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PAIRS TRADING: AN OPTIMAL SELLING RULE UNDER A REGIME SWITCHING MODEL

JINGZHI TIE and QING ZHANG

Department of Mathematics, University of Georgia
Athens, GA 30602, USA
ORCID: 0000-0001-6212-7056, 0000-0002-8846-5587
E-mail: jtie@uga.edu, qz@uga.edu

Abstract. This paper is concerned with an optimal selling rule for pairs stock trading. A pairs position consists of a long position in one stock and a short position in the other. The problem is to find an optimal stopping time to close the pairs position by selling the long position and buying back the short position. In this paper, we consider the optimal pairs-trading selling rule by allowing the stock prices to follow a general geometric Brownian motion with regime switching. The optimal policy is characterized by threshold curves obtained by solving the associated HJB equations (quasi-variational inequalities). Moreover, numerical examples are provided to illustrate optimal policies and value functions.

1. Introduction. Pairs trading is about simultaneously trading of a pair of stocks. A pairs position consists of a long position in one stock and a short position in the other. In this paper, assuming an existing pairs position, our goal is to determine when to fold and close the position.

Pairs trading is closely related to the timing of the optimal investments studied in McDonald and Siegel [MS]. In particular, they considered the optimal timing of investment in an irreversible project. Two main variables in their model are the value of the project and the cost of investing. They demonstrated one should defer the investment until the present value of the benefits from the project exceed the investment cost by a certain margin. Further studies along this line were carried out by Hu and Øksendal [HO] to specify precise optimality conditions and to provide a new proof of the following

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variational inequalities among others. Their results can be easily interpreted in terms of pairs-trade selling rule when treating the project value as the long position and investment cost the short position.

In this paper, we extend these results to incorporate markets with regime switching. We focus on a simple and easily implementable strategy, and its optimality and sufficient conditions for a closed-form solution.

Mathematical trading rules have been studied for many years. For example, Zhang [Z] considered a selling rule determined by two threshold levels, a target price, and a stop-loss limit. In [Z], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [GZ] studied the optimal selling rule under a model with switching Geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Recently, Dai et al. [DZZ] developed a trend-following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton–Jacobi–Bellman (HJB) equations. A similar idea was developed following a confidence interval approach by Iwarere and Barmish [IB]. Besides, Merhi and Zervos [MZ] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean reversion trading, Zhang and Zhang [ZZ] obtained a buy-low and sell-high policy by characterizing the ‘low’ and ‘high’ levels in terms of the mean reversion parameters. Song and Zhang [SZ] studied pairs trading under a mean reversion model. It is shown that the optimal trading rule can be determined by threshold levels that can be obtained by solving a set of algebraic equations. A set of sufficient conditions are also provided to establish the desired optimality. Deshpande and Barmish [DB] introduced a control-theoretic approach. In particular, they were able to relax the requirement for spread functions and showed that their trading algorithm produces positive expected returns. Other related pairs technologies can be found in Elliott et al. [EHM] and Whistler [W]. Recently, we [TZZ] studied an optimal pairs trading rule. The objective is to initiate and close the positions of the pair sequentially to maximize a discounted payoff function. Using a dynamic programming approach, we study the problem under a geometric Brownian motion model and proved that the buying and selling can be determined by two threshold curves in closed form. They also demonstrate the optimality of their trading strategy.

Market models with regime switching are important in market analysis. In this paper, we consider a geometric Brownian motion with regime switching. The market mode is represented by a two-state Markov chain. We focus on the selling part of pairs trading and generalize the results of Hu and Øksendal [HO] by incorporating models with regime switching. We show that the optimal selling rule can be determined by two threshold curves and establish a set of sufficient conditions that guarantee the optimality of the policy. We also include several numerical examples under a different set of parameter values.

This paper is organized as follows. In §2, we formulate the pairs trading problem under consideration. In §3, we study the associated HJB equations and their solutions. We provide a set of sufficient conditions that guarantee the optimality of our selling rule. Numerical examples are given in §4. Some concluding remarks are given in §5.

2. Problem formulation. We consider two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of stock $\mathbf{S}^1$ and $\{X_t^2, t \geq 0\}$ that of stock $\mathbf{S}^2$. They satisfy the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right], \tag{1}$$

where μ_i, $i = 1, 2$, are the return rates, σ_{ij}, $i, j = 1, 2$, the volatility constants, α_t a two-state Markov chain, and (W_t^1, W_t^2) a two-dimensional standard Brownian motion.

Let $\mathcal{M} = \{1, 2\}$ denote the state space for the Markov chain α_t and let $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, with $\lambda_1 > 0$ and $\lambda_2 > 0$, be its generator. We assume α_t and (W_t^1, W_t^2) are independent.

In this paper, we consider a pair selling rule. For simplicity, we assume the corresponding pair's position consists of a one-share long position in stock $\mathbf{S}^1$ and a one-share short position in stock $\mathbf{S}^2$. The problem is to determine an optimal stopping time τ to close the pair's position by selling $\mathbf{S}^1$ and buying back $\mathbf{S}^2$.

Let K denote the transaction cost percentage (e.g., slippage and/or commission) associated with stock transactions. For example, the proceeds to close the pairs position at t is $(1-K)X_t^1 - (1+K)X_t^2$. For ease of notation, let $\beta_{\mathrm{b}} = 1 + K$ and $\beta_{\mathrm{s}} = 1 - K$.

Given the initial state (x_1, x_2), $\alpha = 1, 2$, and the selling time τ, the corresponding reward function

$$J(x_1, x_2, \alpha, \tau) = E\big[e^{-\rho\tau}(\beta_{\mathrm{s}} X_\tau^1 - \beta_{\mathrm{b}} X_\tau^2)\big], \tag{2}$$

where $\rho > 0$ is a given discount factor.

Let $\mathcal{F}_t = \sigma\{(X_r^1, X_r^2, \alpha_r) : r \leq t\}$. The problem is to find an $\{\mathcal{F}_t\}$ stopping time τ to maximize J. Let $V(x_1, x_2, \alpha)$ denote the corresponding value functions:

$$V(x_1, x_2, \alpha) = \sup_\tau J(x_1, x_2, \alpha, \tau). \tag{3}$$

Remark 2.1. The 'one-share' pair position is not as restrictive as it appears. For example, one can consider any pairs with n_1 shares of long position in $\mathbf{S}^1$ and n_2 shares of short position in $\mathbf{S}^2$. To treat this case, one only has to make change of the state variables $(X_t^1, X_t^2) \to (n_1 X_t^1, n_2 X_t^2)$. Due to the nature of GBMs, the corresponding system equation in (1) will remain the same. The modification only affects the reward function in (2) implicitly.

Throughout this paper, we impose the following conditions:

(A1) For $\alpha = 1, 2$, $\rho > \mu_1(\alpha)$ and $\rho > \mu_2(\alpha)$.

Under these conditions, we have the lower and upper bounds for V:

$$\beta_{\mathrm{s}} x_1 - \beta_{\mathrm{b}} x_2 \leq V(x_1, x_2, \alpha) \leq \beta_{\mathrm{s}} x_1. \tag{4}$$

Actually, the lower bound follows from the value function definition

$$V(x_1,x_2,\alpha) \geq J(x_1,x_2,\alpha,0) = \beta_{\rm s}x_1 - \beta_{\rm b}x_2.$$

The upper bound can be obtained from Dynkin's formula

$$J(x_1,x_2,\alpha,\tau) \leq E[e^{-\rho\tau}\beta_{\rm s}X^1_\tau] = \beta_{\rm s}\Big(x_1 + E\int_0^\tau e^{-\rho t}X_t^1(-\rho+\mu_1(\alpha_t))\,dt\Big) \leq \beta_{\rm s}x_1.$$

3. HJB equations. In this paper, we follow the dynamic programming approach and focus on HJB equations. First, for $i=1,2$, let

$$\mathcal{A}_i = \frac{1}{2}\left[a_{11}(i)x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}(i)x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + a_{22}(i)x_2^2\frac{\partial^2}{\partial x_2^2}\right] + \mu_1(i)x_1\frac{\partial}{\partial x_1} + \mu_2(i)x_2\frac{\partial}{\partial x_2} \quad (5)$$

where

$a_{11}(i) = \sigma_{11}^2(i)+\sigma_{12}^2(i)$, $a_{12}(i) = \sigma_{11}(i)\sigma_{21}(i)+\sigma_{12}(i)\sigma_{22}(i)$, and $a_{22}(i) = \sigma_{21}^2(i)+\sigma_{22}^2(i)$.

Formally, the associated HJB equations have the form:

$$\begin{cases} \min\big\{(\rho-\mathcal{A}_1)v(x_1,x_2,1) - \lambda_1(v(x_1,x_2,2)-v(x_1,x_2,1)), \\ \qquad\qquad v(x_1,x_2,1) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\big\} = 0, \\ \min\big\{(\rho-\mathcal{A}_2)v(x_1,x_2,2) - \lambda_2(v(x_1,x_2,1)-v(x_1,x_2,2)), \\ \qquad\qquad v(x_1,x_2,2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\big\} = 0. \end{cases} \quad (6)$$

To solve the HJB equations, we first convert them into equations with a single independent variable by introducing $y = x_2/x_1$ and $v(x_1,x_2,i) = x_1w_i(x_2/x_1)$, for some function $w_i(y)$ and $i=1,2$. Then direct calculation yields

$$\frac{\partial v(x_1,x_2,i)}{\partial x_1} = w_i(y) - yw_i'(y), \quad \frac{\partial v(x_1,x_2,i)}{\partial x_2} = w_i'(y),$$
$$\frac{\partial^2 v(x_1,x_2,i)}{\partial x_1^2} = \frac{y^2w_i''(y)}{x_1}, \quad \frac{\partial^2 v(x_1,x_2,i)}{\partial x_2^2} = \frac{w_i''(y)}{x_1}, \quad \frac{\partial^2 v(x_1,x_2,i)}{\partial x_1\partial x_2} = -\frac{yw_i''(y)}{x_1}.$$

We rewrite $\mathcal{A}_iv(x_1,x_2,i)$ in terms of w_i to obtain

$$\mathcal{A}_iv(x_1,x_2,i) = x_1\big\{\sigma_iy^2w_i''(y) + [\mu_2(i)-\mu_1(i)]yw_i'(y) + \mu_1(i)w_i(y)\big\}.$$

where $\sigma_i = [a_{11}(i) - 2a_{12}(i) + a_{22}(i)]/2$. Let

$$\mathcal{L}_i[w_i(y)] = \sigma_iy^2w_i''(y) + [\mu_2(i)-\mu_1(i)]yw_i'(y) + \mu_1(i)w_i(y), \quad i=1,2.$$

Then, the HJB equations can be given in terms of y and w_i as follows:

$$\begin{cases} \min\big\{(\rho+\lambda_1-\mathcal{L}_1)w_1(y) - \lambda_1w_2(y),\ w_1(y)+\beta_{\rm b}y-\beta_{\rm s}\big\} = 0, \\ \min\big\{(\rho+\lambda_2-\mathcal{L}_2)w_2(y) - \lambda_2w_1(y),\ w_2(y)+\beta_{\rm b}y-\beta_{\rm s}\big\} = 0. \end{cases} \quad (7)$$

In this paper, we only consider the case when $\sigma_i \neq 0$, $i=1,2$. If either $\sigma_1 = 0$ and/or $\sigma_2 = 0$, the problem reduces to a (partial) first order case and can be treated in a similar and much simpler way.

First we consider the equations:

$$(\rho+\lambda_1-\mathcal{L}_1)w_1=\lambda_1 w_2 \quad \text{and} \quad (\rho+\lambda_2-\mathcal{L}_2)w_2=\lambda_2 w_1. \tag{8}$$

Then both w_1 and w_2 satisfy the equation

$$\big[(\rho+\lambda_1-\mathcal{L}_1)(\rho+\lambda_2-\mathcal{L}_2)-\lambda_1\lambda_2\big]w=0.$$

Note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are the classical Euler type operators and therefore the solutions to the above equation is of the form $w=y^\delta$ for some δ. This leads to that δ must satisfy

$$[\rho+\lambda_1-A_1(\delta)][\rho+\lambda_2-A_2(\delta)]-\lambda_1\lambda_2=0, \tag{9}$$

with

$$A_i(\delta)=\sigma_i\delta(\delta-1)+[\mu_2(i)-\mu_1(i)]\delta+\mu_1(i), \quad i=1,2. \tag{10}$$

It is elementary to show the equation (9) has four zeros which can be arranged as follows: $\delta_1\ge\delta_2>1>0>\delta_3\ge\delta_4$.

Let

$$w_1=\sum_{j=1}^4 c_{1j}y^{\delta_j} \quad \text{and} \quad w_2=\sum_{j=1}^4 c_{2j}y^{\delta_j},$$

for some constants c_{ij}. Then, in view of the equations in (8), it follows that

$$c_{1,j}(\rho+\lambda_1-A_1(\delta_j))=\lambda_1 c_{2j} \quad \text{and} \quad c_{2j}(\rho+\lambda_2-A_2(\delta_j))=\lambda_2 c_{1j}.$$

Define

$$\eta_j=\frac{\rho+\lambda_1-A_1(\delta_j)}{\lambda_1}. \tag{11}$$

Then it follows from (9) that

$$\eta_j=\frac{\lambda_2}{\rho+\lambda_2-A_2(\delta_j)}.$$

Therefore, $c_{2j}=\eta_j c_{1j}$, $j=1,2,3,4$. Hence,

$$w_1=\sum_{j=1}^4 c_{1j}y^{\delta_j} \quad \text{and} \quad w_2=\sum_{j=1}^4 \eta_j c_{1j}y^{\delta_j} \tag{12}$$

will be the general solution of (8).

Heuristically, one should close the pairs position when X_t^1 is large and X_t^2 is small. In view of this, we introduce $H_1=\{(x_1,x_2): x_2\le k_1x_1\}$ and $H_2=\{(x_1,x_2): x_2\le k_2x_1\}$, for some k_1 and k_2 so that one should sell when (X_t^1,X_t^2) enters H_i provided $\alpha_t=i$, $i=1,2$.

In this paper, we need consider two cases: $k_1\le k_2$ and $k_2\ge k_1$. By symmetry in $\alpha_t=1$ and $\alpha_t=2$, we only need to consider one of them, say $k_1\le k_2$. We treat two separate cases: $k_1<k_2$ and $k_1=k_2$.

Case 1: $k_1<k_2$. First, we divide $(0,\infty)$ into three intervals:

$$\Gamma_1=(0,k_1], \quad \Gamma_2=(k_1,k_2), \quad \text{and} \quad \Gamma_3=[k_2,\infty).$$

Then, on each of these intervals, the HJB equations can be specified as follows:

$$\begin{cases} \Gamma_1: & w_1(y) = \beta_{\rm s} - \beta_{\rm b} y; & w_2(y) = \beta_{\rm s} - \beta_{\rm b} y; \\ \Gamma_2: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y); & w_2(y) = \beta_{\rm s} - \beta_{\rm b} y; \\ \Gamma_3: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y); & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) = \lambda_2 w_1(y). \end{cases}$$

We are to find solutions on each intervals. First, on Γ_3, recall the linear bounds for value functions given in (4). Recall also that $\delta_1 > 1$ and $\delta_2 > 1$. It follows that the coefficients in (12) for y^{δ_1} and y^{δ_2} must be zero. Therefore,

$$w_1 = C_1 y^{\delta_3} + C_2 y^{\delta_4} \quad \text{and} \quad w_2 = \eta_3 C_1 y^{\delta_3} + \eta_4 C_2 y^{\delta_4}.$$

Next, to find solution on Γ_2, note that a particular solution for

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y) = \lambda_1(\beta_{\rm s} - \beta_{\rm b} y)$$

can be given by $w_1 = a_1 + a_2 y$, with

$$a_1 = \frac{\lambda_1 \beta_{\rm s}}{\rho + \lambda_1 - \mu_1(1)} \quad \text{and} \quad a_2 = -\frac{\lambda_1 \beta_{\rm b}}{\rho + \lambda_1 - \mu_2(1)}. \tag{13}$$

To find a general solution of the above non-homogeneous equation, we only need to solve the homogeneous equation $(\rho + \lambda_1 - \mathcal{L}_1) w_1 = 0$. This is also of Euler type and its solution is of the form y^γ. Then γ must be the roots of the quadratic equation

$$\sigma_1 \gamma(\gamma - 1) + [\mu_2(1) - \mu_1(1)]\gamma + \mu_1(1) - \rho - \lambda_1 = 0.$$

They are given by

$$\begin{cases} \gamma_1 = \dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1} + \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}, \\ \gamma_2 = \dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}. \end{cases} \tag{14}$$

The general solution for w_1 on Γ_2 is given by

$$w_1 = C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + \frac{\lambda_1 \beta_{\rm s}}{\rho + \lambda_1 - \mu_1(1)} - \frac{\lambda_1 \beta_{\rm b}}{\rho + \lambda_1 - \mu_2(1)} y, \tag{15}$$

for some constants C_3 and C_4.

Smooth-fit conditions. Smooth-fit conditions in connection with optimal stopping typically require the value functions to be continuously differentiable. Next we use such smooth-fit conditions to set up equations for parameters C_j, $j = 1, 2, 3, 4$, k_1 and k_2.

First, the continuous differentiability of w_1 at k_1 yields

$$\begin{aligned} \beta_{\rm s} - \beta_{\rm b} k_1 &= C_3 k_1^{\gamma_1} + C_4 k_1^{\gamma_2} + a_1 + a_2 k_1, \\ -\beta_{\rm b} &= C_3 \gamma_1 k_1^{\gamma_1 - 1} + C_4 \gamma_2 k_1^{\gamma_2 - 1} + a_2. \end{aligned} \tag{16}$$

Similarly, we have the equation for w_2 at k_2

$$\begin{aligned} \beta_{\rm s} - \beta_{\rm b} k_2 &= \eta_3 C_1 k_2^{\delta_3} + \eta_4 C_2 k_2^{\delta_4}, \\ -\beta_{\rm b} &= \eta_3 \delta_3 C_1 k_2^{\delta_3 - 1} + \eta_4 \delta_4 C_2 k_2^{\delta_4 - 1}. \end{aligned} \tag{17}$$

Finally, the equations for w_1 at k_2 are given by

$$\begin{aligned} C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 &= C_1 k_2^{\delta_3} + C_2 k_2^{\delta_4}, \\ C_3 \gamma_1 k_2^{\gamma_1 - 1} + C_4 \gamma_2 k_2^{\gamma_2 - 1} + a_2 &= \delta_3 C_1 k_2^{\delta_3 - 1} + \delta_4 C_2 k_2^{\delta_4 - 1}. \end{aligned} \tag{18}$$

We solve equations (16) and (17) for C_j, $j = 1, 2, 3, 4$, in terms of k_1 and k_2 and obtain

$$\begin{cases} C_1 = \dfrac{-\delta_4 \beta_{\mathrm{s}} + (\delta_4 - 1)\beta_{\mathrm{b}} k_2}{\eta_3(\delta_3 - \delta_4) k_2^{\delta_3}}, \\ C_2 = \dfrac{\delta_3 \beta_{\mathrm{s}} + (1 - \delta_3)\beta_{\mathrm{b}} k_2}{\eta_4(\delta_3 - \delta_4) k_2^{\delta_4}}, \\ C_3 = \dfrac{\gamma_2(\beta_{\mathrm{s}} - a_1) + (1 - \gamma_2)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1) k_1^{\gamma_1}}, \\ C_4 = \dfrac{-\gamma_1(\beta_{\mathrm{s}} - a_1) + (\gamma_1 - 1)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1) k_1^{\gamma_2}}. \end{cases}$$

Substituting these into (18), we obtain two equations on k_1 and k_2

$$\begin{aligned} &\frac{\gamma_2(\beta_{\mathrm{s}} - a_1) + (1 - \gamma_2)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1)} \left(\frac{k_2}{k_1}\right)^{\gamma_1} \\ &\quad + \frac{-\gamma_1(\beta_{\mathrm{s}} - a_1) + (\gamma_1 - 1)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1)} \left(\frac{k_2}{k_1}\right)^{\gamma_2} + a_1 + a_2 k_2 \\ &= \frac{-\delta_4 \beta_{\mathrm{s}} + (\delta_4 - 1)\beta_{\mathrm{b}} k_2}{\eta_3(\delta_3 - \delta_4)} + \frac{\delta_3 \beta_{\mathrm{s}} + (1 - \delta_3)\beta_{\mathrm{b}} k_2}{\eta_4(\delta_3 - \delta_4)} \end{aligned}$$

and

$$\begin{aligned} &\frac{\gamma_2(\beta_{\mathrm{s}} - a_1) + (1 - \gamma_2)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1)} \gamma_1 \left(\frac{k_2}{k_1}\right)^{\gamma_1} \\ &\quad + \frac{-\gamma_1(\beta_{\mathrm{s}} - a_1) + (\gamma_1 - 1)(\beta_{\mathrm{b}} + a_2) k_1}{(\gamma_2 - \gamma_1)} \gamma_2 \left(\frac{k_2}{k_1}\right)^{\gamma_2} + a_2 k_2 \\ &= \frac{-\delta_4 \beta_{\mathrm{s}} + (\delta_4 - 1)\beta_{\mathrm{b}} k_2}{\eta_3(\delta_3 - \delta_4)} \delta_3 + \frac{\delta_3 \beta_{\mathrm{s}} + (1 - \delta_3)\beta_{\mathrm{b}} k_2}{\eta_4(\delta_3 - \delta_4)} \delta_4. \end{aligned}$$

We next simplify these equations and obtain

$$\begin{aligned} &[\gamma_2(\beta_{\mathrm{s}} - a_1) + (1 - \gamma_2)(\beta_{\mathrm{b}} + a_2) k_1] \left(\frac{k_2}{k_1}\right)^{\gamma_1} + \gamma_2 a_1 + (\gamma_2 - 1) a_2 k_2 \\ &\quad = \frac{-\delta_4 \beta_{\mathrm{s}} + (\delta_4 - 1)\beta_{\mathrm{b}} k_2}{\eta_3(\delta_3 - \delta_4)} (\gamma_2 - \delta_3) + \frac{\delta_3 \beta_{\mathrm{s}} + (1 - \delta_3)\beta_{\mathrm{b}} k_2}{\eta_4(\delta_3 - \delta_4)} (\gamma_2 - \delta_4) \end{aligned}$$

and

$$\begin{aligned} &[-\gamma_1(\beta_{\mathrm{s}} - a_1) + (\gamma_1 - 1)(\beta_{\mathrm{b}} + a_2) k_1] \left(\frac{k_2}{k_1}\right)^{\gamma_2} + (1 - \gamma_1) a_2 k_2 - \gamma_1 a_1 \\ &\quad = \frac{-\delta_4 \beta_{\mathrm{s}} + (\delta_4 - 1)\beta_{\mathrm{b}} k_2}{\eta_3(\delta_3 - \delta_4)} (\delta_3 - \gamma_1) + \frac{\delta_3 \beta_{\mathrm{s}} + (1 - \delta_3)\beta_{\mathrm{b}} k_2}{\eta_4(\delta_3 - \delta_4)} (\delta_4 - \gamma_1). \end{aligned}$$

To reduce the above equations into linear equations in k_1 and k_2, we let $r = k_2/k_1$. Then, we have

$$\big[\gamma_2(\beta_{\rm s} - a_1) + (1-\gamma_2)(\beta_{\rm b} + a_2)k_1\big]r^{\gamma_1} = \left[\frac{-\delta_4\beta_{\rm s}(\gamma_2-\delta_3)}{\eta_3(\delta_3-\delta_4)} + \frac{\delta_3\beta_{\rm s}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - \gamma_2 a_1\right] + \left[\frac{(\delta_4-1)(\gamma_2-\delta_3)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \frac{(1-\delta_3)\beta_{\rm b}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - (\gamma_2-1)a_2\right]k_2$$

and

$$\big[-\gamma_1(\beta_{\rm s} - a_1) + (\gamma_1-1)(\beta_{\rm b} + a_2)k_1\big]r^{\gamma_2} = \left[\frac{-\delta_4\beta_{\rm s}(\delta_3-\gamma_1)}{\eta_3(\delta_3-\delta_4)} + \frac{\delta_3\beta_{\rm s}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} + \gamma_1 a_1\right] + \left[\frac{(\delta_4-1)(\delta_3-\gamma_1)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \frac{(1-\delta_3)\beta_{\rm b}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} - (1-\gamma_1)a_2\right]k_2.$$

To simplify notation, let

$$\begin{cases} A_1 = \dfrac{-\delta_4\beta_{\rm s}(\gamma_2-\delta_3)}{\eta_3(\delta_3-\delta_4)} + \dfrac{\delta_3\beta_{\rm s}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - \gamma_2 a_1, \\ A_2 = \dfrac{-\delta_4\beta_{\rm s}(\delta_3-\gamma_1)}{\eta_3(\delta_3-\delta_4)} + \dfrac{\delta_3\beta_{\rm s}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} + \gamma_1 a_1, \\ B_1 = \dfrac{(\delta_4-1)(\gamma_2-\delta_3)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \dfrac{(1-\delta_3)\beta_{\rm b}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - (\gamma_2-1)a_2, \\ B_2 = \dfrac{(\delta_4-1)(\delta_3-\gamma_1)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \dfrac{(1-\delta_3)\beta_{\rm b}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} - (1-\gamma_1)a_2. \end{cases}$$

Then, we have

$$\begin{cases} \big(\gamma_2(\beta_{\rm s} - a_1) + (1-\gamma_2)(\beta_{\rm b} + a_2)k_1\big)r^{\gamma_1} = A_1 + B_1 k_2, \\ \big(-\gamma_1(\beta_{\rm s} - a_1) + (\gamma_1-1)(\beta_{\rm b} + a_2)k_1\big)r^{\gamma_2} = A_2 + B_2 k_2. \end{cases}$$

Eliminate k_1 to obtain the equation in r:

$$\frac{A_1 - \gamma_2(\beta_{\rm s}-a_1)r^{\gamma_1}}{(1-\gamma_2)(\beta_{\rm b}+a_2)r^{\gamma_1} - B_1 r} = \frac{A_2 + \gamma_1(\beta_{\rm s}-a_1)r^{\gamma_2}}{(\gamma_1-1)(\beta_{\rm b}+a_2)r^{\gamma_2} - B_2 r}.$$

Let

$$f(r) = \frac{A_1 - \gamma_2(\beta_{\rm s}-a_1)r^{\gamma_1}}{(1-\gamma_2)(\beta_{\rm b}+a_2)r^{\gamma_1} - B_1 r} - \frac{A_2 + \gamma_1(\beta_{\rm s}-a_1)r^{\gamma_2}}{(\gamma_1-1)(\beta_{\rm b}+a_2)r^{\gamma_2} - B_2 r}. \tag{19}$$

We assume

(A2) $f(r)$ has a zero $r_0 > 1$.

Use this r_0 and recall that $k_2 = r_0 k_1$ to obtain

$$\begin{cases} k_1 = \dfrac{A_1 - \gamma_2(\beta_{\rm s}-a_1)r_0^{\gamma_1}}{(1-\gamma_2)(\beta_{\rm b}+a_2)r_0^{\gamma_1} - B_1 r_0} = \dfrac{A_2 + \gamma_1(\beta_{\rm s}-a_1)r_0^{\gamma_2}}{(\gamma_1-1)(\beta_{\rm b}+a_2)r_0^{\gamma_2} - B_2 r_0}, \\ k_2 = r_0 k_1 = \dfrac{A_1 r_0 - \gamma_2(\beta_{\rm s}-a_1)r_0^{\gamma_1+1}}{(1-\gamma_2)(\beta_{\rm b}+a_2)r_0^{\gamma_1} - B_1 r_0} = \dfrac{A_2 r_0 + \gamma_1(\beta_{\rm s}-a_1)r_0^{\gamma_2+1}}{(\gamma_1-1)(\beta_{\rm b}+a_2)r_0^{\gamma_2} - B_2 r_0}. \end{cases} \tag{20}$$

Using these k_1 and k_2, we can express C_1, C_2, C_3, and C_4. Therefore, the solutions w_1 and w_2 are given by

$$w_1(y) = \begin{cases} \beta_{\rm s} - \beta_{\rm b} y & \text{for } y \in \Gamma_1, \\ C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + a_1 + a_2 y & \text{for } y \in \Gamma_2, \\ C_1 y^{\delta_3} + C_2 y^{\delta_4} & \text{for } y \in \Gamma_3; \end{cases}$$

$$w_2(y) = \begin{cases} \beta_{\rm s} - \beta_{\rm b} y & \text{for } y \in \Gamma_1 \cup \Gamma_2, \\ C_1 \eta_3 y^{\delta_3} + C_2 \eta_4 y^{\delta_4} & \text{for } y \in \Gamma_3. \end{cases}$$

Note that the variational inequalities in the HJB equations need to hold. In particular, we need the HJB inequalities to hold:

$$\begin{array}{lll} \Gamma_1: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0; \\ \Gamma_2: & w_1 \geq \beta_{\rm s} - \beta_{\rm b} y, & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0; \\ \Gamma_3: & w_1 \geq \beta_{\rm s} - \beta_{\rm b} y, & w_2 \geq \beta_{\rm s} - \beta_{\rm b} y. \end{array} \quad (21)$$

Next, we simplify these inequalities and establish equivalent conditions.

First, consider the inequalities on Γ_1. Recall that on this interval, both w_1 and w_2 equal $\beta_{\rm b} - \beta_{\rm b} y$. Simple calculation yields that

$$(\rho + \lambda_1 - \mathcal{L}_1)(\beta_{\rm s} - \beta_{\rm b} y) = (\rho + \lambda_1 - \mu_1(1))\beta_{\rm s} - (\rho + \lambda_1 - \mu_2(1))\beta_{\rm b} y.$$

So, $(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0$ leads to $(\rho - \mu_1(1))\beta_{\rm s} - (\rho - \mu_2(1))\beta_{\rm b} y \geq 0$. This is equivalent to

$$k_1 \leq \frac{(\rho - \mu_1(1))\beta_{\rm s}}{(\rho - \mu_2(1))\beta_{\rm b}}.$$

Similarly, if $w_2 = \beta_{\rm s} - \beta_{\rm b} y$, then

$$(\rho + \lambda_2 - \mathcal{L}_2) w_2(y) = (\rho + \lambda_2 - \mu_1(2))\beta_{\rm s} - (\rho + \lambda_2 - \mu_2(2))\beta_{\rm b} y.$$

Therefore, $(\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0$ on Γ_1 is equivalent to

$$k_1 \leq \frac{(\rho - \mu_1(2))\beta_{\rm s}}{(\rho - \mu_2(2))\beta_{\rm b}}.$$

The inequalities on Γ_1 are equivalent to

$$k_1 \leq \min\left\{ \frac{(\rho - \mu_1(1))\beta_{\rm s}}{(\rho - \mu_2(1))\beta_{\rm b}}, \frac{(\rho - \mu_1(2))\beta_{\rm s}}{(\rho - \mu_2(2))\beta_{\rm b}} \right\}. \quad (22)$$

Similarly, the second inequality in (21) on Γ_2 is equivalent to

$$w_1(y) \leq \beta_{\rm s} - \beta_{\rm b} y + \frac{1}{\lambda_2}\big[(\rho - \mu_1(2))\beta_{\rm s} - (\rho - \mu_2(2))\beta_{\rm b} y\big]. \quad (23)$$

To see an equivalent condition for the first inequality on Γ_2, let $\phi(y) = w_1(y) - \beta_{\rm s} + \beta_{\rm b} y$. Then $\phi(k_1) = 0$, $\phi'(k_1) = 0$. Note that $\phi''(y)$ can have at most one zero on Γ_2. This implies $\phi(y) \geq 0$ on Γ_2 is equivalent to $\phi''(k_1) \geq 0$ and $\phi(k_2) \geq 0$. Namely,

$$\begin{cases} \phi''(k_1) = C_3 \gamma_1 (\gamma_1 - 1) k_1^{\gamma_1 - 2} + C_4 \gamma_2 (\gamma_2 - 1) k_1^{\gamma_2 - 2} \geq 0 \text{ and} \\ \phi(k_2) = C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 - \beta_{\rm s} + \beta_{\rm b} y \geq 0. \end{cases} \quad (24)$$

Finally, to see an equivalent condition of the second inequality in (21) on Γ_3, let $\psi(y) = w_2(y) - \beta_s + \beta_b y$. Then, $\psi(k_2) = 0$, $\psi'(k_2) = 0$, $\psi(\infty) = \infty$, and $\psi'(\infty) > 0$. Note also that $\psi(y)$ can have at most one zero on Γ_3. It follows that $\psi(y) \geq 0$ on Γ_3 is equivalent to $\psi''(k_2) \geq 0$. So, the second inequality on Γ_3 in (21) is equivalent to

$$\psi''(k_2) = C_1\eta_3\delta_3(\delta_3 - 1)k_2^{\delta_3-2} + C_2\eta_4\delta_4(\delta_4 - 1)k_2^{\delta_4-2} \geq 0. \tag{25}$$

The other inequality on Γ_3 is $w_1(y) \geq \beta_s - \beta_b y$. Hence,

$$C_1 y^{\delta_3} + C_2 y^{\delta_4} \geq \beta_s - \beta_b y. \tag{26}$$

We assume these equivalent inequalities.

(A3) The inequalities in (22), (23), (24), (25), and (26) hold.

Case 2: $k_1 = k_2$**.** In this case, let $k_0 = k_1 = k_2$. We have $w_1 = w_2 = \beta_s - \beta_b y$ on $(0, k_0]$ and

$$w_1 = C_1 y^{\delta_3} + C_2 y^{\delta_4} \quad \text{and} \quad w_2 = C_1\eta_3 y^{\delta_3} + C_2\eta_4 y^{\delta_4}$$

on $[k_0, \infty)$. Then, smooth-fit conditions imply at k_0

$$\begin{aligned} \beta_s - \beta_b k_0 &= C_1 k_0^{\delta_3} + C_2 k_0^{\delta_4}, \\ -\beta_b &= \delta_3 C_1 k_0^{\delta_3-1} + \delta_4 C_2 k_0^{\delta_4-1}, \\ \beta_s - \beta_b k_0 &= \eta_3 C_1 k_0^{\delta_3} + \eta_4 C_2 k_0^{\delta_4}, \\ -\beta_b &= \eta_3\delta_3 C_1 k_0^{\delta_3-1} + \eta_4\delta_4 C_2 k_0^{\delta_4-1}. \end{aligned}$$

This implies $C_1 = C_1\eta_3$ and $C_2 = C_2\eta_4$. Hence $w_1(y) = w_2(y) = w(y)$. Then $w(y)$ satisfies

$$(\rho + \lambda_1 - \mathcal{L}_1)w(y) = \lambda_1 w(y) \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w(y) = \lambda_2 w(y)$$

for $y > k_0$. This yields $(\rho - \mathcal{L}_1)w(y) = 0$ and $(\rho - \mathcal{L}_2)w(y) = 0$. Since both $\rho - \mathcal{L}_1$ and $\rho - \mathcal{L}_2$ are of Euler type, we have $w(y) = C_1 y^{\gamma_0}$ with γ_0 satisfying the two quadratic equations:

$$\begin{aligned} \sigma_1\gamma_0(\gamma_0 - 1) + [\mu_2(1) - \mu_1(1)]\gamma_0 + \mu_1(1) - \rho = 0 \\ \sigma_2\gamma_0(\gamma_0 - 1) + [\mu_2(2) - \mu_1(2)]\gamma_0 + \mu_1(2) - \rho = 0 \end{aligned}$$

and taking the value:

$$\begin{aligned} \gamma_0 &= \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho - \mu_1(1)}{\sigma_1}} \\ &= \frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2}\right)^2 + \frac{\rho - \mu_1(2)}{\sigma_2}}. \end{aligned} \tag{27}$$

The second equality is the necessary and sufficient condition for $k_1 = k_2$. Now the smooth fitting conditions yield

$$\beta_s - \beta_b k_0 = C_1 k_0^{\gamma_0} \quad \text{and} \quad -\beta_b k_0 = C_1\gamma_0 k_0^{\gamma_0}.$$

We can solve this system and obtain

$$k_0 = \frac{-\gamma_0\beta_s}{(1 - \gamma_0)\beta_b}, \quad C_1 = \frac{-k_0\beta_b}{\gamma_0 k_0^{\gamma_0}} = \frac{(-\gamma_0)^{-\gamma_0}}{(1 - \gamma_0)^{1-\gamma_0}} \cdot \frac{\beta_s^{1-\gamma_0}}{\beta_b^{-\gamma_0}} = \frac{\beta_s^{1-\gamma_0}\beta_b^{\gamma_0}}{(-\gamma_0)^{\gamma_0}(1 - \gamma_0)^{1-\gamma_0}}. \tag{28}$$

Note that $\gamma_0 < 0$, which implies that both k_0 and C_1 are positive. Finally the solution to the HJB equations is given by

$$w_1(y) = w_2(y) = w(y) = \begin{cases} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y & \text{for } y \in (0, k_0], \\ C_1 y^{\gamma_0} & \text{for } y \in (k_0, \infty). \end{cases}$$

Next we prove that the variational inequalities hold in this case. We need to show

$$\begin{aligned} &(\rho - \mathcal{L}_1) w(y) \geq 0 \text{ and } (\rho - \mathcal{L}_2) w(y) \geq 0 && \text{on } (0, k_0], \\ &w(y) \geq \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y && \text{on } (k_0, \infty). \end{aligned} \tag{29}$$

We first prove the second inequality. Let $\phi_0(y) = w(y) - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y = C_1 y^{\gamma_0} - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y$. Then, $\phi_0(k_0) = 0$, $\phi_0'(k_0) = 0$, and $\phi_0''(y) > 0$. It follows that $\phi_0(y)$ is increasing on (k_0, ∞). Therefore, $\phi_0(y) \geq 0$ on this interval.

To show the first inequality in (29), note that on $(0, k_0]$, for $i = 1, 2$,

$$(\rho - \mathcal{L}_i) w(y) = (\rho - \mathcal{L}_i)(\beta_{\mathrm{s}} - \beta_{\mathrm{b}} y) = (\rho - \mu_1(i))\beta_{\mathrm{s}} - (\rho - \mu_2(i))\beta_{\mathrm{b}} y.$$

Therefore,

$$(\rho - \mathcal{L}_i) w(y) \geq 0 \text{ on } (0, k_0] \iff k_0 \leq \frac{(\rho - \mu_1(i))\beta_{\mathrm{s}}}{(\rho - \mu_2(i))\beta_{\mathrm{b}}}.$$

We have

$$k_0 \leq \frac{(\rho - \mu_1(i))\beta_{\mathrm{s}}}{(\rho - \mu_2(i))\beta_{\mathrm{b}}} \iff \frac{-\gamma_0}{1 - \gamma_0} \leq \frac{\rho - \mu_1(i)}{\rho - \mu_2(i)} \iff \gamma_0(\mu_2(i) - \mu_1(i)) \leq \rho - \mu_1(i).$$

If $\mu_2(i) \geq \mu_1(i)$, then using (27), we have $\gamma_0 \leq 1/2$. It follows that

$$(\mu_2(i) - \mu_1(i))\gamma_0 \leq \frac{\mu_2(i) - \mu_1(i)}{2} \leq \rho - \mu_1(i)$$

because of assumption (A1). So $(\rho - \mathcal{L}_i) w(y) \geq 0$ on $(0, k_0]$ in this case.

If $\mu_2(i) < \mu_1(i)$, then we have

$$\begin{aligned} &\gamma_0(\mu_2(i) - \mu_1(i)) \leq \rho - \mu_1(i) \\ &\iff (\mu_2(i) - \mu_1(i)) \sqrt{\left(\frac{1}{2} + \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i}\right)^2 + \frac{\rho - \mu_1(i)}{\sigma_i}} \\ &\qquad \leq \rho - \mu_1(i) + \frac{\mu_1(i) - \mu_2(i)}{2} + \frac{(\mu_1(i) - \mu_2(i))^2}{2\sigma_i} \\ &\iff \sqrt{\left(\frac{1}{2} + \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i}\right)^2 + \frac{\rho - \mu_1(i)}{\sigma_i}} \leq \frac{2\rho - \mu_1(i) - \mu_2(i)}{2(\mu_1(i) - \mu_2(i))} + \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i} \\ &\iff \left(\frac{1}{2} + \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i}\right)^2 + \frac{\rho - \mu_1(i)}{\sigma_i} \leq \left(\frac{2\rho - \mu_1(i) - \mu_2(i)}{2(\mu_1(i) - \mu_2(i))} + \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i}\right)^2 \\ &\iff \frac{\mu_1(i) - \mu_2(i)}{2\sigma_i} + \frac{1}{4} + \frac{\rho - \mu_1(i)}{\sigma_i} \leq \left(\frac{2\rho - \mu_1(i) - \mu_2(i)}{2(\mu_1(i) - \mu_2(i))}\right)^2 + \frac{2\rho - \mu_1(i) - \mu_2(i)}{2\sigma_i} \\ &\iff \frac{1}{4} \leq \left(\frac{2\rho - \mu_1(i) - \mu_2(i)}{2(\mu_1(i) - \mu_2(i))}\right)^2 \iff \frac{1}{2} \leq \frac{2\rho - \mu_1(i) - \mu_2(i)}{2(\mu_1(i) - \mu_2(i))} \iff \rho \geq \mu_1(i), \end{aligned}$$

which holds due to (A1). Therefore, the inequalities in (29) hold.

A verification theorem. We provide a verification theorem for both Cases 1 and 2.

THEOREM 3.1. *In Case* 1*, assume* (A1)*,* (A2)*, and* (A3)*. In Case* 2*, assume* (A1)*. Then,* $v(x_1, x_2, \alpha) = x_1 w_\alpha(x_2/x_1) = V(x_1, x_2, \alpha)$*,* $\alpha = 1, 2$*. Let* $D = \{(x_1, x_2, 1) : x_2 > k_1 x_1\} \cup \{(x_1, x_2, 2) : x_2 > k_2 x_1\}$*. Let* $\tau^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D\}$*. Then* τ^* *is optimal.*

Proof. The proof is similar to that of [GZ, Theorem 2]. We only sketch the main steps for the sake of completeness. First, for any stopping time τ, following Dynkin's formula, we have

$$v(x_1, x_2, \alpha) \geq Ee^{-\rho\tau} v(X_\tau^1, X_\tau^2, \alpha_\tau) \geq Ee^{-\rho\tau}(\beta_{\rm s} X_\tau^1 - \beta_{\rm b} X_\tau^2) = J(x_1, x_2, \alpha, \tau).$$

So, $v(x_1, x_2, \alpha) \geq V(x_1, x_2, \alpha)$. The equality holds when $\tau = \tau^*$. Hence, $v(x_1, x_2, \alpha) = J(x_1, x_2, \alpha, \tau^*) = V(x_1, x_2, \alpha)$. ∎

4. Numerical examples. In this section, we give three examples, one for each case: $k_1 < k_2$, $k_1 = k_2$, or $k_1 > k_2$.

EXAMPLE 4.1 ($k_1 < k_2$). In this example, we take

$$\begin{array}{llll} \mu_1(1) = 0.20, & \mu_2(1) = 0.25, & \mu_1(2) = -0.30, & \mu_2(2) = -0.35, \\ \sigma_{11}(1) = 0.30, & \sigma_{12}(1) = 0.10, & \sigma_{21}(1) = 0.10, & \sigma_{22}(1) = 0.35, \\ \sigma_{11}(2) = 0.40, & \sigma_{12}(2) = 0.20, & \sigma_{21}(2) = 0.20, & \sigma_{22}(2) = 0.45, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

Then, we use the function $f(r)$ in (19) and find the unique zero $r_0 = 1.020254 > 1$. Using this r_0 and (20), we obtain $k_1 = 0.723270$ and $k_2 = 0.737920$. Then, we calculate and get $C_1 = 0.11442$, $C_2 = -0.00001$, $C_3 = 0.29121$, $C_4 = 0.00029$, $\eta_3 = 0.985919$, and $\eta_4 = -1.541271$. With these numbers, we verify all variational inequalities required in (A3). The graphs of the value functions are given in Figure 1.

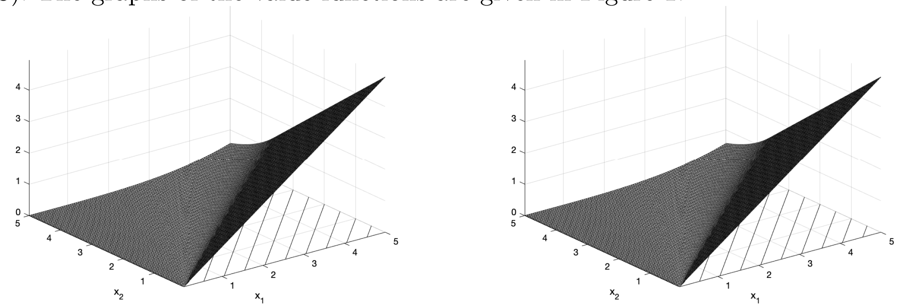

Fig. 1. Value functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

EXAMPLE 4.2 ($k_1 = k_2$). In this example, we take

$$\begin{array}{ll} \mu_1(1) = \mu_1(2) = 0.20, & \mu_2(1) = \mu_2(2) = 0.25, \\ \sigma_{11}(1) = \sigma_{11}(2) = 0.30, & \sigma_{12}(1) = \sigma_{12}(2) = 0.10, \\ \sigma_{21}(1) = \sigma_{21}(2) = 0.10, & \sigma_{22}(1) = \sigma_{22}(2) = 0.35, \\ \lambda_1 = 6.0,\ \lambda_2 = 10.0, & K = 0.001,\ \rho = 0.50. \end{array}$$

Clearly, the second equality in (27) holds, which leads to $k_1 = k_2 = k_0$. Use (28) to obtain $k_0 = 0.705098$ and $C_1 = 0.126431$. This gives the corresponding value function. Its graph is given in Figure 2.

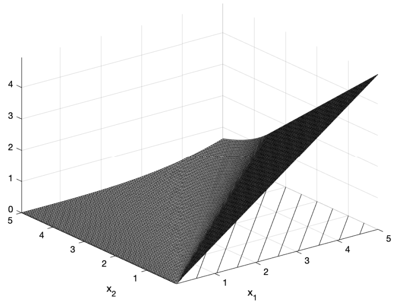

Fig. 2. Value function $V(x_1, x_2) = V(x_1, x_2, 1) = V(x_1, x_2, 2)$

EXAMPLE 4.3 ($k_1 > k_2$). Finally, we take a different set of parameters from those used in Example 4.1:

$$\begin{array}{llll} \mu_1(1) = -0.10, & \mu_2(1) = 0.20, & \mu_1(2) = 0.25, & \mu_2(2) = -0.15, \\ \sigma_{11}(1) = 0.35, & \sigma_{12}(1) = 0.15, & \sigma_{21}(1) = 0.15, & \sigma_{22}(1) = 0.30, \\ \sigma_{11}(2) = 0.20, & \sigma_{12}(2) = 0.10, & \sigma_{21}(2) = 0.10, & \sigma_{22}(2) = 0.15, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

In this example, if we apply the procedure used in Example 4.1 for k_1 and k_2, we notice some of the variational inequalities in (A3) will be violated. This means the condition $k_1 < k_2$ does not apply. Based on the symmetry of the problem in $\alpha = 1$ and $\alpha = 2$, we switch the set of parameters about $\alpha = 1$ and $\alpha = 2$ and obtain $\tilde{k}_1 = 0.379300$ and $\tilde{k}_2 = 0.824070$. The 'new' value functions $(\tilde{V}(x_1, x_2, 1), \tilde{V}(x_1, x_2, 2))$ can be obtained in a similar way. So are the verification of the variational inequalities in (A3). Then, we switch back to obtain $k_1 = \tilde{k}_2 = 0.824070$ and $k_2 = \tilde{k}_1 = 0.379300$. The same for the value functions $(V(x_1, x_2, 1) = \tilde{V}(x_1, x_2, 2)$ and $V(x_1, x_2, 2) = \tilde{V}(x_1, x_2, 1))$. Their graphs are given in Figure 3.

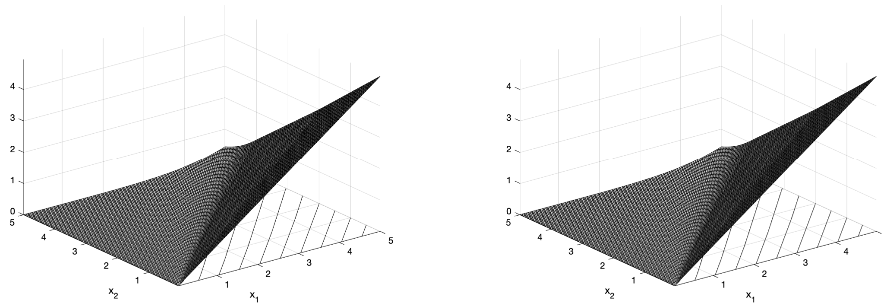

Fig. 3. Value functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

5. Conclusions. The main focus of this paper is on a pairs trade selling rule. It extends the results of McDonald and Siegel [MS] and Hu and Øksendal [HO] by incorporating models with regime switching. It would be interesting to extend the results to include the buying side of optimal timing. Besides, it would also be interesting to consider models in which the market mode α_t is not directly observable. In this case, the Wonham filter can be used for calculation of the conditional probabilities of $\alpha = 1$ given the stock prices up to time t. Some ideas along this line have been used in Dai et al. [DZZ] in connection with trend following trading.

References

[DZZ] M. Dai, Q. Zhang, Q. Zhu, *Trend following trading under a regime switching model*, SIAM J. Financial Math. 1 (2010), 780–810.

[DB] A. Deshpande, B. R. Barmish, *A general framework for pairs trading with a control-theoretic point of view*, in: 2016 IEEE Conference on Control Applications (CCA), Buenos Aires, Argentina, September 19–22, 2016.

[EHM] R. J. Elliott, J. van der Hoek, W. P. Malcolm, *Pairs trading*, Quant. Finance 5 (2005), 271–276.

[GGR] E. Gatev, W. N. Goetzmann, K. G. Rouwenhorst, *Pairs trading: Performance of a relative-value arbitrage rule*, Rev. Financ. Stud. 19 (2006), 797–827.

[GZ] X. Guo, Q. Zhang, *Optimal selling rules in a regime switching model*, IEEE Trans. Automat. Control 50 (2005), 1450–1455.

[HO] Y. Hu, B. Øksendal, *Optimal time to invest when the price processes are geometric Brownian motions*, Finance Stoch.2 (1998), 295–310.

[IB] S. Iwarere, B. R. Barmish, *A confidence interval triggering method for stock trading via feedback control*, in: 2010 American Control Conference (Baltimore, MD, 2010), IEEE, 2010, 6910–6916.

[MS] R. McDonald, D. Siegel, *The value of waiting to invest*, Quart. J. Econom. 101 (1986), 707–727.

[MZ] A. Merhi, M. Zervos, *A model for reversible investment capacity expansion*, SIAM J. Control Optim. 46 (2007), 839–876.

[O] B. Øksendal, *Stochastic Differential Equations*, 6th ed., Springer, Berlin, 2003.

[SZ] Q. S. Song, Q. Zhang, *An optimal pairs-trading rule*, Automatica J. IFAC 49 (2013), 3007–3014.

[TZZ] J. Tie, H. Zhang, Q. Zhang, *An Optimal strategy for pairs-trading under geometric Brownian motions*, J. Optim. Theory Appl. 179 (2018), 654–675.

[W] M. Whistler, *Trading Pairs: Capturing Profits and Hedging Risk with Statistical Arbitrage Strategies*, Wiley, Hobokon, 2004.

[ZZ] H. Zhang, Q. Zhang, *Trading a mean-reverting asset: Buy low and sell high*, Automatica J. IFAC 44 (2008), 1511–1518.

[Z] Q. Zhang, *Stock trading: An optimal selling rule*, SIAM J. Control Optim. 40 (2001), 64–87.