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A NOTE ON $*(x, y, z)$-SIMPLE RINGS

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Abstract. The main goal of this paper is to prove a correct version of one of the main results in the paper *Note on some ideals of associative rings* by M. Filipowicz, M. Kępczyk [Acta Math. Hungar. 142 (2014), 72–79]. Moreover, we give a new proof of Theorem 8 there.

1. Introduction and preliminaries. All rings in this paper are associative but not necessarily with unity.

We write $I \lhd_t R$ ($I \lhd_l R, I \lhd_r R$), if I is a two-sided ideal (left ideal, right ideal) of a ring R.

For a given ring R, we denote by R^1 the ring obtained by adjoining a unity to R and by R^{op} the ring with the opposite multiplication.

In [3] E. R. Puczyłowski introduced the notion of $*$-ideals, which is related to radical theory of rings. He defined $*$-simple rings (i.e. rings without non-trivial $*$-ideals) which are important examples of unequivocal rings. Later, in [2], the authors considered left $*$-ideals. Finally, in [1] a generalization of $*$-ideals and left $*$-ideals was introduced, i.e. the notion of $*(x, y, z)$-ideals, where $x, y, z \in \{l, r, t\}$.

Definition 1.1 ([1, Definition 1]). Let $x, y, z \in \{l, r, t\}$. A subring I of a ring R such that $I \lhd_x R$ is called a $*(x, y, z)$-*ideal* of R, if $I \lhd_z A$ for every ring A such that $R \lhd_y A$.

In our notation, $*$-ideals and left $*$-ideals are $*(t, t, t)$-ideals and $*(l, l, l)$-ideals, respectively.

A ring containing no non-trivial $*(x, y, z)$-ideals is called a $*(x, y, z)$-*simple* ring. The class of $*(x, y, z)$-simple rings will be denoted by $\mathcal{S}(x, y, z)$.

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In the paper we prove that there exist $*(x,y,z)$-simple rings with zero multiplication which are not algebras over a field. This shows that Lemma 7 in [1], needed in the proof of Theorem 8 in [1], is not true. Next, we present a new, correct proof of Theorem 8 without using Lemma 7.

2. Results. We begin this section with an easy to observe property of $*(x,y,z)$-simple rings.

LEMMA 2.1. *Let $x,y \in \{l,r,t\}$. Then $R \in \mathcal{S}(x,y,r)$ if and only if $R^{\mathrm{op}} \in \mathcal{S}(x',y',l)$, where $l' = r$, $r' = l$, $t' = t$.*

Obviously, there are 27 classes $\mathcal{S}(x,y,z)$, where $x,y,z \in \{l,r,t\}$. The next three facts show that there are 12 classes among them which consist of all associative rings.

PROPOSITION 2.2. *Let $x \in \{l,r,t\}$. Then the class $\mathcal{S}(x,r,l)$ is equal to the class of all rings.*

Proof. Assume R is an associative ring and I is a $*(x,r,l)$-ideal of R, where $x \in \{l,r,t\}$. Note that $R \cong \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} <_r \begin{pmatrix} R^1 & 0 \\ R^1 & 0 \end{pmatrix}$. Since $I \cong \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is a $*(x,r,l)$-ideal of R, $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} <_l \begin{pmatrix} R^1 & 0 \\ R^1 & 0 \end{pmatrix}$. Then $\begin{pmatrix} R^1 & 0 \\ R^1 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^1 I & 0 \\ R^1 I & 0 \end{pmatrix} \subseteq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. The last inclusion implies the equality $R^1 I = 0$. Therefore $I = 0$. This means that $R \in \mathcal{S}(x,r,l)$. ∎

Below we present an immediate consequence of the above proposition and Lemma 2.1.

COROLLARY 2.3. *Let $x \in \{l,r,t\}$. Then the class $\mathcal{S}(x,l,r)$ is equal to the class of all rings.*

Note that $*(x,r,t)$-ideals are $*(x,r,l)$-ideals and $*(x,l,t)$-ideals are $*(x,l,r)$-ideals, for every $x \in \{l,r,t\}$. Therefore the inclusions $\mathcal{S}(x,r,l) \subseteq \mathcal{S}(x,r,t)$ and $\mathcal{S}(x,l,r) \subseteq \mathcal{S}(x,l,t)$ hold. Thus by Proposition 2.2 and Corollary 2.3 we obtain the following result.

COROLLARY 2.4. *Let $x \in \{l,r,t\}$. Then*:

(i) *The class $\mathcal{S}(x,r,t)$ is equal to the class of all rings.*
(ii) *The class $\mathcal{S}(x,l,t)$ is equal to the class of all rings.*

REMARK 2.5. Proposition 2.2 and Corollaries 2.3 and 2.4 show that Lemma 7 in [1] saying that a $*(x,y,z)$-simple ring with zero multiplication is an algebra over a field, for every $x,y,z \in \{l,r,t\}$, is not true.

Now we are ready to present a corrected version of Lemma 7. The proof remains the same as that in [1]. We will denote by $\mathcal{A}$ any class of rings appearing in Proposition 2.2 and Corollary 2.3.

COROLLARY 2.6. *Let $x,y,z \in \{l,r,t\}$, $R \in \mathcal{S}(x,y,z)$ and $R \notin \mathcal{A}$. If R is a ring with zero multiplication, then R is an algebra over a field.*

The next two results allow us to prove Theorem 8 in [1] without using Lemma 7.

LEMMA 2.7. *The following equalities hold*:

(i) $\mathcal{S}(l,t,t) = \mathcal{S}(t,t,t)$.
(ii) $\mathcal{S}(r,t,l) = \mathcal{S}(t,t,l)$.
(iii) $\mathcal{S}(r,t,t) = \mathcal{S}(t,t,t)$.
(iv) $\mathcal{S}(l,t,r) = \mathcal{S}(t,t,r)$.

Proof. (i): Two-sided ideals are left ideals, hence $*(t,t,t)$-ideals are $*(l,t,t)$-ideals. This proves the inclusion $\mathcal{S}(l,t,t) \subseteq \mathcal{S}(t,t,t)$.

To show the reverse inclusion, let $R \in \mathcal{S}(t,t,t)$ and J be a $*(l,t,t)$-ideal of R. Obviously $J <_l R \lhd R$, hence $J \lhd R$. This means that J is a $*(t,t,t)$-ideal of $R \in \mathcal{S}(t,t,t)$. Consequently, either $J = 0$ or $J = R$, so $R \in \mathcal{S}(l,t,t)$.

(ii): Note that $*(t,t;l)$-ideals are $*(r,t;l)$-ideals. Therefore the inclusion $\mathcal{S}(r,t;l) \subseteq \mathcal{S}(t,t;l)$ is clear.

Assume $R \in \mathcal{S}(t,t,l)$ and J is a $*(r,t,l)$-ideal of R. Then $J <_r R \lhd R$ and by the assumption we obtain $J <_l R$. Thus J is a $*(t,t,l)$-ideal of $R \in \mathcal{S}(t,t,l)$. Hence, either $J = 0$ or $J = R$. Finally, $\mathcal{S}(t,t,l) \subseteq \mathcal{S}(r,t,l)$.

Statements (iii) and (iv) directly follow from Lemma 2.1 and the above statements (i) and (ii), respectively. ∎

LEMMA 2.8. *The following equalities hold*:

(i) $\mathcal{S}(t,t;l) = \mathcal{S}(t,t;t)$.
(ii) $\mathcal{S}(t,t,r) = \mathcal{S}(t,t,t)$.

Proof. (i): $*(t,t,t)$-ideals are $*(t,t,l)$-ideals, hence $\mathcal{S}(t,t,l) \subseteq \mathcal{S}(t,t,t)$.

To prove the opposite implication let $R \in \mathcal{S}(t,t,t)$. Then by Theorem 1 in [3] we know that R is either a simple ring or an algebra with zero multiplication over a field. If R is a simple ring, then obviously $R \in \mathcal{S}(t,t,l)$. If R is an algebra with zero multiplication over a field, then applying Lemma 6 in [1], we get again that $R \in \mathcal{S}(t,t,l)$.

Applying Lemma 2.1 and the above statement (i) we obtain (ii). ∎

Using the above results we are able to give a new proof of the following characterization from [1].

THEOREM 2.9 ([1, Theorem 8]). *Let $x,y \in \{l,r,t\}$. Then $R \in \mathcal{S}(x,t,y)$, where $x = y = t$ or $x \neq y$ if and only if R is either a simple ring or an algebra with zero multiplication over a field.*

Proof. Assume $x,y \in \{l,r,t\}$ and $x = y = t$ or $x \neq y$. Lemmas 2.7 and 2.8 imply the equality $\mathcal{S}(x,t,y) = \mathcal{S}(t,t,t)$. Now, the required equivalence follows from Theorem 1 in [3]. ∎

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