_2) \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{J_3} \to 0.$$ 

$D_2$ is the divisor associated to the torsion subsheaf $\frac{M_2}{O_Z(-m)}$.

We have

$$\frac{J_2}{J_3} \simeq \wedge^2 \left( \frac{J_2}{I_Z J_2} \right) \otimes M_2 \otimes^{-1},$$

$$\frac{J_2}{J_3} \simeq \omega_Z \otimes (-2(N + 1)) \otimes O_Z(2m - D_1 - D_2).$$

Notice that $(\frac{I_Z}{J_2})^2 \simeq \frac{I_Z^2}{J_2}$. We have

$$0 \to \frac{I_Z^2}{J_3} \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{J_3}.$$ 

Consider the exact sequence [5]. There exists an effective divisor $N_1$ on $Z$ such that

$$\frac{J_2}{J_3} \simeq \left( \frac{I_Z}{J_2} \right)^2 (N_1).$$

Let $M_{r-1} \simeq \frac{I_Z}{J_{(r-1)}}$ which is a rank 1 locally free sheaf on $Z$. As in [4], we also have

$$0 \to O_{P^N}(-m) \to J_r \to J_{(r-1)} \to I_Z.$$ 

After restricting to $Z$ the map $0 \to O_{P^N}(-m) \to J_{(r-1)}$, we have $\mu_{(r-2)} : O_Z(-m) \to \frac{J_{(r-1)}}{I_Z J_{(r-1)}}$.

Consider $\mu_{(r-2)1} : \frac{J_{(r-1)}}{I_Z J_{(r-1)}} \to \text{Coker } \mu_{(r-2)}$. Let $\mu_{(r-2)2} : O_Z(-m) \to \frac{J_{(r-1)}}{I_Z J_{(r-1)}}$. The image of $\mu_{(r-2)}$ maps to zero in $\frac{J_{(r-1)}}{J_{(r-1)}}$.

We have an exact sequence $0 \to \text{Coker } \mu_{(r-2)2} \to \text{Coker } \mu_{(r-2)} \to \frac{J_{(r-1)}}{J_{(r-1)}} \to 0$. We deduce that there exists an effective divisor $D_{(r-1)}$ such that $M_{(r-1)} = O_Z(D_{(r-1)} - m)$.

**Proposition 1.** $\frac{J_{(r-1)}}{J_{(r-1)}} \simeq (\frac{I_Z}{J_2})^{\otimes (r-1)}(N_{(r-2)})$, $r \geq 3$, where $N_{(r-2)}$ is an effective divisor on $Z$. Also, $\wedge^2 \left( \frac{J_{(r-1)}}{I_Z J_{(r-1)}} \right) \simeq \omega_Z (-r+1) \otimes O_Z((-r+1)(r-1)) \otimes O_Z((-r+1)m-(r-1)D_1+D_{(r-1)})$, where $D_1$ and $D_{(r-1)}$ are effective divisors on $Z$. 

**Proof.** For $N = 3$, see [1].

Let $N \geq 3$. We have an exact sequence

$$0 \to \frac{J_r}{I_r J_{(r-1)}} \to \frac{J_{(r-1)}}{I_r J_{(r-1)}} \to \frac{J_{(r-1)}}{J_r} \to 0. \quad (6)$$

Notice that $(\frac{I_r}{J_r})^{(r-1)} \simeq \frac{I_r^{(r-1)}}{I_r J_{(r-2)} J_r}$.

$$0 \to \frac{I_r^{(r-1)}}{I_r J_{(r-2)} J_r} \to \frac{J_{(r-1)}}{I_r J_{(r-2)} J_r} \to \frac{J_{(r-1)}}{J_r} \to 0.$$

Since $I_r J_{(r-2)} J_r \subset J_r J_{(r-2)}$ the map $\frac{J_{(r-1)}}{I_r J_{(r-2)} J_r} \to \frac{J_{(r-1)}}{J_r}$ is surjective. Also $\frac{J_{(r-1)}}{J_r} \to \frac{J_{(r-1)}}{J_r} \to 0$, since $J_r J_{(r-2)} \subset J_r$. There exists an effective divisor $N_{(r-2)}$ on $Z$ such that

$$\frac{J_{(r-1)}}{J_r} \simeq \left(\frac{I_r}{J_r}\right)^{(r-1)} (N_{(r-2)}).$$

From (6) we deduce $\frac{J_{(r-1)}}{J_r} \simeq \wedge^2 \left(\frac{I_r J_{(r-1)}}{I_r J_{(r-1)}}\right) \otimes M_{(r-1)}^{-1}$. Thus,

$$\wedge^2 \left(\frac{J_{(r-1)}}{I_r J_{(r-1)}}\right) \simeq \frac{J_{(r-1)}}{J_r} \otimes M_{(r-1)},$$

$$\simeq \omega_Z^{-1}((N + 1)(r - 1)) \otimes O_Z((r - 2)m - (r - 1)D_1 + D_{(r-1)}).$$

**Proposition 2.** Let $Z$ be an irreducible nonsingular $(N - 2)$-subvariety in $\mathbb{P}^N$, $rZ = F \cap X$, $r \in \mathbb{N}$, $r \geq 2$, where $F$ is a smooth hypersurface of degree $m$, $X$ is a normal hypersurface of degree $n$, $Z \cap \text{Sing}(X) \neq \emptyset$. Then $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-m + n - (N + 1))$. There are effective divisors $V$ and $D_1$ on $Z$ such that $O_Z(V - (r - 1)D_1) \simeq \omega_Z^r(-r m - n + (N + 1)r)$.

**Proof.** Since $rZ = F \cap X$, $J_r$ is a locally complete intersection ideal. We see that $\omega_{rZ} \simeq O_{rZ}(m + n - (N + 1))$.

As we have seen in Section 4 letting $D_1$ be the associated divisor to the torsion subsheaf $\frac{M_1}{O_Z(-m)}$, we have $M_1 = O_Z(D_1 - m)$. We shall see that $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-m + n - (N + 1))$.

Let $S = \frac{J_{(r-1)}}{J_r}$. The exact sequence

$$0 \to S \to O_{\mathbb{P}^N} \to O_{\mathbb{P}^N} \to 0$$

induces the surjective map

$$\omega_{rZ} \simeq \mathcal{E}xt^2(O_{rZ}, \omega_{\mathbb{P}^N}) \to \mathcal{E}xt^2(S, \omega_{\mathbb{P}^N}),$$

since $O_{(r-1)Z}$ is locally Cohen–Macaulay and hence $\mathcal{E}xt^3(O_{(r-1)Z}, \omega_{\mathbb{P}^N}) = 0$.

$\mathcal{E}xt^2(S, \omega_{\mathbb{P}^N})$ is a rank 1 locally free sheaf on $Z$.

$$S \simeq \mathcal{E}xt^2(\mathcal{E}xt^2(S, \omega_{\mathbb{P}^N}), \omega_{\mathbb{P}^N}) \simeq \mathcal{E}xt^2((\omega_{rZ})|_Z, \omega_{\mathbb{P}^N})$$

$$\simeq \mathcal{E}xt^2(O_Z, \omega_{\mathbb{P}^N})(-l) \simeq \omega_Z(-l),$$

where $l = m + n - (N + 1)$.  

The functor $\mathcal{E}xt^2(-, \omega_{\mathbb{P}^N})$ is exact and reflexive on the category of Cohen–Macaulay $O_{\mathbb{P}^N}$-modules of codimension 2. Thus, $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-(m+n-(N+1)))$.

By Proposition 4

\[ \omega_Z(-(m+n-(N+1))) \]
\[ \simeq \omega_Z^{r+1}((r-1)(m-(N+1))) \otimes O_Z(-(r-1)D_1) \otimes O_Z(N_{(r-2)}). \]

Let $V := N_{(r-2)}$. It follows $O_Z(V - (r-1)D_1) \simeq \omega_Z^r(-rm - n + (N+1)r)$. ■

2.

**Definition 3.** A closed subscheme $Z \subset \mathbb{P}^N$ of codimension $k$ is a complete intersection if there are $k$ hypersurfaces (i.e. locally principal subschemes of codimension 1) $F_1, F_2, \ldots, F_k$, such that $Z = F_1 \cap \ldots \cap F_k$, as schemes.

**Proposition 4.** Let $Z \subset \mathbb{P}^N$ be a reduced irreducible subvariety of codimension 2. Let $Y$ be an irreducible hypersurface in $\mathbb{P}^N$, $Z \subset Y$. Let $\omega^O_Z$ be the dualizing sheaf of $Z$. Then there exists a hypersurface $X \subset \mathbb{P}^N$ such that $Z = Y \cap X$ is a scheme-theoretical complete intersection if and only if $Z$ satisfies the following properties

- $\omega^O_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 N_Z|_{\mathbb{P}^N}$.
- $\deg Y$ divides $\deg Z$.
- $\omega^O_Z \simeq O_Z(\deg Y + (\frac{\deg Z}{\deg Y}) - N - 1)$.

**Proof.** For $N = 3$ see [1] and [3]. Let us assume $\deg Y = m$. Let $\omega^O_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 N_Z|_{\mathbb{P}^N}$. Let $\frac{\deg Z}{\deg Y} = n$ and $\omega^O_Z \simeq O_Z(m + n - N - 1)$.

Let $N$ be the normal bundle of $Z$ in $\mathbb{P}^N$. Let $\mathcal{E}$ be a rank 2 vector bundle in $\mathbb{P}^N$ obtained from $Z$ via Hartshorne–Serre correspondence so that $Z \subset \mathbb{P}^N$ is the set of zeroes of a section $s \in H^0(\mathbb{P}^N, \mathcal{E})$ and $\mathcal{E}|Z \simeq N$. Let $\mathcal{E}^*$ be the dual of $\mathcal{E}$. Let $I_Z$ be the ideal sheaf of $Z$ in $\mathbb{P}^N$.

We have an exact sequence

\[ 0 \rightarrow \det \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow I_Z \rightarrow 0 \]  \hspace{1cm} (7)

Since $\omega^O_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 N_Z|_{\mathbb{P}^N}$,

\[ \omega^O_Z \simeq \wedge^2 \left( \frac{I_Z}{I_Z^2} \right)^* (-N - 1). \]

Since $\omega^O_Z \simeq O_Z(m + n - N - 1)$,

\[ \wedge^2 \left( \frac{I_Z}{I_Z^2} \right)^* \simeq O_Z(m + n), \]
\[ \wedge^2 \mathcal{E}^*|_Z \simeq \wedge^2 \left( \frac{I_Z}{I_Z^2} \right) \simeq O_Z(-m - n), \]
\[ \wedge^2 \mathcal{E}^* \simeq O_{\mathbb{P}^N}(-m - n). \]

The first Chern class of $\mathcal{E}^*$ is $-m - n$, the second Chern class is $mn$.

From (7) we have

\[ 0 \rightarrow O_{\mathbb{P}^N}(-m - n) \rightarrow \mathcal{E}^* \rightarrow I_Z \rightarrow 0. \]
Tensoring with $- \otimes O_{\mathbb{P}^N}(m)$, we obtain

$$0 \to O_{\mathbb{P}^N}(-n) \to \mathcal{E}^*(m) \to I_Z(m) \to 0.$$ 

Computing cohomology,

$$H^0(\mathcal{E}^*(m)) \to H^0(I_Z(m)) \to 0,$$

since $H^1(O_{\mathbb{P}^N}(-n)) = 0$. Let $s_1 \in H^0(\mathcal{E}^*(m))$ be an element whose image is equal to $Y$ in $H^0(I_Z(m))$. For every effective divisor $D$ in $\mathbb{P}^N$, there exists a commutative diagram

$$H^0(\mathcal{E}^*(m-D)) \xrightarrow{\alpha} H^0(I_Z(m-D)) \xrightarrow{\beta} H^0(\mathcal{E}^*(m))$$

with vertical arrows $\alpha : H^0(\mathcal{E}^*(m-D)) \to H^0(\mathcal{E}^*(m))$ and $\beta : H^0(I_Z(m-D)) \to H^0(I_Z(m))$ for $s_1 \in H^0(\mathcal{E}^*(m))$ is not in $\text{Im} \, \alpha$ since $Y$ is irreducible. Thus, the scheme of zeroes of $s_1$ is empty in codimension 1 components.

$$c_2(\mathcal{E}^*(m)) = c_2(\mathcal{E}^*) + mc_1(\mathcal{E}^*) + m^2 = 0,$$

so the scheme of zeroes of $s_1$ is empty. Note that $c_1(\mathcal{E}^*(m)) = c_1(\mathcal{E}^*) + 2m = m - n$. Thus, $\mathcal{E}^*(m)$ is decomposable into the direct summands $O_{\mathbb{P}^N}$ and $O_{\mathbb{P}^N}(m-n)$.

Hence, we deduce that $\mathcal{E}^*$ is decomposable into the direct summands $O_{\mathbb{P}^N}(-m)$ and $O_{\mathbb{P}^N}(-n)$. Therefore, there exists a hypersurface $X$ of degree $n$ such that $Z = Y \cap X$.

The necessity condition is obvious since $Z = Y \cap X$ is a scheme-theoretical complete intersection.

References


