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A CLOSEDNESS THEOREM OVER HENSELIAN FIELDS WITH ANALYTIC STRUCTURE AND ITS APPLICATIONS

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Abstract. In this brief note, we present our closedness theorem in geometry over Henselian valued fields with analytic structure. It enables, among others, application of resolution of singularities and of transformation to normal crossings by blowing up in much the same way as over locally compact ground fields. Also given are many applications which, at the same time, provide useful tools in geometry and topology of definable sets and functions. They include several versions of the Łojasiewicz inequality, Hölder continuity of definable functions continuous on closed bounded subsets of the affine space, piecewise continuity of definable functions or curve selection. We also present our most recent research concerning definable retractions and the extension of continuous definable functions. These results were established in several successive papers of ours, and their proofs made, in particular, use of the following fundamental tools: elimination of valued field quantifiers, term structure of definable functions and b-minimal cell decomposition, due to Cluckers–Lipshitz–Robinson, relative quantifier elimination for ordered abelian groups, due to Cluckers–Halupczok, the closedness theorem as well as canonical resolution of singularities and transformation to normal crossings by blowing up due to Bierstone–Milman. As for the last tool, our approach requires its definable version established in our most recent paper within a category of definable, strong analytic manifolds and maps.

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1. A closedness theorem. Fix a Henselian non-trivially valued field K of equicharacteristic zero, with analytic structure (cf. [7, 8, 24, 26]) and the analytic language $\mathcal{L}$ being an analytic expansion of the 3-sorted one (Denef–Pas [29]) or the 2-sorted one (Basarab–Kuhlmann [2, 19]).

Denote by v, $\Gamma = \Gamma_K$, K°, $K^{\circ\circ}$ and $\widetilde{K}$ the valuation, value group, valuation ring, maximal ideal and residue field, respectively. By the K-topology on K^n we mean the topology induced by the valuation v. Throughout the paper the word "definable" will mean "definable with parameters".

We begin by stating below the closedness theorem from [26] along with its immediate consequences. It enables, in particular, application of resolution of singularities and of transformation to normal crossings by blowing up in much the same way as over locally compact ground fields.

Theorem 1.1. *Let K be a Henselian valued field with separated analytic structure in the analytic language $\mathcal{L}$. Given an $\mathcal{L}$-definable subset D of K^n, the canonical projection*

$$\pi : D \times (K^\circ)^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times (K^\circ)^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

We immediately obtain two consequences.

Corollary 1.2. *Let D be an $\mathcal{L}$-definable subset of K^n and $\mathbb{P}^m(K)$ stand for the projective space of dimension m over K. Then the canonical projection*

$$\pi : D \times \mathbb{P}^m(K) \longrightarrow D$$

is definably closed.

Corollary 1.3. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or of $(K^\circ)^m$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.*

This theorem was formulated and proven in the case of separated analytic structures, but remains valid, along with the other results from [26], in the case of strictly convergent analytic structures, because every such structure can be extended in a definitional way (extension by Henselian functions) to a separated analytic structure (cf. [8]). Classical examples of such structures are complete, rank one valued fields with the Tate algebra of strictly convergent power series.

The strategy of proof in the analytic settings generally follows the one in the algebraic case from our earlier papers [23, 25]. Here, however, we apply elimination of valued field quantifiers for the theory $T_{\mathrm{Hen},\mathcal{A}}$ along with b-minimal cell decompositions with centers and term structure of definable functions, due to Cluckers–Lipshitz–Robinson [9, 7], as well as relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts), due to Cluckers–Halupczok [6]. Besides, in the proof of the closedness theorem, we rely on the local behavior of definable functions of one variable and on fiber shrinking, being a relaxed version of curve selection.

We now recall the concept of fiber shrinking. Let A be an $\mathcal{L}$-definable subset of K^n with accumulation point

$$a = (a_1, \dots, a_n) \in K^n$$

and E an $\mathcal{L}$-definable subset of K with accumulation point a_1. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \dots, a_n),$$

i.e. for any neighborhood U of $(a_2, \dots, a_n) \in K^{n-1}$, there is a neighborhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, A is itself a fiber shrinking for the subset A of K at an accumulation point $a \in K$.

Proposition 1.4 (Fiber shrinking). *Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

Its proof came down, via elimination of valued field quantifiers, to Lemma 1.5 below. This lemma, in turn, was reduced by relative quantifier elimination for ordered abelian groups to a problem of piecewise linear geometry.

Lemma 1.5. *Let Γ be an ordered abelian group and P be a definable subset of Γ^n. Suppose that $(\infty, \dots, \infty)$ is an accumulation point of P, i.e. for any $\delta \in \Gamma$ the set*

$$\{x \in P : x_1 > \delta, \dots, x_n > \delta\}$$

is non-empty. Then there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \dots, r_n t + \gamma_n) : t \in \Gamma,\ t \geq 0\} \quad \textit{with} \quad r_1, \dots, r_n \in \mathbb{N},$$

passing through a given point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that $(\infty, \dots, \infty)$ is an accumulation point of the intersection $P \cap L$ as well.

In a similar manner, one can obtain the following lemma, which along with Lemma 1.5 is used in the proofs of the theorem on existence of the limit, the extension theorem and curve selection.

Lemma 1.6. *Let P be a definable subset of Γ^n and*

$$\pi : \Gamma^n \to \Gamma, \quad (x_1, \dots, x_n) \mapsto x_1$$

be the projection onto the first factor. Suppose that ∞ is an accumulation point of $\pi(P)$. Then there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \dots, r_n t + \gamma_n) : t \in \Gamma,\ t \geq 0\} \quad \textit{with} \quad r_1, \dots, r_n \in \mathbb{N},\ r_1 > 0,$$

passing through a given point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that ∞ is an accumulation point of $\pi(P \cap L)$ as well.

Note that non-Archimedean analytic geometry over Henselian valued fields has a long history (see e.g. [12, 13, 20, 15, 14, 21, 22, 9, 7, 8]) and that the concept of fields with analytic structure, introduced by Cluckers–Lipshitz–Robinson [9], unifies to a large extent the research of those works.

Section 2 recalls the main applications of the closedness theorem in non-Archimedean geometry established in our paper [26] with annotations concerning their proofs. Finally, in the last section, we state some results concerning definable retractions and the extension of continuous definable functions. They were established in papers [24, 27, 28]. Their proofs rely basically on our closedness theorem and on canonical resolution of singularities and transformation to normal crossings by blowing up due to Bierstone–Milman [3]. As for the last tool, our approach requires its definable version established in our most recent paper [24] within a category of definable, strong analytic manifolds and maps.

2. Applications of the closedness theorem

2.1. Existence of the limit. Below we state the full version of the theorem on existence of the limit, whose proof applies the closedness theorem (cf. [26, Section 5]). Note that the proof of the latter makes use of a weak version of the former (op. cit., Section 4).

THEOREM 2.1. *Let $f : E \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function on a subset E of K, and suppose that 0 is an accumulation point of E. Then there is a finite partition of E into $\mathcal{L}$-definable sets $E_1, \ldots, E_r$ and points $w_1, \ldots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \to 0} f|E_i\,(x) = w_i \quad \text{for} \quad i = 1, \ldots, r.$$

Moreover, there is a neighborhood U of 0 such that each definable set

$$\big\{(v(x), v(f(x))) : x \in (E_i \cap U) \setminus \{0\}\big\} \subset \Gamma \times (\Gamma \cup \{\infty\}),\ i = 1, \ldots, r,$$

is contained in an affine line with rational slope

$$q \cdot l = p_i \cdot k + \beta_i,\ i = 1, \ldots, r,$$

with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

2.2. Piecewise continuity. The proof of the following theorem on piecewise continuity is by induction on the dimension of the source of a given function, and uses the closedness theorem and the basic properties of dimension (op. cit., Section 5).

THEOREM 2.2. *Let $A \subset K^n$ and $f : A \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.*

We immediately obtain

COROLLARY 2.3. *The conclusion of the above theorem holds for any $\mathcal{L}$-definable function $f : A \to K$.*

2.3. The Łojasiewicz inequalities. We recall the following three versions of the Łojasiewicz inequality (op. cit., Section 6).

I. *Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geq v((g_1(x), \ldots, g_m(x))) \quad \text{for all } x \in A.$$

II. *Let $f, g : A \to K$ be two continuous $\mathcal{L}$-definable functions on a locally closed subset A of K^n. If*

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Finally, put

$$\mathcal{D}(f) := \{x \in A : f(x) \neq 0\} \quad \text{and} \quad \mathcal{Z}(f) := \{x \in A : f(x) = 0\}.$$

III. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f)$ to a (unique) continuous $\mathcal{L}$-definable function on A.*

The proofs rely on elimination of valued field quantifiers, relative quantifier elimination for ordered abelian groups and the closedness theorem whereby the Łojasiewicz inequalities are reduced to some problems of piecewise linear geometry.

2.4. Hölder continuity. It is a direct consequence of the first version of the Łojasiewicz inequality (op. cit., Section 6).

PROPOSITION 2.4. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.*

$$s \cdot v(f(x) - f(z)) + \beta \geq v(x - z)$$

for all $x, z \in A$. Equivalently, there is a $C \in |K|$ such that

$$|f(x) - f(z)|^s \leq C \cdot |x - z|$$

for all $x, z \in A$.

We immediately obtain

COROLLARY 2.5. *Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed bounded subset $A \subset K^n$ is uniformly continuous.*

2.5. Curve selection. In the following general version of curve selection for $\mathcal{L}$-definable sets, the domain of the selected curve is, unlike in the classical version, only an $\mathcal{L}$-definable subset of the valuation ring K° (op. cit., Section 7).

PROPOSITION 2.6. *Let A be an $\mathcal{L}$-definable subset of K^p. If a point $a \in K^p$ lies in the closure (in the K-topology) $\operatorname{cl}(A \setminus \{a\})$ of $A \setminus \{a\}$, then there exist an $\mathcal{L}$-definable map $\varphi : K^\circ \longrightarrow K^p$, given by power series from $A^\dagger_{1,0}(K) := K \otimes_{K^\circ} A_{1,0}(K)$, and an $\mathcal{L}$-definable subset E of K° with accumulation point 0 such that*

$$\varphi(0) = a \quad \text{and} \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

The proof applies elimination of valued field quantifiers, resolution of singularities (cf. [3, 30]) and the closedness theorem.

3. Existence of definable retractions. In the recent papers [24, 27, 28], we provided some theorems on the existence of definable retractions. They immediately yield some definable, non-Archimedean versions of the classical theorems on extending continuous functions as the theorems of Tietze–Urysohn or Dugundji (cf. [27] for a longer discussion of this topic). The algebraic case was treated in [27].

The case of analytic structures, determined on complete rank one valued fields K by separated power series, was established in [28]. The proof was based on the following basic tools: elimination of valued field quantifiers (due to Cluckers–Lipshitz–Robinson [22, 9, 7, 8]), embedded resolution of singularities and transforming an ideal to normal crossings by blowing up (due to Bierstone–Milman [3] or Temkin [30]), the technique of quasi-rational and R-subdomains (due to Lipshitz–Robinson [21]) and our closedness theorem [23, 25, 26]. Note, however, that the technique of resolution of singularities cannot be directly applied to the general settings of Henselian fields with separated analytic structure, because the rings $A^{\dagger}_{k,l}(K)$ of power series with two kinds of variables seem to suffer from lack of good algebraic properties. Only the rings $A^{\dagger}_{n,0}(K)$ and $A^{\dagger}_{0,n}(K)$ of power series with one kind of variables enjoy good algebraic properties; they are, for instance, Noetherian, factorial, normal and excellent. This follows from that they fall under the Weierstrass–Rückert theory (cf. [7, Section 5.2] and [4, Section 5.2]).

To go around the above problem, in our last paper [24] we adapted to the definable settings the canonical algorithm for resolution of singularities due to Bierstone–Milman [3]. That algorithm provides a local invariant such that blowing up its maximum strata leads to desingularization or transformation to normal crossings. Actually, we achieved definable versions of resolution of singularities in the hypersurface case ([24, Theorem 3.1]) and of transforming to normal crossings an ideal generated by a finite number of analytic functions ([24, Theorem 3.4]). This was carried out within a category of definable, strong analytic manifolds and maps. It is a strengthening of the weak concept of analyticity determined by a given separated Weierstrass system, which works well within the definable settings.

Remark 3.1. The definable version of the desingularization algorithm for separated analytic structures is of significance because just those structures enable reasonable elimination of valued field quantifiers in the analytic language; and quantifier elimination together with resolution of singularities are powerful tools of geometry. For a discussion of intricacies of non-Archimedean analytic geometry, the reader is referred to our paper [24, Section 5]. We should emphasize yet another advantage of working with the strong analytic settings, namely that we do not need to appeal to the theory of quasi-affinoid subdomains.

By *strong analytic* functions and manifolds, we mean the analytic ones that are definable in the structure K and remain analytic in each field L elementarily equivalent to K in the analytic language $\mathcal{L}_{\mathcal{A}(K)}$. Examples of such functions and manifolds are those obtained by means of the implicit function theorem and the zero loci of definable strong analytic submersions, respectively.

The definable character of the canonical algorithm means that the local desingularization invariant is definable, i.e. it takes only finitely many values (being finite words composed of non-negative rational numbers) and its equimultiple loci are definable. One of the most basic points, standing behind the finitary character of the algorithm, is that every strong analytic function has only finitely many orders of vanishing. This, in turn, can be deduced via a model-theoretic compactness argument.

Below we present the version from [24] formulated in terms of strong analytic manifolds and maps.

THEOREM 3.2. *Consider definable, strong analytic functions $g_1, \dots, g_r$ on a strong analytic manifold M. Let $X := V(g_1, \dots, g_r)$ be their zero locus and A be a closed definable subset of X. Then there exists a definable retraction $X \to A$.*

We immediately obtain

COROLLARY 3.3. *For each closed $\mathcal{L}$-definable subset A of K^n, there exists an $\mathcal{L}$-definable retraction $K^n \to A$.*

The earlier techniques and approaches to the purely topological versions of the above problems cannot be carried over to the definable settings because, among others, non-Archimedean geometry over non-locally compact fields suffers from lack of definable Skolem functions. Perhaps the strongest, purely topological, non-Archimedean results on retractions are those from the papers [11, 17] recalled below.

THEOREM 3.4.

(1) *Any closed subset A of an ultranormal metrizable space X is a retract of X.*
(2) *Any compact metrizable subset A of an ultraregular space X is a retract of X.*

We conclude the paper with the following three comments. First, it is plausible that the results presented in this paper will also hold in more general settings of certain tame non-Archimedean geometries considered in the papers [16] and [5]. Second, my work in non-Archimedean geometry was inspired by the joint paper with J. Kollár [18], which deals with the very concept and extension of continuous hereditarily rational functions on real and p-adic varieties, and the results of which were further carried over to non-locally compact fields in my papers [23, 25].

Finally, a further research might be towards definable Lipschitz retractions and extending definable Lipschitz continuous functions (perhaps with the same Lipschitz constant) over non-locally compact fields. These as yet open problems may be investigated in analytic structures and in the tame non-Archimedean geometries from the papers [16, 5] as well. Extending Lipschitz continuous functions $f : A \to \mathbb{R}$, with the same Lipschitz constant from a subset A of $\mathbb{R}^n$, goes back to McShane and Whitney. The more difficult case of functions with values in $\mathbb{R}^k$ was achieved by Kirszbraun. Aschenbrenner–Fischer [1] obtained a definable version of Kirszbraun's theorem for definably complete expansions of ordered fields. Recently Cluckers–Martin [10] established a p-adic version of Kirszbraun's theorem. They proved it, along with the existence of a definable Lipschitz retraction (with constant 1) for any closed definable subset A of $\mathbb{Q}_p^n$, proceeding with simultaneous induction on the dimension n of the ambient space. To this end, they introduced a certain form of preparation cell decompositions with Lipschitz continuous centers. Besides, their construction of definable retractions makes use of some definable Skolem functions.

Therefore, their approach cannot be directly carried over to geometry over non-locally compact Henselian fields, where cells are no longer finite in number (but parametrized by residue field variables) and definable Skolem functions do not exist in general. The non-locally compact case will certainly require a new approach and ingenious ideas.

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