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L^p-THEORY OF FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. For forward-backward stochastic differential equations (FBSDEs, for short), under certain conditions, one has the existence and uniqueness of an adapted L^2-solution. A natural question is whether such a uniquely existed adapted L^2-solution is actually an adapted L^p-solution for some $p > 2$, under proper conditions? Such a result has its own interest in the theory of FBSDEs and it also has some important applications in optimal stochastic control theory of FBSDEs. This paper addresses such an issue in certain extent and poses some open questions.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geqslant 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following coupled *forward-backward stochastic differential equation* (FBSDE, for short):

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t))\, dt + \sigma(t, X(t), Y(t), Z(t))\, dW(t), \quad t \in [0, T], \\ dY(t) = -g(t, X(t), Y(t), Z(t))\, dt + Z(t)\, dW(t), \quad t \in [0, T], \\ X(0) = x, \qquad Y(T) = h(X(T)). \end{cases} \tag{1.1}$$

In the above, we suppose

$$\begin{aligned} &b : [0, T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^n, \quad \sigma : [0, T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^{n\times d}, \\ &g : [0, T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^m, \quad h : \mathbb{R}^n\times\Omega\to\mathbb{R}^m, \end{aligned} \tag{1.2}$$

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are some suitable maps. We call $\{b, \sigma, g, h\}$ the *generator* of FBSDE (1.1). By an *adapted solution* of (1.1), we mean a triple $(X(\cdot), Y(\cdot), Z(\cdot))$ of $\mathbb{F}$-adapted processes satisfying the following in the usual Itô's sense:

$$
\begin{aligned}
X(t) &= x + \int_0^t b(s, X(s), Y(s), Z(s))\, ds + \int_0^t \sigma(s, X(s), Y(s), Z(s))\, dW(s), \quad t \in [0, T], \\
Y(t) &= h(X(T)) + \int_t^T g(s, X(s), Y(s), Z(s))\, ds - \int_t^T Z(s)\, dW(s), \quad t \in [0, T].
\end{aligned}
\tag{1.3}
$$

Under proper conditions, for any $x \in \mathbb{R}^n$, FBSDE (1.1) admits a unique adapted solution (X, Y, Z) such that

$$
\mathbb{E}\Big[\sup_{t \in [0,T]} |X(t)|^2 + \sup_{t \in [0,T]} |Y(t)|^2 + \int_0^T |Z(t)|^2\, dt\Big] \leqslant K(1 + |x|^2), \tag{1.4}
$$

hereafter $K > 0$ stands for a generic constant which could be different from line to line. We refer to the above (X, Y, Z) as an *adapted L^2-solution* to FBSDE (1.1), and refer to the corresponding study as the *L^2-theory* of FBSDEs. For relevant details, see, for examples, [16, 26, 28] and references cited therein. In some applications, especially in the derivation of Pontryagin type maximum principle for stochastic optimal controls with recursive utilities (see [26, 13, 14, 8, 9]), one would like to have adapted solution (X, Y, Z) such that

$$
\mathbb{E}\Big[\sup_{t \in [0,T]} |X(t)|^p + \sup_{t \in [0,T]} |Y(t)|^p + \Big(\int_0^T |Z(t)|^2\, dt\Big)^{p/2}\Big] \leqslant K(1 + |x|^p), \tag{1.5}
$$

for some $p > 2$ (in particular, $p \geqslant 4$). We refer to such a triple (X, Y, Z) as an *adapted L^p-solution* to FBSDE (1.1), and refer to the relevant study as the *L^p-theory* of FBSDEs. The purpose of this paper is to revisit some known results and explore further conditions under which the adapted L^2-solution (X, Y, Z) to FBSDE (1.1) is actually its adapted L^p-solution. Our goal is to summarize what are known, to explore further in some extent, and to pose some challenging open questions. We will see from our presentation that the problem under investigation is closely related to quite a few interesting aspects of stochastic (partial) differential equations and stochastic optimal control theory.

The rest of the paper is organized as follows. In Section 2, we will look at the general situation, for which, two results will be presented: When the generator is bounded, then any adapted L^2-solution (on any time duration) will be automatically an adapted L^p-solution for any $p > 2$; under proper conditions, adapted L^p-solution will uniquely exist in small time duration. Section 3 is concerned with the situation of sub-linear generators, for which any adapted L^2-solution (on any time duration) will be automatically an adapted L^p-solution for some $p > 2$. In Section 4, we look at the so-called global decoupling which is a general version of the so-called four-step scheme (introduced in [15], see also [16]). The case of linear FBSDEs are investigated in Section 5. This is closely related to the linear-quadratic stochastic optimal control and Riccati equation (see [20, 19, 21]). In Section 6, we look at the case that the terminal function $h(\cdot)$ is either bounded or very weakly unbounded, allowing the generator of the backward equation having (diagonally) quadratic growth. Finally, some conclusion remarks will be collected in Section 7.

2. Some general considerations. In this section, we will make a general consideration on FBSDE (1.1). To begin with, let us introduce some spaces: Let $\mathbb{H}$ be a Euclidean space (which could be $\mathbb{R}^n$, $\mathbb{R}^{m\times d}$, etc.) whose norm is denoted by $|\cdot|$. Let $p, q \geqslant 1$.

$$L^p_{\mathbb{F}}(\Omega; L^q(0,T;\mathbb{H})) = \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \,|\, \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \quad \mathbb{E}\Big(\int_0^T |\varphi(s)|^q\,ds\Big)^{p/q} < \infty\Big\},$$

$$L^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{H})) = \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \,|\, \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \ t\mapsto\varphi(t) \text{ is continuous}, \quad \mathbb{E}\Big(\sup_{t\in[0,T]}|\varphi(t)|^p\Big) < \infty\Big\}.$$

We can similarly define $L^\infty_{\mathbb{F}}(\Omega; L^q(0,T;\mathbb{H}))$, $L^p_{\mathbb{F}}(\Omega; L^\infty(0,T;\mathbb{H}))$, and so on. For the case $p = q$, we denote $L^p_{\mathbb{F}}(0,T;\mathbb{H}) = L^p_{\mathbb{F}}(\Omega; L^p(0,T;\mathbb{H}))$. Further, in the case $\mathbb{H} = \mathbb{R}$, we omit $\mathbb{H}$, say, $L^p_{\mathbb{F}}(\Omega; L^q(0,T))$, for simplicity. We also define

$$\mathcal{H}^p[0,T] = L^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^m)) \times L^p_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^{m\times d})),$$
$$\mathcal{M}^p[0,T] = L^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^m)) \times L^p_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^{m\times d})).$$

We will see that $\mathcal{H}^p[0,T]$ and $\mathcal{M}^p[0,T]$ are the spaces to which (Y,Z) and (X,Y,Z) will belong, respectively. Note that in the above definition, we may replace $[0,T]$ by any $[a,b]$ with $0\leqslant a<b\leqslant T$.

Before going further, let us present the following example.

EXAMPLE 2.1. Let $a,b,c\in\mathbb{R}$. Consider the following FBSDE on $[0,T]$:

$$\begin{cases} dX(t) = [aZ(t)+b]\,dW(t), \\ dY(t) = Z(t)\,dW(t), \\ X(0) = x, \qquad Y(T) = cX(T). \end{cases} \tag{2.1}$$

Suppose (X,Y,Z) is an adapted solution. Then

$$\begin{aligned} Y(t) &= cX(T) - \int_t^T Z(s)\,dW(s) \\ &= cx + c\int_0^T [aZ(s)+b]\,dW(s) - \int_t^T Z(s)\,dW(s) \\ &= cx + bcW(T) + ac\int_0^t Z(s)\,dW(s) + (ac-1)\int_t^T Z(s)\,dW(s), \qquad t\in[0,T]. \end{aligned}$$

Now, if

$$ac-1=0, \qquad bc\neq 0, \tag{2.2}$$

then the above Y is not $\mathbb{F}$-adapted. Hence, in such a case, FBSDE (2.1) does not admit an $\mathbb{F}$-adapted solution. On the other hand, if

$$ac-1=0, \qquad b=0, \tag{2.3}$$

then (2.1) becomes the following:

$$\begin{cases} dX(t) = aZ(t)\,dW(t), \\ dY(t) = Z(t)\,dW(t), \\ X(0) = x, \qquad Y(T) = cX(T). \end{cases} \tag{2.4}$$

Therefore, for any $Z \in L^p_{\mathbb{F}}(\Omega; L^2(0,T))$, by defining

$$X(t) = x + \int_0^t aZ(s)\,dW(s), \quad Y(t) = cx + \int_0^t Z(s)\,dW(s), \qquad t \in [0,T],$$

one infers that (X,Y,Z) is an adapted solution of FBSDE (2.4), which means that adapted solutions are not unique.

Finally, if

$$ac - 1 \neq 0, \tag{2.5}$$

then from

$$Y(t) = cx + c\int_0^t [aZ(s)+b]\,dW(s) + \int_t^T \big[(ac-1)Z(s) + bc\big]\,dW(s),$$

we see the following must be true:

$$Z(t) = \frac{bc}{1-ac}\,,$$

so that

$$\begin{aligned} X(t) &= x + \int_0^t [aZ(s)+b]\,dW(s) = x + \frac{b}{1-ac}\,W(t),\\ Y(t) &= cx + c\int_0^t [aZ(s)+b]\,dW(s) = cx + \frac{b}{1-ac}\,W(t). \end{aligned}$$

Then (X,Y,Z) is the unique adapted solution of FBSDE (2.1), which is in $\mathcal{M}^p[0,T]$ for any $p > 1$.

In the above,

$$\sigma_z = a, \qquad h_x = c, \qquad h_x\sigma_z = ac.$$

Thus, (2.1) admits a unique adapted solution if and only if $h_x\sigma_z \neq 1$, regardless the size of the time horizon T, namely, no matter how small $T > 0$ is, one still needs condition like $h_x\sigma_z \neq 1$.

We now introduce the following basic assumption.

(H0) The maps

$$\begin{aligned} &b : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^n,\\ &\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^{n\times d},\\ &g : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^{m\times d},\\ &h : \mathbb{R}^n\times\Omega \to \mathbb{R}^m, \end{aligned}$$

are measurable such that

$$(t,\omega) \mapsto (b(t,x,y,z,\omega), \sigma(t,x,y,z,\omega), g(t,x,y,z,\omega)) \text{ is } \mathbb{F}\text{-progressively measurable,}$$

for every $(x,y,z) \in \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$, and $h(x,\cdot)$ is $\mathcal{F}_T$-measurable for each $x \in \mathbb{R}^n$.

Let us recall the Burkholder–Davis–Gundy's inequalities: There are constants $0 < \underline{K}_p < \overline{K}_p$ only depending on $p > 0$ such that

$$\underline{K}_p \mathbb{E}_t\Big(\int_t^T |Z(s)|^2\, ds\Big)^{p/2} \leqslant \mathbb{E}_t\Big(\sup_{r\in[t,T]}\Big|\int_t^r Z(s)\, dW(s)\Big|^p\Big)$$
$$\leqslant \overline{K}_p \mathbb{E}_t\Big(\int_t^T |Z(s)|^2\, ds\Big)^p, \quad (2.6)$$

for any $Z \in L^p_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^{m\times d}))$, where $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot \mid \mathcal{F}_t]$. For $p \geqslant 1$, one possible choice of constants $\underline{K}_p$ and $\overline{K}_p$ is (see [7], p. 285):

$$\underline{K}_p = (7+4\sqrt{2})^{-p} p^{-(p+1)}, \qquad \overline{K}_p = (2\sqrt{6})^p p^{p+1}.$$

Also, it is known that $\underline{K}_2 = 1$, $\overline{K}_2 = 4$.

Now, we present the following first basic result.

THEOREM 2.2. *Let* (H0) *hold and let*

$$\begin{aligned} &|b(t,x,y,z)| \leqslant L_{b0}(t), \quad |\sigma(t,x,y,z)| \leqslant L_{\sigma 0}(t),\\ &|g(t,x,y,z)| \leqslant L_{g0}(t), \quad |h(x)| \leqslant L_{h0}, \end{aligned} \quad (2.7)$$

for all $(t,x,y,z) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$, *with*

$$L_{b0}, L_{g0} \in L^p_{\mathbb{F}}(\Omega; L^1(0,T)), \quad L_{\sigma 0} \in L^p_{\mathbb{F}}(\Omega; L^2(0,T)), \quad L_{h0} \in L^p_{\mathcal{F}_T}(\Omega), \quad (2.8)$$

for some $p > 2$. *Let* $(X,Y,Z) \in \mathcal{M}^2[0,T]$ *be an adapted* L^2*-solution to FBSDE* (1.1). *Then it is actually an adapted* L^p*-solution of* (1.1), *and*

$$\|(X,Y,Z)\|^p_{\mathcal{M}^p[0,T]} \equiv \mathbb{E}\Big[\sup_{t\in[0,T]}|X(t)|^p + \sup_{t\in[0,T]}|Y(t)|^p + \Big(\int_0^T |Z(s)|^2\, ds\Big)^{p/2}\Big]$$
$$\leqslant K\mathbb{E}\Big[|x|^p + L^p_{h0} + \Big(\int_0^T L_{b0}(s)\, ds\Big)^p + \Big(\int_0^T L_{\sigma 0}(s)^2\, ds\Big)^{p/2} + \Big(\int_0^T L_{g0}(s)\, ds\Big)^p\Big]. \quad (2.9)$$

Proof. Suppose that (X,Y,Z) is an adapted L^2-solution of (1.1). From the FSDE

$$X(t) = x + \int_0^t b(s,X(s),Y(s),Z(s))\, ds + \int_0^t \sigma(s,X(s),Y(s),Z(s))\, dW(s), \quad t\in[0,T],$$

we obtain

$$\begin{aligned} \mathbb{E}\big[\sup_{t\in[0,T]}|X(t)|^p\big] &\leqslant 3^{p-1}\mathbb{E}\Big[|x|^p + \Big(\int_0^T L_{b0}(s)\, ds\Big)^p \\ &\quad + \sup_{t\in[0,T]}\Big|\int_0^t \sigma(s,X(s),Y(s),Z(s))\, dW(s)\Big|^p\Big] \\ &\leqslant 3^{p-1}\mathbb{E}\Big[|x|^p + \Big(\int_0^T L_{b0}(s)\, ds\Big)^p + \overline{K}_p\Big(\int_0^T L_{\sigma 0}(s)^2\, ds\Big)^{p/2}\Big], \end{aligned}$$

where $\overline{K}_p > 0$ is the constant appearing in the Burkholder–Davis–Gundy's inequalities (2.6). Next, from the BSDE

$$Y(t) = h(X(T)) + \int_t^T g(s,X(s),Y(s),Z(s))\, ds - \int_t^T Z(s)\, dW(s), \qquad t\in[0,T],$$

we have

$$
\begin{aligned}
|Y(t)| &= \Big|\mathbb{E}_t\Big[h(X(T))+\int_t^T g(s,X(s),Y(s),Z(s))\,ds\Big]\Big|\\
&\leqslant \mathbb{E}_t\Big[|h(X(T))|+\int_t^T |g(s,X(s),Y(s),Z(s))|\,ds\Big]\leqslant \mathbb{E}_t\Big[L_{h0}+\int_0^T L_{g0}(s)\,ds\Big].
\end{aligned}
$$

Then, by Doob's maximal inequality,

$$
\mathbb{E}\big[\sup_{t\in[0,T]}|Y(t)|^p\big]\leqslant \mathbb{E}\Big\{\sup_{t\in[0,T]}\Big|\mathbb{E}_t\Big[L_{h0}+\int_0^T L_{g0}(s)\,ds\Big]\Big|^p\Big\}\leqslant\Big(\frac{p}{p-1}\Big)^p\mathbb{E}\Big(L_{h0}+\int_0^T L_{g0}(s)\,ds\Big)^p.
$$

This yields (with use of (2.6))

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^T|Z(s)|^2\,ds\Big)^{p/2}\leqslant \underline{K}_p^{-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t Z(s)\,dW(s)\Big|^p\Big]\\
&\leqslant \underline{K}_p^{-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^T Z(s)\,dW(s)-\int_t^T Z(s)\,dW(s)\Big|^p\Big]\\
&\leqslant 2^p\underline{K}_p^{-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_t^T Z(s)\,dW(s)\Big|^p\Big]\\
&= 2^p\underline{K}_p^{-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|Y(t)-h(X(T))-\int_t^T g(s,X(s),Y(s),Z(s))\,ds\Big|^p\Big]\\
&\leqslant 2^p\cdot 3^{p-1}\underline{K}_p^{-1}\mathbb{E}\Big[\sup_{t\in[0,T]}|Y(t)|^p+L_{h0}^p+\Big(\int_0^T L_{g0}(s)\,ds\Big)^p\Big]\\
&\leqslant K\mathbb{E}\Big[L_{h0}^p+\Big(\int_t^T L_{g0}(s)\,ds\Big)^p\Big].
\end{aligned}
$$

Then combining the above, we obtain our conclusion. ∎

The above result shows that as long as the generator is bounded in the sense of (2.7)–(2.8), any adapted L^2-solution of (1.1) (on $[0,T]$) must also be an adapted L^p-solution.

For general situations, we introduce the following hypothesis.

(H1) Let (H0) hold and for all $t\in[0,T]$, $x,x'\in\mathbb{R}^n$, $y,y'\in\mathbb{R}^m$, $z,z'\in\mathbb{R}^{m\times d}$,

$$
\begin{cases}
|b(t,x,y,z)-b(t,x',y',z')|\leqslant L_{bx}(t)|x-x'|+L_{by}(t)|y-y'|+L_{bz}(t)|z-z'|,\\
|\sigma(t,x,y,z)-\sigma(t,x',y',z')|\leqslant L_{\sigma x}(t)|x-x'|+L_{\sigma y}(t)|y-y'|+L_{\sigma z}(t)|z-z'|,\\
|g(t,x,y,z)-g(t,x',y',z')|\leqslant L_{gx}(t)|x-x'|+L_{gy}(t)|y-y'|+L_{gz}(t)|z-z'|,\\
|h(x)-h(x')|\leqslant L_{hx}|x-x'|.
\end{cases}
\tag{2.10}
$$

for some processes $L_{bx}(\cdot),L_{by}(\cdot),L_{bz}(\cdot),L_{\sigma x}(\cdot),L_{\sigma y}(\cdot),L_{\sigma z}(\cdot),L_{gx}(\cdot),L_{gy}(\cdot),L_{gz}(\cdot)$, and a random variable L_{hx}. Also, we set

$$
|b(t,0,0,0)|=b_0(t),\quad |\sigma(t,0,0,0)|=\sigma_0(t),\quad |g(t,0,0,0)|=g_0(t),\quad |h(0)|=h_0. \tag{2.11}
$$

Under (H1), one has for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$:

$$\begin{aligned} |b(t,x,y,z)| &\leqslant b_0(t) + L_{bx}(t)|x| + L_{by}(t)|y| + L_{bz}(t)|z|, \\ |\sigma(t,x,y,z)| &\leqslant \sigma_0(t) + L_{\sigma x}(t)|x| + L_{\sigma y}(t)|y| + L_{\sigma z}(t)|z|, \\ |g(t,x,y,z)| &\leqslant g_0(t) + L_{gx}(t)|x| + L_{gy}(t)|y| + L_{gz}(t)|z|, \\ |h(x)| &\leqslant h_0 + L_{hx}|x|. \end{aligned} \tag{2.12}$$

We have the following general result.

Theorem 2.3. *Let* (H1) *hold with*

$$\begin{array}{lll} b_0(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^1(0,T)), & L_{bx}(\cdot), L_{by}(\cdot) \in L^1(0,T), & L_{bz}(\cdot) \in L^2(0,T), \\ \sigma_0(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^2(0,T)), & L_{\sigma x}(\cdot), L_{\sigma y}(\cdot) \in L^2(0,T), & L_{\sigma z}(\cdot) \in L^\infty(0,T), \\ g_0(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^1(0,T)), & L_{gx}(\cdot), L_{gy}(\cdot) \in L^1(0,T), & L_{gz}(\cdot) \in L^2(0,T), \\ h_0 \in L^p_{\mathcal{F}_T}(\Omega), & L_{hx} \in (0,\infty), & \end{array} \tag{2.13}$$

and moreover,

$$\overline{K}_p^{1/p}\Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\,\frac{2p-1}{p-1}\Big)\, L_{hx}\|L_{\sigma z}(\cdot)\|_\infty < 1. \tag{2.14}$$

Then (1.1) *admits a unique adapted L^p-solution $(X,Y,Z) \in \mathcal{M}^p[0,T]$, as long as $T > 0$ is sufficiently small.*

As we pointed out right after Example 2.1, unlike FSDEs or BSDEs, the time duration T being sufficiently small does not even guarantee the existence and uniqueness of adapted (L^2-)solution. Hence, condition (2.14) imposed in the above theorem should be acceptable. Theorem 2.3 is comparable with that in [23]. Note that by [28], when $p = 2$, condition (2.14) can be replaced by

$$L_{hx}\|L_{\sigma z}(\cdot)\|_\infty < 1. \tag{2.15}$$

Also, from Example 2.1, we see that such a condition is very close to a necessary condition.

Proof. First of all, for any $x(\cdot) \in L^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$, by (2.13), the following hold:

$$\mathbb{E}|h(x(T))|^p \leqslant \mathbb{E}\big(h_0 + L_{hx}|x(T)|\big)^p \leqslant 2^{p-1}\mathbb{E}\big(h_0^p + L_{hx}^p|x(T)|^p\big) < \infty,$$

and

$$\begin{aligned} &\mathbb{E}\Big(\int_0^T |g(t,x(t),0,0)|\,dt\Big)^p \leqslant \mathbb{E}\Big[\int_0^T \big(g_0(t) + L_{gx}(t)|x(t)|\big)\,dt\Big]^p \\ &\leqslant 2^{p-1}\mathbb{E}\Big[\Big(\int_0^T g_0(t)\,dt\Big)^p + \Big(\int_0^T L_{gx}(t)|x(t)|\,dt\Big)^p\Big] \\ &\leqslant 2^{p-1}\Big[\mathbb{E}\Big(\int_0^T g_0(t)\,dt\Big)^p + \Big(\int_0^T L_{gx}(t)\,dt\Big)^p \mathbb{E}\big(\sup_{t\in[0,T]}|x(t)|^p\big)\Big] < \infty. \end{aligned}$$

Hence, making use of (2.13) again, by [2] (see also [6]), we may let (Y,Z) to be the unique adapted solution to the following BSDE:

$$Y(t) = h(x(T)) + \int_t^T g(s,x(s),Y(s),Z(s))\,ds - \int_t^T Z(s)\,dW(s), \qquad t \in [0,T],$$

for which

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y(t)|^p + \Big(\int_0^T |Z(t)|^2\,dt\Big)^{p/2}\Big] \leqslant K\mathbb{E}\Big[|h(x(T))|^p + \Big(\int_0^T |g(t,x(t),0,0)|\,dt\Big)^p\Big].$$

With such a pair (Y,Z), we set:

$$X(t)=x+\int_0^t b(s,x(s),Y(s),Z(s))\,ds+\int_0^t \sigma(s,x(s),Y(s),Z(s))\,dW(s),\qquad t\in[0,T]. \quad (2.16)$$

By (2.13), one has

$$\begin{aligned}\mathbb{E}\Big(\int_0^T |b(t,0,Y(t),Z(t))|\,dt\Big)^p &\leqslant \mathbb{E}\Big[\int_0^T \big(b_0(t)+L_{by}(t)|Y(t)|+L_{bz}(t)|Z(t)|\big)\,dt\Big]^p\\ &\leqslant 3^{p-1}\Big[\mathbb{E}\Big(\int_0^T b_0(t)\,dt\Big)^p+\Big(\int_0^T L_{by}(t)\,dt\Big)^p\mathbb{E}\big(\sup_{t\in[0,T]}|Y(t)|^p\big)\\ &\qquad +\Big(\int_0^T L_{bz}(t)^2\,dt\Big)^{p/2}\mathbb{E}\Big(\int_0^T |Z(t)|^2\,dt\Big)^{p/2}\Big]<\infty,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\Big(\int_0^T |\sigma(t,0,Y(t),Z(t))|^2\,dt\Big)^{p/2} &\leqslant \mathbb{E}\Big[\int_0^T \big(\sigma_0(t)+L_{\sigma y}(t)|Y(t)|+L_{\sigma z}(t)|Z(t)|\big)^2\,dt\Big]^{p/2}\\ &\leqslant K\Big[\mathbb{E}\Big(\int_0^T \sigma_0(t)^2\,dt\Big)^{p/2}+\Big(\int_0^T L_{\sigma y}(t)^2\,dt\Big)^{p/2}\mathbb{E}\big(\sup_{t\in[0,T]}|Y(t)|^p\big)\\ &\qquad +\|L_{\sigma z}(\cdot)\|_\infty^p\mathbb{E}\Big(\int_0^T |Z(t)|^2\,dt\Big)^{p/2}\Big]<\infty.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]}|X(t)|^p\big]\leqslant K\mathbb{E}\Big[|x|^p&+\Big(\int_0^T |b(s,0,Y(s),Z(s))|\,ds\Big)^p\\ &+\Big(\int_0^T |\sigma(s,0,Y(s),Z(s))|^2\,ds\Big)^{p/2}\Big]<\infty.\end{aligned}$$

Through the above, we have defined a map $x(\cdot)\mapsto X$, from $L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$ to itself.

Now, let $i=1,2$, $x_i(\cdot)\in L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$ be given and let (Y_i,Z_i) be the corresponding adapted L^p-solution to the following BSDE ([2, 6]):

$$Y_i(t)=h(x_i(T))+\int_t^T g(s,x_i(s),Y_i(s),Z_i(s))\,ds-\int_t^T Z_i(s)\,dW(s).$$

Also, let $X_i(\cdot)$ be defined by

$$X_i(t)=x+\int_0^t b(s,x_i(s),Y_i(s),Z_i(s))\,ds+\int_0^t \sigma(s,x_i(s),Y_i(s),Z_i(s))\,dW(s).$$

Let

$$\begin{aligned}&\widehat{x}(\cdot)=x_1(\cdot)-x_2(\cdot), && \widehat{X}=X_1(\cdot)-X_2(\cdot),\\ &\widehat{Y}=Y_1(\cdot)-Y_2(\cdot), && \widehat{Z}=Z_1(\cdot)-Z_2(\cdot).\end{aligned}$$

Then

$$
\begin{aligned}
\widehat{X}(t) &= \int_0^t \big[b(s,x_1(s),Y_1(s),Z_1(s)) - b(s,X_2(s),Y_2(s),Z_2(s))\big]\,ds \\
&\qquad + \int_0^t \big[\sigma(s,x_1(s),Y_1(s),Z_1(s)) - \sigma(s,X_2(s),Y_2(s),Z_2(s))\big]\,dW(s) \\
&= \int_0^t \big[b_x(s)\widehat{x}(s) + b_y(s)\widehat{Y}(s) + b_z(s)\widehat{Z}(s)\big]\,ds \\
&\qquad + \int_0^t \Big[\sigma_x(s)\widehat{x}(s) + \sigma_y(s)\widehat{Y}(s) + \sigma_z(s)\widehat{Z}(s)\Big]\,dW(s),
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
\widehat{Y}(t) &= h(x_1(T)) - h(x_2(T)) + \int_t^T \big[g(s,x_1(s),Y_1(s),Z_2(s)) - g(s,x_2(s),Y_2(s),Z_2(s))\big]\,ds \\
&\qquad - \int_t^T \big[Z_1(s) - Z_2(s)\big]\,dW(s) \\
&= h_x\widehat{x}(T) + \int_t^T \big(g_x(s)\widehat{x}(s) + g_y(s)\widehat{Y}(s) + g_z(s)\widehat{Z}(s)\big)\,ds - \int_t^T \widehat{Z}(s)\,dW(s),
\end{aligned}
\tag{2.18}
$$

where

$$
b_x(s) = \frac{\big[b(s,x_1(s),Y_1(s),Z_1(s)) - b(s,x_2(s),Y_2(s),Z_2(s))\big]^\top}{|x_1(s)-x_2(s)|^2}\big[x_1(s)-x_2(s)\big]\mathbf{1}_{\{x_1(s)\neq x_2(s)\}},
$$

which is $\mathbb{R}^{n\times n}$-valued. All other coefficients $b_y(s), b_z(s)$, and so on are defined similarly. It follows that

$$
|\widehat{Y}(t)|^p = \Big|\mathbb{E}_t\Big[h_x\widehat{x}(T) + \int_t^T \big(g_x(s)\widehat{x}(s) + g_y(s)\widehat{Y}(s) + g_z(s)\widehat{Z}(s)\big)\,ds\Big]\Big|^p.
$$

Consequently,

$$
\begin{aligned}
\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat{Y}(r)|^p\big) &\leqslant \mathbb{E}_t\Big\{\sup_{r\in[t,T]}\Big|\mathbb{E}_r\Big[L_{hx}|\widehat{x}(T)| \\
&\qquad + \int_t^T \big(L_{gx}(s)|\widehat{x}(s)| + L_{gy}(s)|\widehat{Y}(s)| + L_{gz}(s)|\widehat{Z}(s)|\big)\,ds\Big]\Big|^p\Big\} \\
&\leqslant \Big(\frac{p}{p-1}\Big)^p\mathbb{E}_t\Big[L_{hx}|\widehat{x}(T)| + \int_t^T \big(L_{gx}(s)|\widehat{x}(s)| + L_{gy}(s)|\widehat{Y}(s)| + L_{gz}(s)|\widehat{Z}(s)|\big)\,ds\Big]^p.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Big[\mathbb{E}_t\Big(\sup_{r\in[t,T]}|\widehat{Y}(r)|^p\Big)\Big]^{1/p} \\
&\leqslant \frac{p}{p-1}\Big\{\mathbb{E}_t\Big[L_{hx}|\widehat{x}(T)| + \int_t^T \big(L_{gx}(s)|\widehat{x}(s)| + L_{gy}(s)|\widehat{Y}(s)| + L_{gz}(s)|\widehat{Z}(s)|\big)\,ds\Big]^p\Big\}^{1/p} \\
&\leqslant \frac{p}{p-1}\Big\{\big[\mathbb{E}_t\big(L_{hx}|\widehat{x}(T)|\big)^p\big]^{1/p} + \Big[\mathbb{E}_t\Big(\int_t^T L_{gx}(s)|\widehat{x}(s)|\,ds\Big)^p\Big]^{1/p} \\
&\qquad + \Big[\mathbb{E}_t\Big(\int_t^T L_{gy}(s)|\widehat{Y}(s)|\,ds\Big)^p\Big]^{1/p} + \Big[\mathbb{E}_t\Big(\int_t^T L_{gz}(s)|\widehat{Z}(s)|\,ds\Big)^p\Big]^{1/p}\Big\}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\mathbb{E}_t\Big(\int_t^T |\widehat Z(s)|^2\,ds\Big)^{p/2} \leqslant \underline{K}_p^{-1}\mathbb{E}_t\Big(\sup_{r\in[t,T]}\Big|\int_t^r \widehat Z(s)\,dW(s)\Big|^p\Big)\\
&= \underline{K}_p^{-1}\mathbb{E}_t\Big(\sup_{r\in[t,T]}\Big|\int_t^T \widehat Z(s)\,dW(s) - \int_r^T \widehat Z(s)\,dW(s)\Big|^p\Big)\\
&\leqslant 2^p\underline{K}_p^{-1}\mathbb{E}_t\Big(\sup_{r\in[t,T]}\Big|\int_r^T \widehat Z(s)\,dW(s)\Big|^p\Big)\\
&= 2^p\underline{K}_p^{-1}\mathbb{E}_t\Big[\sup_{r\in[t,T]}\Big|\widehat Y(r) - h_x\widehat x(T) - \int_r^T \big(g_x(s)\widehat x(s) + g_y(s)\widehat Y(s) + g_z(s)\widehat Z(s)\big)\,ds\Big|^p\Big]\\
&\leqslant 2^p\underline{K}_p^{-1}\mathbb{E}_t\Big[\sup_{r\in[t,T]}|\widehat Y(r)| + L_{hx}|\widehat x(T)|\\
&\qquad + \int_t^T \big(L_{gx}(s)|\widehat x(s)| + L_{gy}(s)|\widehat Y(s)| + L_{gz}(s)|\widehat Z(s)|\big)\,ds\Big]^p.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big[\mathbb{E}_t\Big(\int_t^T |\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p} \leqslant 2\underline{K}_p^{-1/p}\Big\{\mathbb{E}_t\Big[\sup_{r\in[t,T]}|\widehat Y(r)| + L_{hx}|\widehat x(T)|\\
&\qquad + \int_r^T \big(L_{gx}(s)|\widehat x(s)| + L_{gy}(s)|\widehat Y(s)| + L_{gz}(s)|\widehat Z(s)|\big)\,ds\Big]^p\Big\}^{1/p}\\
&\leqslant 2\underline{K}_p^{-1/p}\Big\{\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat Y(r)|^p\big)\big]^{1/p}\\
&\qquad + \Big[\mathbb{E}_t\Big(L_{hx}|\widehat x(T)| + \int_r^T \big(L_{gx}(s)|\widehat x(s)| + L_{gy}(s)|\widehat Y(s)| + L_{gz}(s)|\widehat Z(s)|\big)\,ds\Big)^p\Big]^{1/p}\Big\}\\
&\leqslant \frac{2}{\underline{K}_p^{1/p}}\,\frac{2p-1}{p-1}\Big\{\mathbb{E}_t\Big[L_{hx}|\widehat x(T)| + \int_r^T\big(L_{gx}(s)|\widehat x(s)| + L_{gy}(s)|\widehat Y(s)| + L_{gz}(s)|\widehat Z(s)|\big)ds\Big]^p\Big\}^{1/p}\\
&\leqslant 2\underline{K}_p^{-1/p}\,\frac{2p-1}{p-1}\Big\{\Big(L_{hx} + \int_t^T L_{gx}(s)\,ds\Big)\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat x(r)|^p\big)\big]^{1/p}\\
&\qquad + \Big(\int_t^T L_{gy}(s)\,ds\Big)\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat Y(r)|^p\big)\big]^{1/p}\\
&\qquad + \Big(\int_t^T L_{gz}(s)^2\,ds\Big)^{1/2}\Big[\mathbb{E}_t\Big(\int_t^T |\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\Big\}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat Y(r)|^p\big)\big]^{1/p} + \Big[\mathbb{E}_t\Big(\int_t^T |\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\leqslant \Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)\Big(L_{hx} + \int_t^T L_{gx}(s)\,ds\Big)\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat x(r)|^p\big)\big]^{1/p}\\
&\qquad + \Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\,\frac{2p-1}{p-1}\Big)\Big[\Big(\int_t^T L_{gy}(s)\,ds\Big) \vee \Big(\int_t^T L_{gz}(s)^2\,ds\Big)^{1/2}\Big]\\
&\times \Big\{\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat Y(r)|^p\big)\big]^{1/p} + \Big[\mathbb{E}_t\Big(\int_t^T |\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\Big\}.
\end{aligned}
$$

When $T-t>0$ is small enough, one has

$$
\begin{aligned}
&\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat Y(r)|^p\big)\big]^{1/p}+\Big[\mathbb{E}_t\Big(\int_t^T|\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\leqslant\Big(\frac{p}{p-1}+2\underline{K}_p^{-1/p}\,\frac{2p-1}{p-1}\Big)\Big(L_{hx}+\int_t^T L_{gx}(s)\,ds\Big)\\
&\qquad\times\Big\{1-\Big(\frac{p}{p-1}+2\underline{K}_p^{-1/p}\,\frac{2p-1}{p-1}\Big)\Big[\Big(\int_t^T L_{gy}(s)\,ds\Big)\vee\Big(\int_t^T L_{gz}(s)^2\,ds\Big)^{1/2}\Big]\Big\}^{-1}\\
&\qquad\times\big[\mathbb{E}_t\big(\sup_{r\in[t,T]}|\widehat x(r)|^p\big)\big]^{1/p}.
\end{aligned}
$$

In particular, with $t=0$ and $T>0$ small enough, we have

$$
\begin{aligned}
&\big[\mathbb{E}\big(\sup_{r\in[0,T]}|\widehat Y(r)|^p\big)\big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T|\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\leqslant\Big(\frac{p}{p-1}+2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)\Big(L_{hx}+\int_0^T L_{gx}(s)\,ds\Big)\\
&\qquad\times\Big\{1-\Big(\frac{p}{p-1}+2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)\Big[\Big(\int_0^T L_{gy}(s)\,ds\Big)\vee\Big(\int_0^T L_{gz}(s)^2\,ds\Big)^{1/2}\Big]\Big\}^{-1}\\
&\qquad\times\big[\mathbb{E}\big(\sup_{r\in[0,T]}|\widehat x(r)|^p\big)\big]^{1/p}.
\end{aligned}
$$

Also, from (2.17), we have

$$
\begin{aligned}
&\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat X(t)|^p\big)\big]^{1/p}\leqslant\Big\{\mathbb{E}\Big[\int_0^T\big(L_{bx}(s)|\widehat x(s)|+L_{by}(s)|\widehat Y(s)|+L_{bz}(s)|\widehat Z(s)|\big)\,ds\Big]^p\Big\}^{1/p}\\
&\qquad+\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\Big(\sigma_x(s)\widehat x(s)+\sigma_y(s)\widehat Y(s)+\sigma_z(s)\widehat Z(s)\Big)\,dW(s)\Big|^p\Big]\Big\}^{1/p}\\
&\leqslant\Big(\int_0^T L_{bx}(s)\,ds\Big)\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat x(t)|^p\big)\big]^{1/p}+\Big(\int_0^T L_{by}(s)\,ds\Big)\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat Y(t)|^p\big)\big]^{1/p}\\
&\qquad+\Big(\int_0^T L_{bz}(s)^2\,ds\Big)^{1/2}\Big[\mathbb{E}\Big(\int_0^T\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\qquad+\overline{K}_p^{1/p}\Big\{\Big[\mathbb{E}\Big(\int_0^T L_{\sigma x}(s)^2|\widehat x(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\qquad+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma y}(s)^2|\widehat Y(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma z}(s)^2|\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\Big\}\\
&\leqslant\Big[\int_0^T L_{bx}(s)\,ds+\overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma x}(s)^2\,ds\Big)^{1/2}\Big]\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat x(t)|^p\big)\big]^{1/p}\\
&\qquad+\Big[\int_0^T L_{by}(s)\,ds+\overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma y}(s)^2\,ds\Big)^{1/2}\Big]\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat Y(t)|^p\big)\big]^{1/p}\\
&\qquad+\Big[\Big(\int_0^T L_{bz}(s)^2\,ds\Big)^{1/2}+\overline{K}_p^{1/p}\|L_{\sigma z}(\cdot)\|_\infty\Big]\Big[\mathbb{E}\Big(\int_0^T|\widehat Z(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Big[\int_0^T L_{bx}(s)\,ds + \overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma x}(s)^2\,ds\Big)^{1/2}\Big]\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat{x}(t)|^p\big)\big]^{1/p}\\
&\quad+\Big\{\Big[\int_0^T L_{by}(s)\,ds + \overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma y}(s)^2\,ds\Big)^{1/2}\Big]\\
&\qquad\vee\Big[\Big(\int_0^T L_{bz}(s)^2\,ds\Big)^{1/2} + \overline{K}_p^{1/p}\|L_{\sigma z}(\cdot)\|_\infty\Big]\Big\}\\
&\quad\times\Big\{\big[\mathbb{E}\Big(\sup_{t\in[0,T]}|\widehat{Y}(t)|^p\Big)\big]^{1/p} + \Big[\mathbb{E}\Big(\int_0^T|\widehat{Z}(s)|^2\,ds\Big)^{p/2}\Big]^{1/p}\Big\}\\
&\leqslant\Big\{\int_0^T L_{bx}(s)\,ds + \overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma x}(s)^2\,ds\Big)^{1/2}\\
&\quad+\Big\{\Big[\int_0^T L_{by}(s)\,ds + \overline{K}_p^{1/p}\Big(\int_0^T L_{\sigma y}(s)^2\,ds\Big)^{1/2}\Big]\\
&\quad\vee\Big[\Big(\int_0^T L_{bz}(s)^2\,ds\Big)^{1/2} + \overline{K}_p^{1/p}\|L_{\sigma z}(\cdot)\|_\infty\Big]\Big\}\\
&\quad\times\Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)\Big(L_{hx} + \int_0^T L_{gx}(s)\,ds\Big)\\
&\quad\times\Big\{1 - \Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)\Big[\Big(\int_0^T L_{gy}(s)\,ds\Big)\vee\Big(\int_0^T L_{gz}(s)^2\,ds\Big)^{1/2}\Big]\Big\}^{-1}\Big\}\\
&\quad\times\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat{x}(t)|^p\big)\big]^{1/p} \equiv (K_0 + K(T))\big[\mathbb{E}\big(\sup_{t\in[0,T]}|\widehat{x}(t)|^p\big)\big]^{1/p},
\end{aligned}
$$

where

$$
K_0 = \overline{K}_p^{1/p}\Big(\frac{p}{p-1} + 2\underline{K}_p^{-1/p}\frac{2p-1}{p-1}\Big)L_{hx}\|L_{\sigma z}(\cdot)\|_\infty,\qquad \lim_{T\to 0}K(T) = 0.
$$

Hence, if (2.14) holds, then for $T > 0$ small enough, FBSDE admits an adapted L^p-solution on $[0,T]$. ■

It is seen that even if assuming (2.14), we could still only obtain the adapted L^p-solution for the FBSDE (1.1) in some small time duration.

3. Sub-linear growth generators.

We have seen that when the generator is uniformly bounded in (x, y, z), to get an adapted L^p-solution from an adapted L^2-solution is quite easy. However, in Theorem 2.3, we have to assume not only condition (2.14), but also the time duration has to be small enough. In this section, we would like to look at the case that the generator is sublinearly growing in (x, y, z). We will see that for such cases, we will need neither restriction on the time duration T, nor the condition (2.14). The basic idea will be similar to the bounded generator case, but with more careful estimates.

To be more precise, let us introduce the following assumption.

(H2) Let (H0) hold. Let

$$
\begin{aligned}
&|b(s,x,y,z) - b(s,x',y,z)| \leqslant L_{bx}(s)|x - x'|,\\
&|\sigma(s,x,y,z) - \sigma(s,x',y,z))| \leqslant L_{\sigma x}(s)|x - x'|,\\
&|g(s,x,y,z) - g(s,x,y',z')| \leqslant L_{gy}(s)|y - y'| + L_{gz}(s)|z - z'|,
\end{aligned}
\tag{3.1}
$$

for all $s \in [0,T]$, $x,x' \in \mathbb{R}^n$, $y,y' \in \mathbb{R}^m$, $z,z' \in \mathbb{R}^{m\times d}$, and

$$\begin{aligned}
&|b(s,0,y,z)| \leqslant L_{b0}(s) + L_{by}(s)|y|^\alpha + L_{bz}(s)|z|^\alpha,\\
&|\sigma(s,0,y,z)| \leqslant L_{\sigma 0}(s) + L_{\sigma y}(s)|y|^\alpha + L_{\sigma z}(s)|z|^\alpha,\\
&|g(s,x,0,0)| \leqslant L_{g0}(s) + L_{gx}(s)|x|^\alpha, \qquad |h(x)| \leqslant L_{h0} + L_{hx}(s)|x|^\alpha,
\end{aligned} \tag{3.2}$$

for all $(s,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, where $\alpha \in (0,1)$ and for some $1 < p < \frac{2}{\alpha}$,

$$\begin{cases}
L_{bx}(\cdot), L_{gy}(\cdot) \in L^1(0,T), & L_{\sigma x}(\cdot), L_{gz}(\cdot) \in L^2(0,T),\\
L_{b0}(\cdot), L_{g0}(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^1(0,T)), & L_{by}(\cdot), L_{gx}(\cdot) \in L^{2p/(2-\alpha p)}_{\mathbb{F}}(\Omega; L^1(0,T)),\\
L_{bz}(\cdot) \in L^{p/(1-\alpha)}_{\mathbb{F}}(\Omega; L^{2/(2-\alpha)}(0,T)), & L_{\sigma 0}(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^2(0,T)),\\
L_{\sigma y}(\cdot) \in L^{2p/(2(2-\alpha p))}_{\mathbb{F}}(\Omega; L^2(0,T)), & L_{\sigma z}(\cdot) \in L^{p/(1-\alpha)}_{\mathbb{F}}(\Omega; L^{2/(1-\alpha)}(0,T)),\\
L_{h0} \in L^p_{\mathcal{F}_T}(\Omega), & L_{hx} \in L^{2p/(2-\alpha p)}_{\mathcal{F}_T}(\Omega).
\end{cases} \tag{3.3}$$

We see that the smaller the $\alpha > 0$ is, the larger the $p > 2$ could be. Also, if all the involved processes and random variables appearing in (3.3) are bounded, then (3.3) holds. We have the following result.

THEOREM 3.1. *Let* (H2) *hold. Let* $(X,Y,Z) \in \mathcal{M}^2[0,T]$ *be an adapted* L^2*-solution to FBSDE* (1.1). *Then it must be an adapted* L^p*-solution.*

Proof. Let $(X,Y,Z) \in \mathcal{M}^2[0,T]$ be an adapted L^2-solution to the FBSDE. Then X is a strong solution to the FSDE (for given $(Y,Z) \in \mathcal{H}^2[0,T]$)

$$\begin{cases}
dX(t) = b(t,X(t),Y(t),Z(t))\,dt + \sigma(t,X(t),Y(t),Z(t))\,dW(t), \qquad t \in [0,T],\\
X(0) = x.
\end{cases}$$

Since $1 < p < \frac{2}{\alpha}$, we have

$$\begin{aligned}
&\Big[\mathbb{E}\Big(\int_0^T |b(s,0,Y(s),Z(s))|\,ds\Big)^p\Big]^{1/p}\\
&\leqslant \Big\{\mathbb{E}\Big[\int_0^T \Big(L_{b0}(s) + L_{by}(s)|Y(s)|^\alpha + L_{bz}(s)|Z(s)|^\alpha\Big)\,ds\Big]^p\Big\}^{1/p}\\
&\leqslant \Big[\mathbb{E}\Big(\int_0^T L_{b0}(s)\,ds\Big)^p\Big]^{1/p} + \Big[\mathbb{E}\Big(\int_0^T L_{by}(s)|Y(s)|^\alpha\,ds\Big)^p\Big]^{1/p} + \Big[\mathbb{E}\Big(\int_0^T L_{bz}(s)|Z(s)|^\alpha\,ds\Big)^p\Big]^{1/p}\\
&\leqslant \Big[\mathbb{E}\Big(\int_0^T L_{b0}(s)\,ds\Big)^p\Big]^{1/p} + \Big\{\mathbb{E}\Big[\Big(\int_0^T L_{by}(s)\,ds\Big)^p\Big(\sup_{s\in[0,T]} |Y(s)|^{\alpha p}\Big)\Big]\Big\}^{1/p}\\
&\quad + \Big\{\mathbb{E}\Big[\Big(\int_0^T L_{bz}(s)^{2/(2-\alpha)}\,ds\Big)^{(2-\alpha)p/2}\Big(\int_0^T |Z(s)|^2\,ds\Big)^{\alpha p/2}\Big]\Big\}^{1/p}\\
&= \Big[\mathbb{E}\Big(\int_0^T L_{b0}(s)\,ds\Big)^p\Big]^{1/p} + \Big[\mathbb{E}\Big(\int_0^T L_{by}(s)\,ds\Big)^{\frac{2p}{2-\alpha p}}\Big]^{\frac{2-\alpha p}{2p}}\Big[\mathbb{E}\Big(\sup_{s\in[0,T]} |Y(s)|^2\Big)\Big]^{\alpha/2}\\
&\quad + \Big[\mathbb{E}\Big(\int_0^T L_{bz}(s)^{2/(2-\alpha)}\,ds\Big)^{(2-\alpha)p/(2(1-\alpha))}\Big]^{(1-\alpha)/p}\Big[\mathbb{E}\Big(\int_0^T |Z(s)|^2\,ds\Big)^{p/2}\Big]^{\alpha/p}.
\end{aligned}$$

Similarly, we have (with $1<p<\frac{2}{\alpha}$)

$$
\begin{aligned}
&\Big[\mathbb{E}\Big(\int_0^T|\sigma(s,0,Y(s),Z(s))|^2\,ds\Big)^{p/2}\Big]^{1/p}\\
&\leqslant\Big\{\mathbb{E}\Big[\int_0^T\big(L_{\sigma0}(s)+L_{\sigma y}(s)|Y(s)|^\alpha+L_{\sigma z}(s)|Z(s)|^\alpha\big)^2\,ds\Big]^{p/2}\Big\}^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{\sigma0}(s)^2\,ds\Big)^{p/2}\Big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma y}(s)^2|Y(s)|^{2\alpha}\,ds\Big)^{p/2}\Big]^{1/p}\\
&\qquad+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma z}(s)^2|Z(s)|^{2\alpha}\,ds\Big)^{p/2}\Big]^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{\sigma0}(s)^2\,ds\Big)^{p/2}\Big]^{1/p}+\Big\{\mathbb{E}\Big[\Big(\int_0^T L_{\sigma y}(s)^2\,ds\Big)^{p/2}\big(\sup_{s\in[0,T]}|Y(s)|\big)^{\alpha p}\Big]\Big\}^{1/p}\\
&\qquad+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma z}(s)^{2/(1-\alpha)}\,ds\Big)^{(1-\alpha)p/2}\Big(\int_0^T|Z(s)|^2\,ds\Big)^{\alpha p/2}\Big]^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{\sigma0}(s)^2\,ds\Big)^{p/2}\Big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma y}(s)^2\,ds\Big)^{\frac{p}{2-\alpha p}}\Big]^{\frac{2-\alpha p}{2p}}\big[\mathbb{E}\big(\sup_{s\in[0,T]}|Y(s)|^2\big)\big]^\alpha\\
&\qquad+\Big[\mathbb{E}\Big(\int_0^T L_{\sigma z}(s)^{2/(1-\alpha)}\,ds\Big)^{p/2}\Big]^{(1-\alpha)/p}\Big[\mathbb{E}\Big(\int_0^T|Z(s)|^2\,ds\Big)^{p/2}\Big]^{\alpha/p}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\mathbb{E}\big[\sup_{t\in[0,T]}|X(t)|^p\big]\leqslant K\mathbb{E}\Big[|x|^p&+\Big(\int_0^T|b(s,0,Y(s),Z(s))|\,ds\Big)^p\\
&+\Big(\int_0^T|\sigma(s,0,Y(s),Z(s))|^2\,ds\Big)^{p/2}\Big]<\infty.
\end{aligned}
$$

We now look at the BSDE (for given $X\in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$). Note that

$$
\begin{aligned}
&\Big[\mathbb{E}\Big(\int_0^T|g(s,X(s),0,0)|\,ds\Big)^p\Big]^{1/p}\leqslant\Big\{\mathbb{E}\Big[\int_0^T\big(L_{g0}(s)+L_{gx}(s)|X(s)|^\alpha\big)\,ds\Big]^p\Big\}^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{g0}(s)\,ds\Big)^p\Big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T L_{gx}(s)|X(s)|^\alpha\,ds\Big)^p\Big]^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{g0}(s)\,ds\Big)^p\Big]^{1/p}+\Big\{\mathbb{E}\Big[\Big(\int_0^T L_{gx}(s)\,ds\Big)^p\big(\sup_{s\in[0,T]}|X(s)|\big)^{\alpha p}\Big]\Big\}^{1/p}\\
&\leqslant\Big[\mathbb{E}\Big(\int_0^T L_{g0}(s)\,ds\Big)^p\Big]^{1/p}+\Big[\mathbb{E}\Big(\int_0^T L_{gx}(s)\,ds\Big)^{\frac{2p}{2-\alpha p}}\Big]^{\frac{2-\alpha p}{2p}}\big[\mathbb{E}\big(\sup_{s\in[0,T]}|X(s)|^2\big)\big]^{\alpha/2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[\mathbb{E}\Big(|h(X(T))|^p\Big)\big]^{1/p}\leqslant\big[\mathbb{E}\Big(L_{h0}+L_{hx}|X(T)|^\alpha\Big)^p\big]^{1/p}\\
&\leqslant\big[\mathbb{E}L_{h0}^p\big]^{1/p}+\big[\mathbb{E}\Big(L_{hx}^p|X(T)|^{\alpha p}\Big)\big]^{1/p}\leqslant\big[\mathbb{E}L_{h0}^p\big]^{1/p}+\Big[\mathbb{E}L_{hx}^{\frac{2p}{2-\alpha p}}\Big]^{\frac{2-\alpha p}{2p}}\Big[\mathbb{E}|X(T)|^2\Big]^{\alpha/2}.
\end{aligned}
$$

Hence, our conclusion follows. ∎

4. Global decoupling. In this section, we look at another approach, inspired by the so-called *Four-Step Scheme* ([15]). More precisely, suppose (X, Y, Z) is an adapted solution to FBSDE (1.1). We assume that the backward component Y admits the following representation, in terms of the forward component X:

$$Y(t) = v(t, X(t)), \qquad t \in [0, T], \tag{4.1}$$

for some random field $v(\cdot\,,\cdot)$. Let $(v(\cdot\,,\cdot), q(\cdot\,,\cdot))$ satisfy the following:

$$v^k(t, x) = v^k(T, x) - \int_t^T p^k(s, x)\, ds - \int_t^T \sum_{i=1}^d q^{ik}(s, x)\, dW_i(s), \qquad t \in [0, T],\ 1 \leqslant k \leqslant d, \tag{4.2}$$

with $p^k(\cdot\,,\cdot)$, $q^{ik}(\cdot\,,\cdot)$ being proper $\mathbb{R}$-valued random fields. By Itô–Ventzell's formula,

$$\begin{aligned} v^k(t, X(t)) &= v^k(T, X(T)) - \int_t^T \Big\{ p^k(s, X(s)) \\ &+ \frac{1}{2}\mathrm{tr}\left[v^k_{xx}(s, X(s))(\sigma\sigma^\top)(s, X(s), Y(s), Z(s))\right] \\ &+ v^k_x(s, X(s)) b(s, X(s), Y(s), Z(s)) \\ &+ \mathrm{tr}\left[q^k_x(s, X(s))\sigma(s, X(s), Y(s), Z(s))\right] \Big\}\, ds \\ &- \int_t^T \left[q^k(s, X(s)) + v^k_x(s, X(s))\sigma(s, X(s), Y(s), Z(s))\right] dW(s) \end{aligned} \tag{4.3}$$

Comparing with the BSDE

$$Y(t) = h(X(T)) + \int_t^T g(s, X(s), Y(s), Z(s))\, ds - \int_t^T Z(s)\, dW(s), \tag{4.4}$$

we see that the following should be satisfied

$$\begin{cases} Z(t) = q(t, X(t)) + v_x(t, X(t))\sigma(t, X(t), v(t, X(t)), Z(t)), \\ p^k(t, X(t)) = -g^k(t, X(t), v(t, X(t)), Z(t)) \\ \qquad\qquad - \dfrac{1}{2}\mathrm{tr}\left[v^k_{xx}(t, X(t))(\sigma\sigma^\top)(t, X(t), v(t, X(t)), Z(t))\right] \\ \qquad\qquad - v^k_x(t, X(t)) b(t, X(t), v(t, X(t)), Z(t)) \\ \qquad\qquad - \mathrm{tr}\left[q^k_x(t, X(t))\sigma(t, X(t), v(t, X(t)), Z(t))\right], \\ v(T, X(T)) = h(X(T)). \end{cases} \tag{4.5}$$

Hence, we obtain the following system of *backward stochastic partial differential equations* (BSPDE, for short) coupled with an algebraic equation:

$$\begin{cases} v^k(t, x) = h^k(x) + \displaystyle\int_t^T \Big\{ g^k(s, x, v(s, x), z) + \frac{1}{2}\mathrm{tr}\left[v^k_{xx}(s, x)(\sigma\sigma^\top)(s, x, v(s, x), z)\right] \\ \qquad\qquad + v^k_x(s, x) b(s, x, v(s, x), z) + \mathrm{tr}\left[q^k_x(s, x)\sigma(s, x, v(s, x), z)\right] \Big\}\, ds \\ \qquad\qquad - \displaystyle\int_t^T q^k(s, x)\, dW(s), \qquad (t, x) \in [0, T] \times \mathbb{R}^n, \text{ a.s.} \\ z = q(t, x) + v_x(t, x)\sigma(t, x, v(t, x), z), \qquad (t, x) \in [0, T] \times \mathbb{R}^n. \end{cases} \tag{4.6}$$

Suppose the above BSPDE admits an adapted strong solution $(v(\cdot\,,\cdot),q(\cdot\,,\cdot))$ (with enough regularity). As a consequence,

$$Z(t)=\zeta(t,X(t)),\qquad t\in[0,T].$$

Then we consider the following FSDE:

$$\begin{cases} dX(t)=b\big(t,X(t),v(t,X(t)),\zeta(t,X(t))\big)\,dt\\ \qquad\qquad +\sigma\big(t,X(t),v(t,X(t)),\zeta(t,X(t))\big)\,dW(t),\quad t\in[0,T],\\ X(0)=x.\end{cases}\tag{4.7}$$

When the above FSDE admits a (strong) solution X, then we define Y by (4.1) and determine Z by the first equation in (4.5), the triple (X,Y,Z) turns out to be an adapted solution to (1.1). Consequently, if the solution X to (4.7) is in $L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$, then by the usual arguments for BSDEs ([2, 6]), we obtain an adapted L^p-solution (X,Y,Z) to (1.1). We refer to the above procedure as the *global decoupling*, which is essentially the random coefficient version of the so-called *Four-Step Scheme* introduced in [15] (see [16, 17] also).

Let us make an observation. For convenience, let $d=1$. Then in order the last equation in (4.6) to admit a unique solution z, we need that

$$\det\big[I-v_x(t,x)\sigma_z(t,x,v(t,x),z)\big]\neq 0.$$

In particular, at $t=T$, we need

$$\det\big[I-h_x(x)\sigma_z(T,x,h(x),z)\big]\neq 0.$$

This is comparable with the solvability condition $1-ac\neq 0$ presented in Example 2.1. See also the comments at the end of Section 2.

Let us now look at some important special cases.

1. *The random field σ is independent of z.* In this case, our FBSDE reads

$$\begin{cases} dX(t)=b(t,X(t),Y(t),Z(t))\,dt+\sigma(t,X(t),Y(t))\,dW(t),\quad t\in[0,T],\\ dY(t)=-g(t,X(t),Y(t),Z(t))\,dt+Z(t)\,dW(t),\quad t\in[0,T],\\ X(0)=x,\qquad Y(T)=h(X(T)).\end{cases}\tag{4.8}$$

and the last equation in (4.6) becomes

$$z=q(t,x)+v_x(t,x)\sigma(t,x,v(t,x)).\tag{4.9}$$

Consequently, the BSPDE in (4.6) takes a simpler form:

$$\begin{aligned} v^k(t,x)=h^k(x)+\int_t^T\Big\{&g^k(s,x,v(s,x),q(s,x)+v_x(s,x)\sigma(s,x,v(s,x)))\\ &+\frac{1}{2}\mathrm{tr}\,\big[v^k_{xx}(s,x)(\sigma\sigma^\top)(s,x,v(s,x))\big]\\ &+v^k_x(s,x)b\big(s,x,v(s,x),q(s,x)+v_x(s,x)\sigma(s,x,v(s,x))\big)\\ &+\mathrm{tr}\,\big[q^k_x(s,x)\sigma(s,x,v(s,x))\big]\Big\}\,ds-\int_t^T q^k(s,x)\,dW(s),\\ &\qquad\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.}\end{aligned}\tag{4.10}$$

Some relevant results can be found in [5, 22]. But, they are not enough to realize the above global decoupling.

2. *Coefficients are deterministic and σ is independent of z.* In this case, BSPDE becomes a system of parabolic PDEs:

$$\begin{cases} v^k_t(t,x) + \frac{1}{2}\mathrm{tr}\left[v^k_{xx}(t,x)\sigma\sigma^\top(t,x,v(t,x))\right] + v^k_x(t,x)b(t,x,v(t,x)) \\ \qquad + g^k(t,x,v(t,x),v_x(t,x)\sigma(t,x,v(t,x))) = 0, \\ \qquad\qquad (t,x)\in[0,T]\times\mathbb{R}^n, \quad 1\leqslant k\leqslant n, \\ v(T,x) = h(x), \qquad x\in\mathbb{R}^n. \end{cases} \tag{4.11}$$

For such a system, by [12] (see also [15]), we know that under proper conditions it admits a unique classical solution $v(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, with $v(\cdot\,,\cdot), v_x(\cdot\,,\cdot), v_{xx}(\cdot\,,\cdot)$ all being bounded. Then (Y,Z) are bounded and hence X is in L^p for any $p>1$. This means that the global decoupling which is now the four-step scheme can be realized.

The results presented above are mainly formal and far from satisfactory. We pose the following open problem.

Open Problem 1.
(a) Under what suitable (nontrivial) conditions, is BSPDE (4.6) well-posed with good enough regularity?
(b) Find conditions under which BSPDE (4.10) is well-posed with satisfied regularity.
(c) How can one realize the global decoupling when σ is degenerate?

5. Linear FBSDEs and Riccati equation. In this section, we are going to look at linear FBSDEs. Although they are special cases of general nonlinear FBSDEs, the general theory does not cover such special cases.

Consider the following linear FBSDE on $[0,T]$:

$$\begin{cases} dX(t) = \Big[A_0(t)X(t) + B_0(t)Y(t) + \sum_{\ell=1}^d C_{0\ell}(t)Z_\ell(t) + b(t)\Big]\,dt \\ \qquad + \sum_{k=1}^d \Big[A_k(t)X(t) + B_k(t)Y(t) + \sum_{\ell=1}^d C_{k\ell}(t)Z_\ell(t) + \sigma_k(t)\Big]\,dW_k(t), \\ dY(t) = -\Big[\widehat{A}(t)X(t) + \widehat{B}(t)Y(t) + \sum_{\ell=1}^d \widehat{C}_\ell(t)Z_\ell(t) + g(t)\Big]\,dt + \sum_{k=1}^d Z_k(t)\,dW_k(t), \\ X(0) = x, \qquad Y(T) = HX(T) + h. \end{cases} \tag{5.1}$$

In the above, $A_0(\cdot), B_0(\cdot)$ and so on, are called coefficients, and $b(\cdot), \sigma_k(\cdot), g(\cdot), h$ are called non-homogeneous terms. Here, we introduce the following assumption.

(H3) Let the coefficients satisfy:

$$\begin{cases} A_k(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n}), & 0\leqslant k\leqslant d, \\ B_k(\cdot), C_{k\ell}(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times m}), & 0\leqslant k\leqslant d,\ 1\leqslant \ell\leqslant d, \\ \widehat{A}(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m\times n}), \quad \widehat{B}(\cdot), \widehat{C}_\ell(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m\times m}), & 1\leqslant \ell\leqslant d, \\ H\in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{m\times n}), \end{cases} \tag{5.2}$$

and the non-homogeneous terms satisfy

$$\begin{cases} b(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \sigma_k(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n)), \quad 1\leqslant k\leqslant d,\\ g(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^m)), \quad h\in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^m). \end{cases} \tag{5.3}$$

Suppose that (X,Y,Z) is an adapted solution to (5.1), and that

$$Y(t)=P(t)X(t)+\eta(t), \tag{5.4}$$

where $P(\cdot)$ satisfies

$$\begin{cases} dP(t)=\Gamma(t)\,dt+\sum_{k=1}^d\Lambda_k(t)\,dW_k(t),\\ P(T)=H, \end{cases} \tag{5.5}$$

with $\Gamma(\cdot)$ and $\Lambda_k(\cdot)$ undetermined, and $\eta(\cdot)$ satisfies

$$\begin{cases} d\eta(t)=\alpha(t)\,dt+\sum_{k=1}^d\zeta_k(t)\,dW_k(t),\\ \eta(T)=h, \end{cases} \tag{5.6}$$

with $\alpha(\cdot)$ and $\zeta_k(\cdot)$ undetermined. Then (suppressing t)

$$\begin{aligned} &-\Big[\widehat AX+\widehat BY+\sum_{k=1}^d\widehat C_kZ_k+g\Big]\,dt+\sum_{k=1}^dZ_k\,dW_k=dY=d(PX+\eta)\\ &=\Big[\Gamma X+P\Big(A_0X+B_0Y+\sum_{k=1}^dC_{0k}Z_k+b\Big)\\ &\qquad+\sum_{k=1}^d\Lambda_k\Big(A_kX+B_kY+\sum_{\ell=1}^dC_{k\ell}Z_\ell+\sigma_k\Big)+\alpha\Big]\,dt\\ &\qquad+\sum_{k=1}^d\Big[P\Big(A_kX+B_kY+\sum_{\ell=1}^dC_{k\ell}Z_\ell+\sigma_k\Big)+\Lambda_kX+\zeta_k\Big]\,dW_k. \end{aligned}$$

Hence, one should have

$$\begin{aligned} Z_k&=PA_kX+PB_k(PX+\eta)+P\sum_{\ell=1}^dC_{k\ell}Z_\ell+P\sigma_k+\Lambda_kX+\zeta_k\\ &=\sum_{\ell=1}^dPC_{k\ell}Z_\ell+\big[P(A_k+B_kP)+\Lambda_k\big]X+P(B_k\eta+\sigma_k)+\zeta_k, \quad 1\leqslant k\leqslant d. \end{aligned}$$

Or, equivalently,

$$\begin{pmatrix} Z_1\\ Z_2\\ \vdots\\ Z_d\end{pmatrix}=\begin{pmatrix} PC_{11} & PC_{12} & \cdots & PC_{1d}\\ PC_{21} & PC_{22} & \cdots & PC_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ PC_{d1} & PC_{d2} & \cdots & PC_{dd}\end{pmatrix}\begin{pmatrix} Z_1\\ Z_2\\ \vdots\\ Z_d\end{pmatrix}+\begin{pmatrix} [P(A_1+B_1P)+\Lambda_1]X+P(B_1\eta+\sigma_1)+\zeta_1\\ [P(A_2+B_2P)+\Lambda_2]X+P(B_2\eta+\sigma_2)+\zeta_2\\ \vdots\\ [P(A_d+B_dP)+\Lambda_d]X+P(B_d\eta+\sigma_d)+\zeta_d\end{pmatrix}.$$

Thus, we should have

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_d \end{pmatrix} = \begin{pmatrix} I-PC_{11} & -PC_{12} & \cdots & -PC_{1d} \\ -PC_{21} & I-PC_{22} & \cdots & -PC_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ -PC_{d1} & -PC_{d2} & \cdots & I-PC_{dd} \end{pmatrix}^{-1} \begin{pmatrix} [P(A_1+B_1P)+\Lambda_1]X+P(B_1\eta+\sigma_1)+\zeta_1 \\ [P(A_2+B_2P)+\Lambda_2]X+P(B_2\eta+\sigma_2)+\zeta_2 \\ \vdots \\ [P(A_d+B_dP)+\Lambda_d]X+P(B_d\eta+\sigma_d)+\zeta_d \end{pmatrix},$$

and assume the inverse exists. The existence of such an inverse is comparable with the invertibility of the matrix $I - h_x(x)\sigma_z(s, x, y, z)$ mentioned at the end of Section 2. Let us write the above as

$$\begin{aligned} Z_k(t) = \sum_{\ell=1}^{d} \Gamma_{k\ell}(t, P(t)) \Big[\big\{ P(t)\big[A_\ell(t) + B_\ell(t)P(t)\big] + \Lambda_\ell(t) \big\} X(t) \\ + P(t)\big[B_\ell(t)\eta(t) + \sigma_\ell(t)\big] + \zeta_\ell(t) \Big], \qquad 1 \leqslant k \leqslant d, \end{aligned} \tag{5.7}$$

where

$$\big(\Gamma_{k\ell}(t, P)\big)_{d\times d} = \begin{pmatrix} I-PC_{11} & -PC_{12} & \cdots & -PC_{1d} \\ -PC_{21} & I-PC_{22} & \cdots & -PC_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ -PC_{d1} & -PC_{d2} & \cdots & I-PC_{dd} \end{pmatrix}^{-1}.$$

Then, comparing the drift terms, one has

$$\begin{aligned} 0 &= \widehat{A}X + \widehat{B}(PX+\eta) + \Gamma X + PA_0X + PB_0(PX+\eta) + Pb + \alpha + g \\ &\quad + \sum_{k=1}^{d} \widehat{C}_k \sum_{\ell=1}^{d} \Gamma_{k\ell}(t, P) \big\{ [P(A_\ell + B_\ell P) + \Lambda_\ell]X + P(B_\ell\eta + \sigma_\ell) + \zeta_\ell \big\} \\ &\quad + \sum_{k=1}^{d} PC_{0k} \sum_{\ell=1}^{d} \Gamma_{k\ell}(t, P) \big\{ [P(A_\ell + B_\ell P) + \Lambda_\ell]X + P(B_\ell\eta + \sigma_\ell) + \zeta_\ell \big\} \\ &\quad + \sum_{k=1}^{d} \Lambda_k \Big[A_kX + B_k(PX+\eta) \\ &\quad + \sum_{\ell=1}^{d} C_{k\ell} \sum_{j=1}^{d} \Gamma_{\ell j}(t, P) \big\{ [P(A_j + B_jP) + \Lambda_j]X + P(B_j\eta + \sigma_j) + \zeta_j \big\} + \sigma_k \Big] \\ &= \Big[\Gamma + PA_0 + \widehat{B}P + PB_0P + \widehat{A} \\ &\quad + \sum_{k=1}^{d} (\widehat{C}_k + PC_{0k}) \sum_{\ell=1}^{d} \Gamma_{k\ell}(t, P) \big[P(A_\ell + B_\ell P) + \Lambda_\ell \big] \\ &\quad + \sum_{k=1}^{d} \Lambda_k \Big(A_k + B_kP + \sum_{\ell=1}^{d} \sum_{j=1}^{d} C_{j\ell}\Gamma_{j\ell}(t, P) \big[P(A_\ell + B_\ell P) + \Lambda_\ell \big] \Big) \Big] X \end{aligned}$$

$$
\begin{aligned}
&+\Big[\widehat B+PB_0+\sum_{k=1}^d(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)PB_\ell\\
&+\sum_{k=1}^d\Lambda_k\Big(B_k+\sum_{\ell=1}^dC_{k\ell}\sum_{j=1}^d\Gamma_{\ell j}(t,P)PB_j\Big)\Big]\eta\\
&+\sum_{k=1}^d\Big[(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)+\Lambda_k\sum_{\ell=1}^d\sum_{j=1}^dC_{kj}\Gamma_{j\ell}(t,P)\Big]\zeta_\ell\\
&+\sum_{k=1}^d\sum_{\ell=1}^d\Big[(\widehat C_k+PC_{0k})\Gamma_{k\ell}(t,P)+\sum_{j=1}^d\Lambda_kC_{kj}\Gamma_{j\ell}(t,P)+\Lambda_\ell\Big]\sigma_\ell+Pb+\alpha+g.
\end{aligned}
$$

Hence, we seek an adapted solution $(P(\cdot),\Lambda(\cdot))$ to the following *Riccati BSDE*:

$$
\begin{cases}
dP=-\Big[PA_0+\widehat BP+PB_0P\\
\qquad+\displaystyle\sum_{k=1}^d(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)\big[P(A_\ell+B_\ell P)+\Lambda_\ell\big]+\widehat A\\
\qquad+\displaystyle\sum_{k=1}^d\Lambda_k\Big(A_k+B_kP+\sum_{\ell=1}^d\sum_{j=1}^dC_{j\ell}\Gamma_{j\ell}(t,P)\big[P(A_\ell+B_\ell P)+\Lambda_\ell\big]\Big)\Big]\,dt\\
\qquad\qquad+\displaystyle\sum_{k=1}^d\Lambda_k\,dW_k(t),\\
P(T)=H.
\end{cases}
\tag{5.8}
$$

Then we look for an adapted solution $(\eta(\cdot),\zeta(\cdot))$ to the following BSDE:

$$
\begin{cases}
d\eta=-\Big\{\Big[\widehat B+PB_0+\displaystyle\sum_{k=1}^d(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)PB_\ell\\
\qquad+\displaystyle\sum_{k=1}^d\Lambda_k\Big(B_k+\sum_{\ell=1}^dC_{k\ell}\sum_{j=1}^d\Gamma_{\ell j}(t,P)PB_j\Big)\Big]\eta\\
\qquad+\displaystyle\sum_{k=1}^d\Big[(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)+\Lambda_k\sum_{\ell=1}^d\sum_{j=1}^dC_{kj}\Gamma_{j\ell}(t,P)\Big]\zeta_\ell\\
\qquad+\displaystyle\sum_{k=1}^d\sum_{\ell=1}^d\Big[(\widehat C_k+PC_{0k})\Gamma_{k\ell}(t,P)P+\sum_{j=1}^d\Lambda_kC_{kj}\Gamma_{j\ell}(t,P)P+\Lambda_k\Big]\sigma_\ell\\
\qquad+Pb+g\Big\}\,dt+\displaystyle\sum_{k=1}^d\zeta_k\,dW_k,\\
\eta(T)=h.
\end{cases}
\tag{5.9}
$$

If the above Riccati BSDE admits an adapted solution $(P(\cdot),\Lambda(\cdot))$ and the above linear BSDE admits an adapted solution $(\eta(\cdot),\zeta(\cdot))$, then we have the representation

$$\begin{aligned}
Y &= PX+\eta,\\
Z_k &= \Big(\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)\big[P(A_\ell+B_\ell P)+\Lambda_\ell\big]\Big)X+\Big(\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)PB_\ell\Big)\eta\\
&\qquad +\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)P\sigma_\ell+\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)\zeta_\ell\\
&\equiv \Phi_k X+\Psi_k\eta+\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)P\sigma_\ell+\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)\zeta_\ell,\quad 1\leqslant k\leqslant d.
\end{aligned}$$

As a result, the forward equation in FBSDE (1.1) can be written as

$$\begin{aligned}
dX &= \Big[A_0X+B_0(PX+\eta)\\
&\qquad +\sum_{k=1}^d C_{0k}\Big(\Phi_kX+\Psi_k\eta+\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)P\sigma_\ell+\sum_{\ell=1}^d \Gamma_{k\ell}(t,P)\zeta_\ell\Big)+b\Big]\,dt\\
&\qquad +\sum_{k=1}^d\Big[A_kX+B_k(PX+\eta)\\
&\qquad +\sum_{\ell=1}^d C_{k\ell}\Big(\Phi_\ell X+\Psi_\ell\eta+\sum_{j=1}^d \Gamma_{\ell j}(t,P)P\sigma_j+\sum_{j=1}^d \Gamma_{\ell j}(t,P)\zeta_j\Big)+\sigma_k\Big]\,dW_k\\
&= \Big[\Big(A_0+B_0P+\sum_{k=1}^d C_{0k}\Phi_k\Big)X+\Big(B_0+\sum_{k=1}^d C_{0k}\Psi_k\Big)\eta\\
&\qquad +\sum_{k=1}^d\sum_{\ell=1}^d C_{0k}\Gamma_{k\ell}(t,P)(P\sigma_\ell+\zeta_\ell)+b\Big]\,dt\\
&\qquad +\sum_{k=1}^d\Big[\Big(A_k+B_kP+\sum_{\ell=1}^d C_{k\ell}\Phi_\ell\Big)X+\Big(B_k+\sum_{\ell=1}^d C_{k\ell}\Psi_\ell\Big)\eta\\
&\qquad +\sum_{\ell=1}^d\sum_{j=1}^d C_{k\ell}\Gamma_{\ell j}(t,P)(P\sigma_j+\zeta_j)\sigma_k\Big]\,dW_k.
\end{aligned}$$

Hence, if

$$\begin{aligned}
&A_i+B_iP+\sum_{\ell=1}^d C_{ik}\Phi_k\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n}),\qquad 0\leqslant i\leqslant d,\\
&\Big(B_0+\sum_{k=1}^d C_{0k}\Psi_k\Big)\eta+\sum_{k=1}^d\sum_{\ell=1}^d C_{0k}\Gamma_{k\ell}(t,P)(P\sigma_\ell+\zeta_\ell)+b\in L^p_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\quad (5.10)\\
&\Big(B_k+\sum_{\ell=1}^d C_{k\ell}\Psi_\ell\Big)\eta+\sum_{\ell=1}^d\sum_{j=1}^d C_{k\ell}\Gamma_{\ell j}(t,P)(P\sigma_j+\zeta_j)+\sigma_k\in L^p_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n)),
\end{aligned}$$

then $X\in L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$, and thus, (X,Y,Z) is an adapted L^p-solution of (1.1).

Let us look at the case that all the coefficients are deterministic. In this case, $P(\cdot)$ is deterministic and $\Lambda(\cdot)=0$. Then

$$\begin{pmatrix} Z_1\\ Z_2\\ \vdots\\ Z_d\end{pmatrix}=\begin{pmatrix} I-PC_{11} & -PC_{12} & \cdots & -PC_{1d}\\ -PC_{21} & I-PC_{22} & \cdots & -PC_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ -PC_{d1} & -PC_{d2} & \cdots & I-PC_{dd}\end{pmatrix}^{-1}\begin{pmatrix} P(A_1+B_1P)X+P(B_1\eta+\sigma_1)+\zeta_1\\ P(A_2+B_2P)X+P(B_2\eta+\sigma_2)+\zeta_2\\ \vdots\\ P(A_d+B_dP)X+P(B_d\eta+\sigma_d)+\zeta_d\end{pmatrix},$$

Riccati BSDE becomes the following terminal value problem of an ordinary differential equation for a matrix-valued function:

$$\begin{cases} \dot P=-\Big[PA_0+\widehat BP+PB_0P\\ \qquad\qquad+\displaystyle\sum_{k=1}^d(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)\big[P(A_\ell+B_\ell P)\big]+\widehat A\Big],\quad t\in[0,T],\\ P(T)=H.\end{cases}\tag{5.11}$$

If this equation admits a solution $P(\cdot)$, it has to be Lipschitz continuous. Then the following BSDE will admit a unique adapted solution $(\eta(\cdot),\zeta(\cdot))\in\mathcal H^p[0,T]$:

$$\begin{cases} d\eta=-\Big\{\Big[\widehat B+PB_0+\displaystyle\sum_{k=1}^d(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)PB_\ell\Big]\eta\\ \qquad+\displaystyle\sum_{k=1}^d\Big[(\widehat C_k+PC_{0k})\sum_{\ell=1}^d\Gamma_{k\ell}(t,P)\Big]\zeta_\ell\\ \qquad+\displaystyle\sum_{k=1}^d\sum_{\ell=1}^d\big[(\widehat C_k+PC_{0k})\Gamma_{k\ell}(t,P)P\big]\sigma_\ell+Pb+g\Big\}\,dt+\sum_{k=1}^d\zeta_k\,dW_k,\\ \eta(T)=h,\end{cases}\tag{5.12}$$

provided (5.3) holds. Consequently, (5.10) holds and (5.1) admits an adapted L^p-solution $(X,Y,Z)\in\mathcal M^p[0,T]$.

Now, let us look at a situation of linear FBSDE for which adapted L^p-solution actually uniquely exists.

Consider the following controlled linear FSDE:

$$\begin{cases} dX(s)=\big[A(s)X(s)+B(s)u(s)+b(s)\big]\,ds\\ \qquad\qquad+\displaystyle\sum_{k=1}^d\big[C_k(s)X(s)+D_k(s)u(s)+\sigma_k(s)\big]\,dW_k(s),\quad s\in[t,T],\\ X(t)=x,\end{cases}\tag{5.13}$$

with the cost functional

$$\begin{aligned} J(t,x;u(\cdot))=\mathbb E\Big[\int_t^T\Big(&\langle Q(s)X(s),X(s)\rangle+2\langle S(s)X(s),u(s)\rangle+\langle R(s)u(s),u(s)\rangle\\ &+2\langle q(s),X(s)\rangle+2\langle\rho(s),u(s)\rangle\Big)\,ds+\langle HX(T),X(T)\rangle+2\langle h,X(T)\rangle\Big].\end{aligned}\tag{5.14}$$

A standard linear-quadratic (LQ, for short) optimal control problem can be stated as follows.

Problem (LQ). For any given $(t,x) \in [0,T) \times \mathbb{R}^n$, find a control $\bar{u}(\cdot) \in \mathcal{U}[t,T] \equiv L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, called an *optimal open-loop control*, such that

$$J(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) \equiv V(t,x).$$

Suppose the following condition is satisfied:

$$J(t,x;u(\cdot)) \geqslant \delta \mathbb{E}\int_t^T |u(s)|^2\,ds, \qquad \forall\, u(\cdot) \in \mathcal{U}[t,T], \tag{5.15}$$

for some $\delta > 0$, then Problem (LQ) admits a unique optimal (open-loop) control $\bar{u}(\cdot)$. Note that (5.15) is implied by the following standard condition for LQ problems:

$$H \geqslant 0, \qquad Q(\cdot) - S(\cdot)^\top R(\cdot)^{-1} S(\cdot) \geqslant 0, \qquad R(\cdot) \geqslant \delta I, \tag{5.16}$$

for some $\delta > 0$. Now, let (5.15) hold. Then the uniform convexity of the functional $u(\cdot) \mapsto J(t,x;u(\cdot))$ implies that Problem (LQ) admits a unique open-loop optimal control $\bar{u}(\cdot)$. Take any $u(\cdot) \in \mathcal{U}[t,T]$, and let $\bar{X} = X(\cdot\,;t,x,\bar{u}(\cdot))$ and $X = X(\cdot\,;t,x,u(\cdot))$, then

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \frac{J(t,x;\bar{u}(\cdot)+\varepsilon u(\cdot)) - J(t,x;\bar{u}(\cdot))}{\varepsilon} \\
&= 2\mathbb{E}\Big[\int_t^T \big(\langle Q(s)\bar{X}(s), X(s)\rangle + \langle S(s)\bar{X}(s), u(s)\rangle + \langle S(s)X(s), \bar{u}(s)\rangle \\
&\qquad + \langle R(s)\bar{u}(s), u(s)\rangle + \langle q(s), X(s)\rangle + \langle \rho(s), u(s)\rangle\big)\,ds \\
&\qquad + \langle H\bar{X}(T), X(T)\rangle + \langle h, X(T)\rangle\Big] \\
&= 2\mathbb{E}\Big[\int_t^T \big(\langle Q(s)\bar{X}(s) + S(s)^\top\,\bar{u}(s) + q(s), X(s)\rangle \\
&\qquad + \langle R(s)\bar{u}(s) + S(s)\bar{X}(s) + \rho(s), u(s)\rangle\big)\,ds + \langle H\bar{X}(T) + h, X(T)\rangle\Big].
\end{aligned} \tag{5.17}$$

Now, let (Y,Z) be the adapted solution to the following linear BSDE:

$$\begin{cases}
dY(s) = -\Big[A(s)^\top Y(s) + \displaystyle\sum_{k=1}^d C_k(s)^\top Z_k(s) + Q(s)\bar{X}(s) \\
\qquad\qquad + S(s)^\top \bar{u}(s) + q(s)\Big]\,ds + \displaystyle\sum_{k=1}^d Z_k(s)\,dW_k(s), \quad s \in [t,T], \\
Y(T) = H\bar{X}(T) + h.
\end{cases} \tag{5.18}$$

Then by duality, (5.17) becomes

$$\mathbb{E}\int_t^T \langle B(s)^\top Y(s) + \sum_{k=1}^d D_k(s)^\top Z_k(s) + S(s)\bar{X}(s) + R(s)\bar{u}(s) + \rho(s), u(s)\rangle\,ds = 0,$$

for every $u(\cdot) \in \mathcal{U}[t,T]$. Hence,

$$B(s)^\top Y(s) + \sum_{k=1}^d D_k(s)^\top Z_k(s) + S(s)\bar{X}(s) + R(s)\bar{u}(s) + \rho(s) = 0, \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{5.19}$$

This is called a *stationarity condition*, which is also the *Pontryagin type maximum condition.* Consequently, we obtain the following optimality system (bars are dropped):

$$
\begin{cases}
dX(s) = [A(s)X(s) + B(s)u(s) + b(s)]\, ds \\
\qquad\qquad + \displaystyle\sum_{k=1}^{d} [C_k(s)X(s) + D_k(s)u(s) + \sigma_k(s)]\, dW_k(s), \quad s \in [t,T], \\
dY(s) = -\Big[A(s)^\top Y(s) + \displaystyle\sum_{k=1}^{d} C_k(s)^\top Z_k(s) + Q(s)X(s) + S(s)^\top u(s) + q(s)\Big]\, ds \\
\qquad\qquad + \displaystyle\sum_{k=1}^{d} Z_k(s)\, dW_k(s), \quad s \in [t,T], \\
X(t) = x, \qquad Y(T) = HX(T) + h, \\
B(s)^\top Y(s) + \displaystyle\sum_{k=1}^{d} D_k(s)^\top Z_k(s) + S(s)X(s) + R(s)u(s) + \rho(s) = 0, \\
\qquad\qquad \text{a.e. } s \in [t,T], \text{ a.s..}
\end{cases}
\tag{5.20}
$$

This is a coupled FBSDE with a special structure. Let us take a closer look at the above. First, suppose

$$
R(s) \geqslant \delta I, \qquad s \in [0,T], \text{ a.s.,}
$$

for some $\delta > 0$. Then from the stationary condition, one has

$$
u(s) = -R(s)^{-1}\Big[S(s)X(s) + B(s)^\top Y(s) + \sum_{k=1}^{d} D_k(s)^\top Z_k(s) + \rho(s)\Big].
$$

Thus, we end up with the following coupled FBSDE (s is suppressed):

$$
\begin{cases}
dX = \Big[(A - BR^{-1}S)X - BR^{-1}B^\top Y - \displaystyle\sum_{k=1}^{d} BR^{-1}D_k^\top Z_k + b - BR^{-1}\rho\Big]\, ds \\
\qquad + \displaystyle\sum_{k=1}^{d} \Big[(C_k - D_k R^{-1}S)X - D_k R^{-1}B^\top Y \\
\qquad - \displaystyle\sum_{\ell=1}^{d} D_k R^{-1} D_\ell^\top Z_\ell + \sigma_k - D_k R^{-1}\rho\Big]\, dW_k, \\
dY = -\Big[(Q - S^\top R^{-1}S)X + (A^\top - S^\top R^{-1}B^\top)Y \\
\qquad + \displaystyle\sum_{k=1}^{d} (C_k^\top - S^\top R^{-1}D_k^\top)Z_k - S^\top R^{-1}\rho + q\Big]\, ds + \displaystyle\sum_{k=1}^{d} Z_k\, dW_k, \quad s \in [t,T], \\
X(t) = x, \qquad Y(T) = HX(T) + h.
\end{cases}
\tag{5.21}
$$

From what we have so far, the above coupled FBSDE admits a unique adapted L^2-solution $(X,Y,Z) \in \mathcal{M}^2[t,T]$. We would like to claim that it is actually the adapted L^p-solution provided some reasonable conditions are satisfied. To see that, we let

$$
Y = PX + \eta,
$$

where P is a deterministic differentiable symmetric matrix function and η is a stochastic process satisfying (5.6). Then by (5.20),

$$\begin{aligned}
-\Big[A^\top Y+\sum_{k=1}^d C_k^\top Z_k+QX+S^\top u+q\Big]\,ds+\sum_{k=1}^d Z_k\,dW_k=dY=d(PX+\eta)\\
=[\dot P X+P(AX+Bu+b)+\alpha]\,ds+\sum_{k=1}^d[P(C_kX+D_ku+\sigma_k)+\zeta_k]\,dW_k.
\end{aligned}$$

Then it should be true that

$$Z_k=P(C_kX+D_ku+\sigma_k)+\zeta_k,\qquad 1\leqslant k\leqslant d.$$

Now, making use of the stationarity condition, we have

$$\begin{aligned}
0&=B^\top Y+\sum_{k=1}^d D_k^\top Z_k+SX+Ru+\rho\\
&=B^\top(PX+\eta)+\sum_{k=1}^d D_k^\top[P(C_kX+D_ku+\sigma_k)+\zeta_k]+SX+Ru+\rho\\
&=\Big(R+\sum_{k=1}^d D_k^\top PD_k\Big)u+\Big(B^\top P+\sum_{k=1}^d D_k^\top PC_k+S\Big)X+B^\top\eta+\sum_{k=1}^d D_k^\top(P\sigma_k+\zeta_k)+\rho.
\end{aligned}$$

Hence,

$$\begin{aligned}
u&=-\Big(R+\sum_{k=1}^d D_k^\top PD_k\Big)^{-1}\Big(B^\top P+\sum_{k=1}^d D_k^\top PC_k+S\Big)X\\
&\quad-\Big(R+\sum_{k=1}^d D_k^\top PD_k\Big)^{-1}\Big(B^\top\eta+\sum_{k=1}^d D_k^\top(P\sigma_k+\zeta_k)+\rho\Big)\\
&\equiv-\widehat\Theta X-\Big(R+\sum_{k=1}^d D_k^\top PD_k\Big)^{-1}\Big(B^\top\eta+\sum_{k=1}^d D_k^\top\zeta_k+\sum_{k=1}^d D_k^\top P\sigma_k+\rho\Big),
\end{aligned}$$

provided the inverse in the above exists. As in [19], we see that $P(\cdot)$ should satisfy the Riccati equation

$$\begin{cases}
\dot P+PA+A^\top P+\displaystyle\sum_{k=1}^d C_k^\top P(s)C_k+Q\\
-\Big(PB+\displaystyle\sum_{k=1}^d C_k^\top PD_k+S^\top\Big)\Big(R+\displaystyle\sum_{k=1}^d D_k^\top PD_k\Big)^{-1}\Big(B^\top P+\displaystyle\sum_{k=1}^d D_k^\top PC_k+S\Big)=0,\\
\qquad\qquad\text{a.e. } s\in[0,T],\\
P(T)=H,
\end{cases}\tag{5.22}$$

and (η,ζ) should be the adapted solution to the following BSDE:

$$\begin{cases} d\eta=-\Big((A+B\widehat\Theta)^\top\eta+\sum_{k=1}^d(C_k+D_k\widehat\Theta)\zeta_k+\sum_{k=1}^d(C_k+D_k\widehat\Theta)P\sigma_k+\widehat\Theta^\top\rho+Pb+q\Big)\,ds\\ \qquad\qquad+\sum_{k=1}^d\zeta_k\,dW_k(s),\qquad s\in[t,T],\\ \eta(T)=h.\end{cases}\tag{5.23}$$

As a matter of fact, by [20, 19, 21], we know that when (5.16) holds, Problem (LQ) is actually the so-called closed-loop solvable, which implies that the Riccati equation (5.22) admits a unique solution $P(\cdot)$ on $[0,T]$ such that

$$P(\cdot)\geqslant 0,$$

and BSDE (5.23) admits a unique adapted solution $(\eta(\cdot),\zeta(\cdot))$. The above implies

$$R(\cdot)+\sum_{k=1}^d D_k(\cdot)^\top P(\cdot)D_k(\cdot)\geqslant\delta I.$$

Hence, by the continuity of $P(\cdot)$, all the involved coefficients in BSDE (5.23) are bounded. Consequently, as long as

$$b,\sigma_k,q\in L^p_{\mathbb F}(\Omega;L^1(0,T;\mathbb R^n)),\quad \rho\in L^p_{\mathbb F}(\Omega;L^1(0,T;\mathbb R^m)),\quad h\in L^p_{\mathcal F_T}(\Omega;\mathbb R^m),\tag{5.24}$$

one has $(\eta(\cdot),\zeta(\cdot))\in\mathcal H^p[0,T]$. On the other hand, we have the representation

$$\begin{cases} Y=PX+\eta,\\ Z_k=P(C_k-D_k\widehat\Theta)X\\ \qquad+D_k\Big(R+\sum_{\ell=1}^d D_\ell^\top PD_\ell\Big)^{-1}\Big(B^\top\eta+\sum_{\ell=1}^d D_\ell^\top(\zeta_\ell+P\sigma_\ell)+\rho\Big)+P\sigma_k+\zeta_k,\end{cases}$$

with X being the solution to the following closed-loop system:

$$\begin{cases} dX=\Big[(A-B\widehat\Theta)X\\ \qquad-B\Big(R+\sum_{\ell=1}^d D_\ell^\top PD_\ell\Big)^{-1}\Big(B^\top\eta+\sum_{\ell=1}^d D_\ell^\top(\zeta_\ell+P\sigma_\ell)+\rho\Big)+b\Big]\,dt\\ \qquad+\sum_{k=1}^d\Big[(C_k-D_k\Theta)X\\ \qquad-D_k\Big(R+\sum_{\ell=1}^d D_\ell^\top PD_\ell\Big)^{-1}\Big(B^\top\eta+\sum_{\ell=1}^d D_\ell^\top(\zeta_\ell+P\sigma_\ell)+\rho\Big)+\sigma_k\Big]\,dW_k,\\ X(0)=x\end{cases}\tag{5.25}$$

Hence, provided the following, in addition to (5.24),

$$\sigma\in L^p_{\mathbb F}(\Omega;L^2(0,T;\mathbb R^{m\times d})),\qquad \rho\in L^p_{\mathbb F}(\Omega;L^2(0,T;\mathbb R^m)),\tag{5.26}$$

we have $X\in L^p_{\mathbb F}(\Omega;C([0,T];\mathbb R^n)$. Then by looking at the BSDE in (5.21), we see that (X,Y,Z) is an adapted L^p-solution to the FBSDE (5.21).

Now, if the FBSDE is of form (5.1) which is rewritten here (suppressing t):

$$\begin{cases} dX = \Big[A_0X + B_0Y + \sum_{k=1}^d C_{0k}Z_k + b_0\Big]\,dt \\ \qquad\qquad + \sum_{k=1}^d \Big[A_kX + B_kY + \sum_{\ell=1}^d C_{k\ell}Z_\ell + \sigma_{0k}\Big]\,dW_k, \\ dY = -\Big[\widehat{A}X + \widehat{B}Y(t) + \sum_{k=1}^d \widehat{C}_kZ_k + g_0\Big]\,dt + \sum_{k=1}^d Z_k\,dW_k, \\ X(0) = x, \qquad Y(T) = HX(T) + h, \end{cases} \tag{5.27}$$

then in order the above results to work, comparing with FBSDE (5.21), we see that the coefficients should have the following representation:

$$\begin{aligned} &A_0 = A - BR^{-1}S, && B_0 = -BR^{-1}B^\top, && C_{0k} = -BR^{-1}D_k^\top, \\ &A_k = C_k - D_kR^{-1}S, && B_k = -D_kR^{-1}B^\top, && C_{k\ell} = -D_kR^{-1}D_\ell^\top, \\ &\widehat{A} = Q - S^\top R^{-1}S, && \widehat{B} = A^\top - S^\top R^{-1}B^\top, && \widehat{C}_k = C_k^\top - S^\top R^{-1}D_k^\top, \\ &b_0 = b - BR^{-1}\rho, && \sigma_{0k} = \sigma_k - D_kR^{-1}\rho, && g_0 = -S^\top R^{-1}\rho + q, \end{aligned}$$

for some $A, B, b, C_k, D_k, \sigma_k, Q, S, R, q, \rho$. To meet the standard condition (5.16), it suffices to let

$$R = I, \qquad S = 0, \qquad \widehat{A} \geqslant 0, \qquad \rho = 0, \qquad H \geqslant 0.$$

Then the above becomes

$$\begin{aligned} &A_0 = A, && B_0 = -BB^\top, && C_{0k} = -BD_k^\top, && b_0 = b, \\ &A_k = C_k, && B_k = -D_kB^\top, && C_{k\ell} = -D_kD_\ell^\top, && \sigma_{0k} = \sigma_k, \\ &\widehat{A} = Q, && \widehat{B} = A^\top, && \widehat{C}_k = C_k^\top, && g_0 = q. \end{aligned}$$

Hence, the general linear FBSDE (with deterministic coefficients) (5.27) becomes

$$\begin{cases} dX = \Big[AX - BB^\top Y - \sum_{k=1}^d BD_k^\top Z_k + b\Big]\,dt \\ \qquad\qquad + \sum_{k=1}^d \Big[C_kX - D_kB^\top Y - \sum_{\ell=1}^d D_kD_\ell^\top Z_\ell + \sigma_k\Big]\,dW_k, \\ dY = -\Big[QX + A^\top Y + \sum_{k=1}^d C_k^\top Z_k + q\Big]\,dt + \sum_{k=1}^d Z_k\,dW_k, \\ X(0) = x, \qquad Y(T) = HX(T) + h, \end{cases} \tag{5.28}$$

with

$$Q(\cdot) \geqslant 0, \qquad H \geqslant 0. \tag{5.29}$$

According to the above, we see that this FBSDE has a unique adapted L^p-solution (X, Y, Z), as long as

$$\begin{aligned} &b(\cdot), q(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n)), \\ &\sigma(\cdot) \in L^p_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^{n\times d})), \quad h \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \end{aligned} \tag{5.30}$$

Let us state such a result in the following theorem.

THEOREM 5.1. *Let* (5.29)–(5.30) *hold. Then the linear FBSDE* (5.28) *admits a unique adapted* L^p*-solution* (X, Y, Z).

Let us take a closer look at the linear FBSDE (5.28). For convenience, let $d = 1$. Then (5.28) reads

$$\begin{cases} dX = \big[AX - BB^\top Y - BD^\top Z + b\big]\,dt + \big[CX - DB^\top Y - DD^\top Z + \sigma\big]\,dW, \\ dY = -\big[QX + A^\top Y + C^\top Z + q\big]\,dt + Z\,dW, \\ X(0) = x, \qquad Y(T) = HX(T) + h. \end{cases} \tag{5.31}$$

If we let $\theta = (x, y, z)$ and

$$F(\theta) = \begin{pmatrix} -Qx - A^\top y - C^\top z \\ Ax - BB^\top y - BD^\top z \\ Cx - DB^\top y - DD^\top z \end{pmatrix}, \qquad \forall\,\theta \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$$

then

$$\langle F(\theta), \theta\rangle = -\langle Qx, x\rangle - |B^\top y + D^\top z|^2 \leqslant 0, \qquad \forall\,\theta \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$$

giving the dissipation of the map $\theta \mapsto F(\theta)$. This is the exactly the monotonicity condition introduced in [10] (see also [24, 18, 26]), where the existence and uniqueness of adapted L^2-solution of the FBSDE was established. Here, for the deterministic coefficient case, we have shown that the adapted L^p-solution uniquely exists.

Now, we pose the following open problem.

OPEN PROBLEM 2.
(a) Under what suitable conditions, does Riccati BSDE (5.8) admit an adapted solution $(P(\cdot), \Lambda(\cdot))$ so that BSDE (5.9) admits an adapted solution $(\eta(\cdot), \zeta(\cdot)) \in \mathcal{H}^p[0,T]$ and (5.10) holds?
(b) For the deterministic coefficient case, under what conditions, does (5.11) admit a solution $P(\cdot)$ on $[0, T]$?
(c) How can the results of the FBSDE obtained from Problem (LQ) with deterministic coefficients be extended to the random coefficients case?

6. Results via quadratic BSDEs. In this section, we consider a class of FBSDE (1.1) for which any adapted L^2-solutions must be adapted L^p-solutions for any $p > 2$. To begin with, we introduce the following set:

$$\mathcal{Z}[0,T] = \Big\{ Z \in L^p_{\mathbb{F}}(\Omega; L^2(0,T; \mathbb{R}^{m\times d})) \;\Big|\; \operatorname*{ess\,sup}_{\omega\in\Omega}\ \sup_{t\in[0,T]} \Big[\mathbb{E}_t\Big(\int_t^T |Z(s)|^2\,ds\Big)^{p/2}\Big] < \infty \Big\}.$$

Recall that for any $Z \in \mathcal{Z}[0,T]$, $t \mapsto \int_0^t Z(s)\,dW(s)$ is called a *BMO martingale*. According to [28], if

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\ \sup_{t\in[0,T]} \mathbb{E}_\tau\Big[\int_t^T |Z(s)|^2\,ds\Big] \leqslant C_0,$$

then for any $\varepsilon \in (0, \frac{1}{C_0})$ and any $\tau \in \mathcal{T}[0,T]$, the set of all $\mathbb{F}$-stopping times valued in $[0,T]$, one has

$$\mathbb{E}_\tau\Big[\exp\Big(\varepsilon\int_\tau^T |Z(s)|^2\,ds\Big)\Big] \leqslant \frac{1}{\varepsilon C_0},$$

which implies

$$\mathbb{E}\Big(\int_\tau^T |Z(s)|^2\, ds\Big)^{p/2} < \infty, \qquad \forall\, p \geqslant 1.$$

Now, let us introduce the following assumption.

(H4) The map $g = (g^1, \dots, g^m) : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^m$ admits the following representation:

$$g^i(t,x,y,z) = g_0^i(t,x,z^i) + g_1^i(t,x,y,z), \tag{6.1}$$

(with $1 \leqslant i \leqslant m$)

$$\begin{aligned}
&|g_0^i(t,x,z^i)| \leqslant K|z^i|^2, \qquad\qquad \forall\,(t,x,z^i) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^d,\\
&|g_0^i(t,x,z^i) - g_0^i(t,x,\bar z^i)| \leqslant K(1+|z^i|+|\bar z^i|)|z^i-\bar z^i|,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(t,x)\in[0,T]\times\mathbb{R}^n,\ z^i,\bar z^i\in\mathbb{R}^{m\times d},\\
&|g_1^i(t,x,y,z)| \leqslant K(1+|y|), \quad \forall\,(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d},\\
&|g_1^i(t,x,y,z) - g_1^i(t,x,\bar y,\bar z)| \leqslant K\big(|y-\bar y|+|z-\bar z|\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(t,x)\in[0,T]\times\mathbb{R}^n,\ y,\bar y\in\mathbb{R}^m,\ z,\bar z\in\mathbb{R}^{m\times d}.
\end{aligned} \tag{6.2}$$

Note that in the above, $z \mapsto g^i(t,x,y,z)$ is quadratic in z^i, and bounded in z^j, $j \neq i$. Due to this, one refers to such a generator of BSDE as diagonally quadratic. Clearly, if in (H1), the functions $L_{gx}(\cdot), L_{gy}(\cdot), L_{gz}(\cdot)$ are bounded, then (H4) holds with $g_0^i(\cdot) = 0$. Hence, the following result is a substantial extension of Theorem 2.2 in some sense.

THEOREM 6.1. *Let* (H4) *hold, and let* $b : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^n$ *and* $\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\Omega \to \mathbb{R}^{n\times d}$ *satisfy* (H1) *with the first two lines in* (2.13). *Further, suppose*

$$h(\cdot) \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \tag{6.3}$$

Then any adapted L^2-solution (X,Y,Z) of FBSDE (1.1) *must be an adapted L^p-solution for any $p > 2$.*

Proof. Note that the first and the third conditions in (6.2) mean that g_0^i and g_1^i are bounded in x. Thus, if (X,Y,Z) is an adapted L^2-solution to (1.1), then with this fixed X, (Y,Z) is the unique adapted solution to the BSDE in (1.1) and by [11], we see that

$$\|Y\|_\infty + \Big\|\sup_{t\in[0,T]}\Big(\int_\tau^T |Z(s)|^2\, ds\Big)^{p/2}\Big\|_\infty < \infty. \tag{6.4}$$

Consequently, for any $p > 2$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y(t)|^p + \Big(\int_0^T |Z(s)|^2\, ds\Big)^{p/2}\Big] < \infty.$$

Hence, by the FSDE in (1.1), we see that

$$\mathbb{E}\big[\sup_{t\in[0,T]}|X(t)|^p\big] < \infty.$$

This proves our conclusion. ■

For the case $m = 1$, condition (6.3) can be relaxed. To see this, we introduce the following assumption.

(H5) Let (H0) hold, with $m = 1$, such that the first two lines in (2.10) hold with the first two lines in (2.13) being true for some $p > 2$ and $(x, y, z) \mapsto g(t, x, y, z)$ is continuous satisfying

$$|g(t,x,y,z)| \leqslant \alpha + \beta|y| + \gamma/2|z|^2, \qquad \forall\,(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d}, \tag{6.5}$$

for some $\alpha, \beta, \gamma > 0$. Further, $L_{hx} = 0$, and

$$\mathbb{E}\big[e^{\lambda h_0}\big] < \infty, \tag{6.6}$$

for some $\lambda > \gamma e^{\beta T}$.

Note that condition (6.6) allows $h(\cdot)$ to be unbounded. We have the following result.

THEOREM 6.2. *Let* (H1) *and* (H5) *hold. Suppose* (X, Y, Z) *is the unique adapted* L^2*-solution of* (1.1)*. Then it is an adapted* L^p*-solution of* (1.1)*.*

Again note that condition (6.5) implies that $x \mapsto g(t, x, y, z)$ is bounded. Hence, by making use of Theorem 2 in [3], similarly to the proof of Theorem 6.1, we can prove Theorem 6.2.

Note that if (see (2.12))

$$\begin{aligned}|g(t,x,y,z)| \leqslant g_0(t) + L_{gx}(t)|x| + L_{gy}(t)|y| + L_{gz}(t)|z|,&\\ \forall\,(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d},&\end{aligned} \tag{6.7}$$

with

$$|g_0(t)| \leqslant \alpha, \quad L_{gx}(t) = 0, \quad L_{gy}(t) \leqslant \beta, \quad L_{gz}(t) \leqslant \gamma_0, \qquad \forall\, t \in [0,T], \tag{6.8}$$

then, for any $\gamma > 0$ (could be arbitrarily small),

$$|g(t,x,y,z)| \leqslant \alpha + \frac{\gamma_0}{2\gamma} + \beta|y| + \frac{\gamma}{2}|z|^2.$$

Hence, as long as there exists a $\lambda > 0$ such that (6.6) holds, we have Theorem 6.2 with (H5) replaced by (6.7)–(6.8).

Relevant to the above results, we have the following open problem.

OPEN PROBLEM 3.
(a) What if the third condition in (6.2) is weakened to

$$|g_1^i(t,x,y,z)| \leqslant K(1 + |y| + |z|), \quad \forall\,(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega? \tag{6.9}$$

(b) What if the BSDE is not diagonally quadratic?
(c) What if the terminal condition is not bounded for the case $m > 1$? Even for the case $m = 1$, we still do not know if $x \mapsto h(x)$ could be linearly growing. Also, can $L_{gx}(\cdot) \neq 0$ in (6.8)?

7. Conclusion. In this paper, we have investigated the problem of when an adapted L^2-solution of an FBSDE is an adapted L^p-solution (for $p > 2$). We have explored several important cases for which the answer is affirmative. We admit that the problem is far from having a satisfactory answer. Therefore, we pose several open questions, and hope interested audience will be get involved. We will continue the exploration in this direction and will report some further results when they become available. To conclude the paper, we would like to pose the following open problem which is the main motivation of the current paper, and the solution of which will have a significant impact on the theory of FBSDEs and their applications.

Open Problem 4. For linear FBSDE (5.1) with (H3), i.e., all the coefficients are bounded and random, if an adapted L^2-solution (X, Y, Z) uniquely exists, is it an adapted L^p-solution for $p \geqslant 4$?

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