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SUPER-REPLICATION ON ILLIQUID MARKETS — SEMISTATIC APPROACH

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Abstract. We investigate the pricing-hedging duality for path dependent European options under model uncertainty in discrete time. The super-replicating portfolio consists of a dynamically traded illiquid risky stock and a static position in vanilla options which can be exercised at maturity. We provide the minimal super-replication price as the supremum of penalized expectations of the payoff over all probability measures which are consistent with observed market prices.

1. Introduction. The problem of the robust super-replication has been an active field of research in mathematical finance over recent years. In contrast to the classical approach, one does not postulate a fixed probability measure $\mathbb{P}$ to describe the future evolution of stock prices. Rather than having a single probabilistic model, one takes into account a whole collection $\mathcal{P}$ of possible models, each model being represented by a probability measure. In discrete time, the duality results for super-replication with respect to the family of probability measure or super-replication in a path-wise sense was shown by, for example, Bouchard and Nutz in [4], Bayraktar and Zhou in [3], Burzoni, Frittelli, Hou, Maggis and Obłój in [6], Cheredito, Kupper and Tangpi in [7]. In the robust approach, one includes finitely many options whose price are known at time zero, which may be available for trading. Pioneering work in the semistatic approach was done by Hobson in [10]; we refer to the papers Hobson [11], Dolinsky and Soner [9], Burzoni, Frittelli and Maggis in [5], Acciaio, Beiglböck, Penkner and Schachermayer in [1]. In this setup, additional market instruments reduce the set of martingale measure, which may be used for pricing.

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The aim of the paper is to study the combined effects of model uncertainty and illiquidity costs following the semistatic approach. As in [2], we consider the discrete time model with uncertain volatility where the only assumption on the stock's price evolutions is that the stock's returns are bounded. The illiquidity effects are captured by convex transaction costs. In our version of the duality for European options with continuous payoff, we may reduce the super-hedging cost by including (liquid) derivatives in the super-replicating portfolio. We provide the minimal super-replication price as the supremum of penalized expectations of the payoff over all admissible probability measures, i.e. measures consistent with observed market prices.

The paper consists of two sections devoted to the pricing-hedging duality with illiquidity costs in discrete models and general uncertain volatility models respectively. In Section 2 we extend and modify the result of [8]. In the semistatic setting, we give the formula for the minimal super-replication price for the case of a finite space of possible scenarios. The main tool that is used in this section is the Kuhn–Tucker theory for convex optimization. Theorem 1.3, which is a general duality result for the super-replication prices of path dependent European options, is proved in Section 3. This is a modification of Bank, Dolinsky and Gökay result from [2], which allows us to reduce the super-replication price by including additional derivatives that may be available for trading.

We consider a discrete time financial market on which one can buy or sell three classes of instruments:

- The risk free asset (the *savings account*) with price B_n, $n = 0, 1, \dots, N$: its return is constant and initially known. The savings account will be used as a *numéraire* and without loss of generality we can assume that the bank interest rate is equal to zero.
- The risky asset (*stock*): their returns are not known in advance. Our sole assumption is that stock price returns are in the range specified by fixed volatility bounds. The stock price at time n will be denoted by $S_n > 0$ for $n = 0, 1, \dots, N$. The log-return for period n will be denoted by $X_n := \ln(\frac{S_n}{S_{n-1}})$, so we can write
$$S_n = s_0 \exp\Big(\sum_{m=1}^{n} X_m\Big) \tag{1}$$
for $n = 1, \dots, N$ where s_0 is the deterministic price of stock at time 0.
- The set of vanilla options $\{\varphi_i : \mathbb{R}_+^{N+1} \to \mathbb{R},\ i \in I\}$ written on the underlying asset with fixed maturity N and initially known prices. Without loss of generality we can assume that they can be bought at price 0.

Following [2], we assume that
$$\underline{\sigma} \le |X_n| \le \overline{\sigma}, \quad n = 1, \dots, N,$$
for some constants $0 \le \underline{\sigma} \le \overline{\sigma} < \infty$. Consider the path-space
$$\Omega_{\underline{\sigma},\overline{\sigma}} := \{\omega = (x_1, x_2 \dots, x_N) \in \mathbb{R}^N : \underline{\sigma} \le |x_n| \le \overline{\sigma},\ n = 1, 2, \dots, N\},$$
the canonical functions $X_n : \Omega_{\underline{\sigma},\overline{\sigma}} \to \mathbb{R}$
$$X_n(\omega) := x_n$$

for $n = 1, \dots, N$ and the canonical filtration

$$\mathcal{F}_n := \sigma(X_m : m \le n), \quad n = 0, 1, \dots, N.$$

Notice that by (1) we can view the stocks price evolution as the stochastic process defined on the probability space $(\Omega_{\underline{\sigma},\overline{\sigma}}, \mathcal{F}_N, \mathbb{P})$ for any probability measure $\mathbb{P}$ defined on $\mathcal{F}_N$. Moreover, the canonical filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0,1,\dots,N}$ coincides with the filtration generated by S.

Next, we introduce a cost function

$$g : \{0, 1, \dots, N\} \times \Omega_{\underline{\sigma},\overline{\sigma}} \times \mathbb{R} \to \mathbb{R}_+$$

where $g(n, \omega, \beta)$ denotes the costs of trading $\beta \in \mathbb{R}$ worth of stock at time n with the returns given by ω. The cost function g is assumed to be

(i) nonnegative, $\mathbb{F}$-adapted, and such that $g_n(0) = 0$ for $n = 0, 1, \dots, N$,
(ii) convex with respect to $\beta \in \mathbb{R}$ for any fixed $\omega \in \Omega_{\underline{\sigma},\overline{\sigma}}$,
(iii) continuous with respect to $\omega \in \Omega_{\underline{\sigma},\overline{\sigma}}$ for any fixed $\beta \in \mathbb{R}$.

The convexity of the cost function captures the fact that the unit prices of stock depend not only on the sign (buy or sell) but also on the quantity of a trade.

Let $G_n : \Omega_{\underline{\sigma},\overline{\sigma}} \times \mathbb{R} \to \mathbb{R}_+$, $n = 0, 1, \dots, N$, be the Legendre–Fenchel transform of g_n, that is,

$$G(\omega, \alpha) := \sup_{\beta \in \mathbb{R}} \{\alpha\beta - g(\omega, \beta)\}, \ \forall\, (\omega, \alpha) \in \Omega_{\underline{\sigma},\overline{\sigma}} \times \mathbb{R}.$$

A semistatic trading strategy π consists of

- the dynamic part which is a pair (v, γ), where v denotes the initial capital and $\gamma : \{1, 2, \dots, N\} \times \Omega_{\underline{\sigma},\overline{\sigma}} \to \mathbb{R}$ is a $(\mathcal{F}_n)$-predictable process describing the number $\gamma(n, \omega) = \gamma_n(\omega)$ of shares held at the beginning of any period $n - 1$ with the stocks price evolution given by $\omega \in \Omega_{\underline{\sigma},\overline{\sigma}}$,
- the static part: constants $\delta_1, \delta_2, \dots, \delta_l$ and indices $i_1, i_2, \dots, i_l \in I$, where δ_l is the number of i_l-th option bought at time 0.

By $\mathcal{A}(v)$ we denote the set of all semistatic strategies starting with the initial capital v. The wealth process $V^\pi = (V_n^\pi(\omega))_{n=0,\dots,N}$ generated by the trading strategy $\pi \in \mathcal{A}(v)$ is given by

$$\begin{aligned}
V_0^\pi &= v \\
V_{n+1}^\pi &= V_n^\pi + \gamma_{n+1}(S_{n+1} - S_n) - g_n((\gamma_{n+1} - \gamma_n)S_n), \quad n = 0, 1, \dots, N-2 \\
V_N^\pi &= V_{N-1}^\pi + \gamma_N(S_N - S_{N-1}) - g_{N-1}((\gamma_N - \gamma_{N-1})S_{N-1}) + \sum_{l=1}^{L} \delta_l \varphi_{i_l}(S)
\end{aligned}$$

with $\gamma_0 = 0$.

Consider a European option $\Phi : \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ (possible path-dependent) which pays off at time N:

$$\Phi(S_0, S_1, \dots, S_N) : \Omega_{\underline{\sigma},\overline{\sigma}} \to \mathbb{R}_+.$$

In a pointwise framework the formal definition of the super-replication price is stated as follows.

DEFINITION 1.1. The *super-replication price* $p_{\underline{\sigma},\overline{\sigma}}(\Phi)$ of a contingent claim Φ is defined as

$$p_{\underline{\sigma},\overline{\sigma}}(\Phi) := \inf\{v \in \mathbb{R} \,|\, \exists\, \pi \in \mathcal{A}(v) : V_N^{\pi}(\omega) \geq \Phi(S(\omega)) \;\forall\, \omega \in \Omega_{\underline{\sigma},\overline{\sigma}}\}.$$

Naturally, including options that can be bought at price 0 constrains the set of probability measure.

DEFINITION 1.2. The set of *admissible measures* is defined as

$$\mathcal{P}_{(\varphi_i)_{i\in I}} := \{\mathbb{P} \in \mathcal{P}_{\underline{\sigma},\overline{\sigma}} : \mathbb{E}_{\mathbb{P}}[\varphi_i(S)] = 0,\; i \in I\},$$

where $\mathcal{P}_{\underline{\sigma},\overline{\sigma}}$ denotes the set of all Borel probability measures on $\Omega_{\underline{\sigma},\overline{\sigma}}$.

Our main result provides a dual characterization of super-replication prices.

THEOREM 1.3. *The super-replication price of any European option with continuous payoff Φ is given by*

$$p_{\underline{\sigma},\overline{\sigma}}(\Phi) = \sup_{\mathbb{P}\in\mathcal{P}_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\bigg[\Phi(S) - \sum_{n=0}^{N-1} G_n\bigg(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\bigg)\bigg].$$

This theorem is proved in Section 3.

2. Discrete models. Let us start with the special case where the set of all possible evolution of stock prices is finite. Consider the N-step multinomial market model with a finite path-space

$$\Omega^k := \Big\{x = (x_1, \ldots, x_N) \in \Omega_{\underline{\sigma},\overline{\sigma}} : |x_n| = \frac{j}{k}\,\underline{\sigma} + \Big(1 - \frac{j}{k}\Big)\overline{\sigma} \text{ for some } j \in \{0, \ldots, k\}\Big\}.$$

Let $\mathcal{P}^k_{(\varphi_i)_{i\in I}} \subset \mathcal{P}_{(\varphi_i)_{i\in I}}$ denote the set of those discrete probability measures that are supported by Ω^k such that $\mathbb{E}_{\mathbb{P}}[\varphi_i(S)] = 0$, $i \in I$.

We now provide a pricing-hedging duality in discrete models.

THEOREM 2.1. *Let $p^k_{\underline{\sigma},\overline{\sigma}}(\Phi)$ denote the super-replication price of a contingent claim with payoff Φ on Ω^k, that is,*

$$p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) = \inf\big\{v \in \mathbb{R} : V_N^{(v,\gamma)}(\omega) \geq \Phi(S(\omega)) \;\forall\, \omega \in \Omega^k_{\underline{\sigma},\overline{\sigma}} \text{ for some strategy } \gamma\big\}.$$

Then for any continuous payoff function Φ

$$p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) = \sup_{\mathbb{P}\in\mathcal{P}^k_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\bigg[\Phi(S) - \sum_{n=0}^{N-1} G_n\bigg(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\bigg)\bigg].$$

Proof. The proof is a semistatic modification of the analogous result in [8] where the binomial model with one risky asset and transaction costs was considered. Consider a tree whose paths are sequences $u = (u_1, \ldots, u_n) \in (A^k)$ for $0 \leq n \leq N$ where

$$A^k := \Big\{a : |a| = \frac{j}{k}\underline{\sigma} + \Big(1 - \frac{j}{k}\Big)\overline{\sigma} \text{ for } j = 0, 1, \ldots, k\Big\}$$

for some $k \in \mathbb{N}$. The set of all paths is denoted by U. The root of the tree which corresponds to empty path is denoted by $\emptyset$. Each path of the form $u = (u_1, \ldots, u_n) \in (A^k)^n$ has $2(k+1)$ immediate successors $u = (u_1, \ldots, u_n, a)$ for $a \in A^k$, and one immediate predecessor $u = (u_1, \ldots, u_{n-1})$. Let $l(u)$ be the number of elements in the sequence u.

By $u^+ = \{(u_1, \dots, u_{l(u)}, a), a \in A\}$ we denote the set of all immediate successors of u and by $u^- = (u_1, \dots, u_{l(u)-1})$ — the unique immediate predecessor of u. By T we denote the set of all paths with length N and for $u \in U \setminus T$ by $T(u) := \{w \in T : u_n = w_n$ for $1 \le n \le l_u\}$ — the set of all paths with length N such that the first l_u elements coincide with u, that is the subtree which start with u.

Finally, we define the functions $S : U \to \mathbb{R}$ by

$$S(u) = s_0 \exp\Big(\sum_{m=1}^{l_u} u_m\Big).$$

In our model, the super-replication price $p^k_{\underline{\sigma},\overline{\sigma}}(\Phi)$ is the solution of an ordinary convex program of the following form:

$$\text{minimize } Y(\emptyset) \tag{2}$$

over all $(\beta, \gamma, \delta, Y) \in \mathbb{R}^{U\setminus\{T\}} \times \mathbb{R}^U \times \mathbb{R}^I \times \mathbb{R}^U$ subject to constraints

$$\gamma(\emptyset) = 0, \tag{3}$$

$$[\gamma(u) - \gamma(u^-)]S(u^-) - \beta(u^-) = 0 \quad \forall\, u \in U \setminus \{\emptyset\}, \tag{4}$$

$$Y(u) + g(l(u^-), u^-, \beta(u^-)) - \gamma(u)(S(u) - S(u^-)) - Y(u^-) \le 0 \quad \forall\, u \in U \setminus \{\emptyset\}, \tag{5}$$

$$\Phi(S(u)) \le Y(u) + \sum_{j=1}^{J} \delta_j \varphi_{i_j}(S(u)) \quad \forall\, u \in T. \tag{6}$$

Following the Lagrange multipliers theory we transform the optimization problem given by (2)–(6) into the optimization problem involving fewer constraints but more variables. Define the Lagrangian $L : \mathbb{R}^U \times \mathbb{R}_+^{U\setminus\{\emptyset\}} \times \mathbb{R}_+^T \times \mathbb{R}^{U\setminus T} \times \mathbb{R}^U \times \mathbb{R}^I \times \mathbb{R}^U \to \mathbb{R}$ by

$$\begin{aligned}
&L(\Upsilon, \Psi, \Theta, \beta, \gamma, \delta, Y) \\
&\quad = Y(\emptyset) + \Upsilon(\emptyset)\gamma(\emptyset) + \sum_{u\in U\setminus\{\emptyset\}} \Upsilon(u)\big[(\gamma(u) - \gamma(u^-))S(u^-) - \beta(u^-)\big] \\
&\qquad + \sum_{u\in U\setminus\{\emptyset\}} \Psi(u)\big[Y(u) + g(l(u^-), u^-, \beta(u^-)) - \gamma(u)(S(u) - S(u^-)) - Y(u^-)\big] \\
&\qquad + \sum_{u\in T} \Theta(u)\Big[\Phi(S(u)) - Y(u) - \sum_{j=1}^{J} \delta_j \varphi_{i_j}(S(u))\Big].
\end{aligned}$$

Firstly note that

$$\begin{aligned}
&\sum_{u\in U\setminus\{\emptyset\}} \Upsilon(u)[(\gamma(u) - \gamma(u^-))S(u^-) - \beta(u^-)] \\
&\qquad = \sum_{u\in U\setminus\{\emptyset\}} \Upsilon(u)\gamma(u)S(u^-) - \sum_{u\in U\setminus T} \sum_{\tilde u\in u^+} \Upsilon(\tilde u)\gamma(u)S(u) \\
&\qquad\qquad - \sum_{u\in\emptyset^+} \Upsilon(u)\gamma(\emptyset)S(\emptyset) - \sum_{u\in U\setminus T} \sum_{\tilde u\in u^+} \Upsilon(\tilde u)\beta(u).
\end{aligned}$$

By applying the similar transformation of the next sum we get

$$\begin{aligned}
&L(\Upsilon,\Psi,\Theta,\beta,\gamma,\delta,Y)\\
&\quad= Y(\emptyset)+\Upsilon(\emptyset)\gamma(\emptyset)+\sum_{u\in U\setminus\{\emptyset\}}\Upsilon(u)\gamma(u)S(u^-)-\sum_{u\in U\setminus T}\sum_{\tilde u\in u^+}\Upsilon(\tilde u)\gamma(u)S(u)\\
&\qquad+\sum_{u\in\emptyset^+}\Upsilon(u)\gamma(\emptyset)S(\emptyset)-\sum_{u\in U\setminus T}\sum_{\tilde u\in u^+}\Upsilon(\tilde u)\beta(u)\\
&\qquad+\sum_{u\in U\setminus(\{\emptyset\}\cup T)}\Psi(u)Y(u)+\sum_{u\in T}\Psi(u)Y(u)+\sum_{u\in U\setminus T}\sum_{\tilde u\in u^+}\Psi(\tilde u)g(l(u),u,\beta(u))\\
&\qquad-\sum_{u\in U\setminus\{\emptyset\}}\Psi(u)\gamma(u)(S(u)-S(u^-))-\sum_{u\in U\setminus T}\sum_{\tilde u\in u^+}\Psi(\tilde u)Y(u)-\sum_{u\in\emptyset^+}\Psi(u)Y(\emptyset)\\
&\qquad+\sum_{u\in T}\Theta(u)\Phi(S(u))-\sum_{u\in T}\Theta(u)Y(u)-\sum_{u\in T}\Theta(u)\Big[\sum_{j=1}^{J}\delta_j\varphi_{i_j}(S(u))\Big].
\end{aligned}$$

Now rearrange the above expression in the following way

$$\begin{aligned}
&L(\Upsilon,\Psi,\Theta,\beta,\gamma,\delta,Y)\\
&\quad= Y(\emptyset)\Big(1-\sum_{u\in\emptyset^+}\Psi(u)\Big)+\sum_{u\in U\setminus(\{\emptyset\}\cup T)}Y(u)\Big(\Psi(u)-\sum_{\tilde u\in u^+}\Psi(\tilde u)\Big)\\
&\qquad+\sum_{u\in T}Y(u)[\Psi(u)-\Theta(u)]+\gamma(\emptyset)\Big[\Upsilon(\emptyset)-\sum_{u\in\emptyset^+}\Upsilon(u)S(\emptyset)\Big]\\
&\qquad+\sum_{u\in U\setminus\{\emptyset\}}\gamma(u)\Big[\Upsilon(u)S(u^-)-\sum_{\tilde u\in u^+}\Upsilon(\tilde u)S(u)-\Psi(u)(S(u)-S(u^-))\Big] \qquad (7)\\
&\qquad+\sum_{u\in T}\Theta(u)\Phi(S(u))-\sum_{u\in T}\Theta(u)\Big[\sum_{j=1}^{J}\delta_j\varphi_{i_j}(S(u))\Big]\\
&\qquad+\sum_{u\in U\setminus T}\sum_{\tilde u\in u^+}\big[\Psi(\tilde u)g\big(l(u),u,\beta(u)\big)-\Upsilon(\tilde u)\beta(u)\big].
\end{aligned}$$

By Theorem 28.4 in [12], we conclude that the value of the optimization problem (2)–(6) is equal to

$$p^k_{\underline{\sigma},\overline{\sigma}}(\Phi)=\sup_{(\Upsilon,\Psi,\Theta)\in\mathbb{R}^U\times\mathbb{R}_+^{U\setminus\{\emptyset\}}\times\mathbb{R}_+^T}\ \inf_{(\beta,\gamma,\delta,Y)\in\mathbb{R}^{U\setminus T}\times\mathbb{R}^U\times\mathbb{R}^I\times\mathbb{R}^U}L(\Upsilon,\Psi,\Theta,\beta,\gamma,Y). \qquad (8)$$

Using (7) and (8) we conclude that

$$\begin{aligned}
p^k_{\underline{\sigma},\overline{\sigma}}(\Phi)=\sup_{(\Upsilon,\Psi,\Theta)\in D}\ \inf_{\beta\in\mathbb{R}^{U\setminus T}}\Big(&\sum_{u\in T}\Theta(u)\Phi(S(u))\\
&+\sum_{u\in U\setminus T}\Big(\sum_{\tilde u\in u^+}\Psi(\tilde u)g(l(u),u,\beta(u))-\sum_{\tilde u\in u^+}\Upsilon(\tilde u)\beta(u)\Big)\Big) \qquad (9)
\end{aligned}$$

where $D \subset \mathbb{R}^U \times \mathbb{R}_+^{U\setminus\{\emptyset\}} \times \mathbb{R}_+^T$ satisfies the constraints

$$\sum_{u\in\emptyset^+} \Psi(u) = 1, \tag{10}$$

$$\sum_{\tilde u\in u^+} \Psi(\tilde u) = \Psi(u) \quad \forall\, u \in U \setminus (\{\emptyset\} \cup T), \tag{11}$$

$$\Upsilon(u)S(u^-) = \sum_{\tilde u\in u^+} \Upsilon(\tilde u)S(u) + \Psi(u)(S(u) - S(u^-)) \quad \forall\, u \in U \setminus \{\emptyset\}\ , \tag{12}$$

$$\Theta(u)\varphi_i(S(u)) = 0 \quad \forall\, u \in T \quad \forall\, i \in I, \tag{13}$$

$$\Psi(u) = \Theta(u) \quad \forall\, u \in T. \tag{14}$$

We will prove by induction on n that (11) and (12) imply that for any $u \in U \setminus T$

$$\sum_{\tilde u\in u^+} \Upsilon(\tilde u)S(u) = \sum_{\tilde u\in T(u)} \Psi(\tilde u)S(\tilde u) - \Psi(u)S(u). \tag{15}$$

It follows that for any $(\Upsilon, \Psi, \Theta) \in D$

$$\frac{\sum_{\tilde u\in u^+} \Upsilon(\tilde u)}{\sum_{\tilde u\in u^+} \Psi(\tilde u)} = \frac{\sum_{\tilde u\in T(u)} \Psi(\tilde u)S(\tilde u)}{\Psi(u)S(u)} - 1 \tag{16}$$

for any $u \in U \setminus T$. This together with (9), (14) and (16) yields that

$$\begin{aligned} p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) &= \sup_{\Psi\in\bar D} \inf_{\beta\in\mathbb{R}^{U\setminus T}} \Bigg(\sum_{u\in T} \Psi(u)\Bigg[\Phi(S(u)) \\ &\quad + \sum_{u\in U\setminus T}\Bigg(g(l(u), u, \beta(u)) - \Bigg(\frac{\sum_{\tilde u\in T(u)} \Psi(\tilde u)S(\tilde u)}{\Psi(u)S(u)} - 1\Bigg)\beta(u)\Bigg)\Bigg]\Bigg) \\ &= \sup_{\Psi\in\bar D}\Bigg(\sum_{u\in T} \Psi(u)\Bigg[\Phi(S(u)) - \sum_{u\in U\setminus T} G\Bigg(l(u), u, \frac{\sum_{\tilde u\in T(u)} \Psi(\tilde u)S(\tilde u)}{\Psi(u)S(u)} - 1\Bigg)\Bigg]\Bigg) \end{aligned}$$

where $\bar D$ denotes the set of functions $\Psi \in \mathbb{R}_+^{U\setminus\{\emptyset\}}$ which satisfy (10), (11) and

$$\Psi(u)\varphi_i(S(u)) = 0 \quad \forall\, u \in T \quad \forall\, i \in I.$$

Since we can identify any $\Psi \in \bar D$ with the function $\Psi : U \setminus \{\emptyset\} \to \mathbb{R}_+$ with probability measure $\mathbb{P}$ defined on Ω^k such that $\mathbb{E}_{\mathbb{P}}[\varphi_i(S)] = 0$, $i \in I$, we conclude that

$$p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) = \sup_{\mathbb{P}\in\mathcal{P}^k_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\Bigg[\Phi(S) - \sum_{n=0}^{N-1} G_n\Bigg(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\Bigg)\Bigg].$$

It remains to prove (15). Note that for $u \in T$ (i.e. $l(u) = N$) by (12) we have

$$\Upsilon(u)S(u^-) = \Psi(u)(S(u) - S(u^-)). \tag{17}$$

As a first step, we take $u \in U \setminus T$ such that $l(u) = N - 1$. By (12)

$$\Upsilon(u)S(u^-) = \Psi(u)(S(u) - S(u^-)) + \sum_{a\in A^k} \Upsilon(u, a)S(u)$$

and by (17), for any $a \in A^k$,

$$\Upsilon(u, a)S(u) = \Psi(u, a)(S(u, a) - S(u)).$$

Hence,

$$\begin{aligned}\Upsilon(u)S(u^-) &= \Psi(u)(S(u)-S(u^-)) + \sum_{a\in A^k}\Psi(u,a)(S(u,a)-S(u))\\ &= \Psi(u)(S(u)-S(u^-)) + \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) - \sum_{\tilde u\in u^+}\Psi(\tilde u)S(u)\\ &= \Psi(u)(S(u)-S(u^-)) + \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) - \Psi(u)S(u),\end{aligned}$$

where the last equality holds by (11). On the other hand, by (12):

$$\Upsilon(u)S(u^-) = \Psi(u)(S(u)-S(u^-)) + \sum_{\tilde u\in u^+}\Upsilon(\tilde u)S^i(u),$$

which means that

$$\sum_{\tilde u\in u^+}\Upsilon(\tilde u)S(u) = \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) - \Psi(u)S(u)$$

for any u such that $l(u)=N-1$. Assume (15) for u with the length $k+1$. We want to show (15) for any u such that $l(u)=k$. From (12) and the induction assumption, we get

$$\begin{aligned}\Upsilon(u)S(u^-) &= \Psi(u)(S(u)-S(u^-)) + \sum_{a\in A^k}\Upsilon(u,a)S(u) = \Psi(u)(S(u)-S(u^-))\\ &\quad + \sum_{a\in A^k}\Psi(u,a)(S(u,a)-S(u)) + \sum_{a\in A^k}\sum_{\tilde u\in T(u,a)}\Psi(\tilde u)S(\tilde u) - \sum_{a\in A^k}\Psi(u,a)S(u,a).\end{aligned}$$

Notice that

$$\sum_{a\in A^k}\sum_{\tilde u\in T(u,a)}\Psi(\tilde u)S(\tilde u) = \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) \quad\text{and}\quad \sum_{a\in A^k}\Psi(u,a)S(u) = \Psi(u)S(u),$$

hence

$$\Upsilon(u)S(u^-) = \Psi(u)(S(u)-S(u^-)) + \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) - \Psi(u)S(u),$$

so (by using (12) again)

$$\sum_{\tilde u\in u^+}\Upsilon(\tilde u)S(u) = \sum_{\tilde u\in T(u)}\Psi(\tilde u)S(\tilde u) - \Psi(u)S(u)$$

for u of length k. This completes the proof of (15). ■

3. Proof of the main result. In this section, we carry out the proof of the Theorem 1.3.

We first prove one of the inequalities of the duality result. Let $\pi=(v,\gamma,\delta)$ be the semistatic super-replicating strategy for the payoff Φ. Then

$$\begin{aligned}v &\ge \Phi(S) - \sum_{n=0}^{N-1}\gamma_{n+1}(S_{n+1}-S_n) + \sum_{n=0}^{N-1}g_n((\gamma_{n+1}-\gamma_n)S_n) - \sum_{l=1}^{L}\delta_l\varphi_{i_l}(S)\\ &= \Phi(S) - \sum_{n=0}^{N-1}(\gamma_{n+1}-\gamma_n)(S_N-S_n) + \sum_{n=0}^{N-1}g_n((\gamma_{n+1}-\gamma_n)S_n) - \sum_{l=1}^{L}\delta_l\varphi_{i_l}(S)\end{aligned}$$

since $\gamma_0 = 0$. Let $\mathbb{P}$ be an admissible probability measure in $\mathcal{P}_{(\varphi_i)_{i\in I}}$. By taking the conditional expectations with respect to $\mathbb{P}$ and using the definition of the dual functions G_n, $n = 0, 1, \dots, N$, we get

$$v \geq \mathbb{E}_{\mathbb{P}}\Big[\Phi(S) - \sum_{n=0}^{N-1}((\gamma_{n+1} - \gamma_n)S_n)\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n} + \sum_{n=0}^{N-1} g_n((\gamma_{n+1} - \gamma_n)S_n)\Big]$$
$$\geq \mathbb{E}_{\mathbb{P}}\Big[\Phi(S) - \sum_{n=0}^{N-1} G_n\Big(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\Big)\Big].$$

Since $\mathbb{P} \in \mathcal{P}_{(\varphi_i)_{i\in I}}$ is arbitrary and v is an arbitrary initial wealth of the super-replicating strategy, we have

$$p_{\underline{\sigma},\overline{\sigma}}(\Phi) \geq \sup_{\mathbb{P}\in\mathcal{P}_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\Big[\Phi(S) - \sum_{n=0}^{N-1} G_n\Big(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\Big)\Big].$$

We next observe that by applying the above inequality and the duality result for discrete models (Theorem 2.1) we obtain

$$\begin{aligned} p_{\underline{\sigma},\overline{\sigma}}(\Phi) &\geq \sup_{\mathbb{P}\in\mathcal{P}_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\Big[\Phi(S) - \sum_{n=0}^{N-1} G_n\Big(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\Big)\Big] \\ &\geq \sup_{\mathbb{P}\in\mathcal{P}^k_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\Big[\Phi(S) - \sum_{n=0}^{N-1} G_n\Big(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\Big)\Big] = p^k_{\underline{\sigma},\overline{\sigma}}(\Phi). \end{aligned} \tag{18}$$

Thus, to complete our proof it remains to prove the following.

LEMMA 3.1. *Suppose that the payoff function Φ is continuous. Then*

$$\liminf_{k\uparrow\infty} p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) \geq p_{\underline{\sigma},\overline{\sigma}}(\Phi). \tag{19}$$

Proof. The same result for the case where the trading strategy consists of dynamic part only (without taking the static position in options) was proved in [2].

For any $k \in \mathbb{N}$, by definition of the minimal super-replication price of the payoff Φ in the k-th discrete model, there exists the dynamic trading strategy $\hat{\gamma}^k$ and static position $\hat{\delta}^k = (\hat{\delta}^k_1, \hat{\delta}^k_2, \dots, \hat{\delta}^k_l)$ in options $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_l}$ such that

$$p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) + \frac{1}{k} + \sum_{n=0}^{N-1} \hat{\gamma}^k_{n+1}(\hat{\omega})(S_{n+1}(\hat{\omega}) - S_n(\hat{\omega}))$$
$$- \sum_{n=0}^{N-1} g_n((\hat{\gamma}^k_{n+1}(\hat{\omega}) - \hat{\gamma}^k_n(\hat{\omega}))S_n(\hat{\omega})) + \sum_{l=1}^{L} \hat{\delta}^k_l \varphi_{i_l}(S(\hat{\omega})) \geq \Phi(S(\hat{\omega}))$$

for any $\hat{\omega} \in \Omega^k_{\underline{\sigma},\overline{\sigma}}$.

We will prove the uniform boundedness of the sequence $(\hat{\gamma}^k)_{k=1,2,\dots}$, that is,

$$|\hat{\gamma}^k| \leq C \tag{20}$$

for some constant $C > 0$. By induction on n, we will show that for any strategy $\hat{\pi}^k = (p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) + \frac{1}{k}, \hat{\gamma}^k, \hat{\delta}^k)$ there are constants $A > 0$ and $M^k > 0$ such that

$$\begin{aligned} V_n^{\hat{\pi}^k}(\hat{\omega}) &\le A(1+e^{\overline{\sigma}})^n - M^k \\ \left|\hat{\gamma}^k_{n+1}(\hat{\omega})\right| &\le \frac{A(1+e^{\overline{\sigma}})^n}{(1-e^{-\overline{\sigma}})S_n(\hat{\omega})} \end{aligned} \tag{21}$$

for any $\hat{\omega} \in \Omega^k$ and $n = 0, \dots, N-1$. The bound for the absolute value of $\hat{\gamma}^k$ does not depend on k and $S_n(\hat{\omega}) \ge s_0 e^{-\overline{\sigma}N}$ for any $n = 0, \dots, N$, so (20) holds for $C := (A(1+e^{\overline{\sigma}})^n)/(s_0(1-e^{-\overline{\sigma}})e^{-\overline{\sigma}N})$. By continuity of S, φ_i, $i \in I$, and boundedness of Ω^k, there is $M^k > 0$ such that

$$\sum_{l=1}^{L} \hat{\delta}^k_l \varphi_{i_l}(S) \ge -M^k.$$

First we prove the statement for $n = 0$. Note that $v^k = p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) + \frac{1}{k} \le A - M^k$, $k = 1, 2, \dots$, for some $A > 0$. Since each strategy $\hat{\pi}^k$ super-replicates payoff $\Phi \ge 0$, the profit or loss generated by the dynamic strategy $\hat{\gamma}^k$ up to time n is not less than $-M^k$ on Ω^k for any $n = 1, 2, \dots, N$. It follows that

$$\begin{aligned} v^k + \hat{\gamma}^k_1 s_0(e^{\overline{\sigma}} - 1) &\ge -M^k \\ v^k + \hat{\gamma}^k_1 s_0(e^{-\overline{\sigma}} - 1) &\ge -M^k \end{aligned}$$

which implies

$$|\hat{\gamma}^k_1| \le \frac{A}{s_0(1-e^{-\overline{\sigma}})}\,.$$

Now suppose that the assertion holds for n. Observe that

$$\begin{aligned} V^{\hat{\pi}^k}_{n+1} &\le V^{\hat{\pi}^k}_n + \hat{\gamma}^k_{n+1} S_n(e^{\overline{\sigma}} - 1) - g_n((\hat{\gamma}^k_{n+1} - \hat{\gamma}^k_n)S_n) \\ &\le V^{\hat{\pi}^k}_n + \hat{\gamma}^k_{n+1} S_n(e^{\overline{\sigma}} - 1) \end{aligned}$$

so by the inductive assumption we have

$$\begin{aligned} V^{\hat{\pi}^k}_{n+1} &\le A(1+e^{\overline{\sigma}})^n - M^k + \frac{A(1+e^{\overline{\sigma}})^n}{(1-e^{-\overline{\sigma}})S_n(\hat{\omega})}\, S_n(e^{\overline{\sigma}} - 1) \\ &= A(1+e^{\overline{\sigma}})^{n+1} - M^k. \end{aligned} \tag{22}$$

Moreover, since the wealth generated by the strategy $\hat{\gamma}^k$ at time $n+2$ is bounded below for any $\hat{\omega} \in \Omega^k$ we get

$$\begin{aligned} -M^k &\le V^{\hat{\pi}^k}_{n+1} + \hat{\gamma}^k_{n+2} S_{n+1}(e^{\overline{\sigma}} - 1) \\ -M^k &\le V^{\hat{\pi}^k}_{n+1} + \hat{\gamma}^k_{n+2} S_{n+1}(e^{-\overline{\sigma}} - 1), \end{aligned}$$

which together with (22) gives

$$\left|\hat{\gamma}^k_{n+2}(\hat{\omega})\right| \le \frac{A(1+e^{\overline{\sigma}})^{n+1}}{(1-e^{-\overline{\sigma}})S_{n+1}(\hat{\omega})}\,. \tag{23}$$

This completes the proof of (21).

Consider the dynamic strategies $\gamma^k := \hat{\gamma}^k \circ p^k$ where $p^k : \Omega \ni \omega = (x_1, \ldots, x_N) \to \hat{\omega} = (\hat{x}_1, \ldots, \hat{x}_N) \in \Omega^k$ is the projection given by

$$\hat{x}_n = \max\Big\{x : x \leq x_n \text{ and } |x| = \frac{j}{k}\underline{\sigma} + \Big(1 - \frac{j}{k}\Big)\overline{\sigma} \text{ for some } j \in \{0, 1, \ldots, k\}\Big\}$$

and static strategies $\delta^k := \hat{\delta}^k$. Observe that

$$\begin{aligned} V_N^{(v,\gamma^k,\delta^k)}&(\omega) - V_N^{(v,\hat{\gamma}^k,\hat{\delta}^k)}(\hat{\omega}) \\ &= \sum_{n=0}^{N-1} \hat{\gamma}_{n+1}^k(\hat{\omega})\big((S_{n+1}(\omega) - S_n(\omega)) - \big(S_{n+1}(\hat{\omega}) - S_n(\hat{\omega})\big)\big) \\ &\quad - \sum_{n=0}^{N-1} \big(g_n\big(\omega, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\omega)\big) - g_n\big(\hat{\omega}, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\hat{\omega})\big)\big) \\ &\quad + \sum_{l=1}^{L} \hat{\delta}_l^k\big(\varphi_{i_l}(S(\omega)) - \varphi_{i_l}(S(\hat{\omega}))\big) \end{aligned}$$

for any initial capital v. In view of (22), for the first sum we get

$$\begin{aligned} \Big|\sum_{n=0}^{N-1} \hat{\gamma}_{n+1}^k(\hat{\omega})\big((S_{n+1}(\omega) - S_n(\omega)) - (S_{n+1}(\hat{\omega}) - S_n(\hat{\omega}))\big)\Big| \\ \leq C\sum_{n=0}^{N-1}\big|S_{n+1}(\omega) - S_{n+1}(\hat{\omega})\big| + C\sum_{n=0}^{N-1}\big|S_n(\omega) - S_n(\hat{\omega})\big| \end{aligned}$$

and by continuity of S this bound tends to 0 as $|\omega - \hat{\omega}| \to 0$. Similarly, the absolute value of the second sum does not exceed

$$\begin{aligned} &\sum_{n=0}^{N-1}\big|g_n\big(\omega, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\omega)\big) - g_n\big(\hat{\omega}, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\omega)\big)\big| \\ &\quad + \sum_{n=0}^{N-1}\big|g_n\big(\hat{\omega}, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\omega)\big) - g_n\big(\hat{\omega}, (\hat{\gamma}_{n+1}^k(\hat{\omega}) - \hat{\gamma}_n^k(\hat{\omega}))S_n(\hat{\omega})\big)\big|, \end{aligned}$$

which tend to 0 as $|\omega - \hat{\omega}| \to 0$ because S_n and g_n are continuous and the sequence $\hat{\gamma}^k$ is uniformly bounded. Because φ_i, $i \in I$, are also continuous, the last sum tends to 0. It follows that there are $\epsilon_k \to 0$ as $k \to \infty$ such that

$$V_N^{(v,\hat{\gamma}^k,\hat{\delta}^k)}(\hat{\omega}) \leq V_N^{(v,\gamma^k,\delta^k)}(\omega) + \epsilon_k \tag{24}$$

for all $|\omega - \hat{\omega}| \leq \frac{1}{k}$ and $v \in \mathbb{R}$. By assumption, Φ is also continuous, so that there are $\eta_k \to 0$ as $k \to \infty$ such that for all $|\omega - \hat{\omega}| \leq \frac{1}{k}$

$$\Phi(S(\omega)) \leq \Phi(S(\hat{\omega})) + \eta_k. \tag{25}$$

The trading strategy $\hat{\pi}^k = (p_{\underline{\sigma},\overline{\sigma}}^k(\Phi) + \frac{1}{k}, \hat{\gamma}^k, \hat{\delta}^k)$ super-replicates payoff Φ on Ω_k, so together with (24) and (25) we get

$$\begin{aligned} \Phi(S(\omega)) \leq \Phi(S(\hat{\omega})) + \eta_k &\leq V_N^{(p_{\underline{\sigma},\overline{\sigma}}^k(\Phi)+1/k,\hat{\gamma}^k,\hat{\delta}^k)}(\hat{\omega}) + \eta_k \\ &\leq V_N^{(p_{\underline{\sigma},\overline{\sigma}}^k(\Phi)+1/k,\gamma^k,\delta^k)}(\omega) + \epsilon_k + \eta_k, \end{aligned}$$

which means that $p_{\underline{\sigma},\overline{\sigma}}(\Phi) \leq p^k_{\underline{\sigma},\overline{\sigma}}(\Phi) + \frac{1}{k} + \epsilon_k + \eta_k$ and passing to the limit we get our claim. ■

It is immediate from the series of inequalities (18) and Lemma 3.1 that

$$p_{\underline{\sigma},\overline{\sigma}}(\Phi) = \sup_{\mathbb{P}\in\mathcal{P}_{(\varphi_i)_{i\in I}}} \mathbb{E}_{\mathbb{P}}\Bigg[\Phi(S) - \sum_{n=0}^{N-1} G_n\bigg(\frac{\mathbb{E}_{\mathbb{P}}(S_N \,|\, \mathcal{F}_n) - S_n}{S_n}\bigg)\Bigg].$$

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