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Linear Diophantine equations in Piatetski-Shapiro sequences

by

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1. Introduction. Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. For a non-integral $\alpha > 0$, the sequence $(\lfloor n^\alpha \rfloor)_{n=1}^\infty$ is called the *Piatetski-Shapiro sequence with exponent* α. Let $\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$. We say that an equation $f(x_1, \dots, x_n) = 0$ is *solvable* in $\mathrm{PS}(\alpha)$ if there are infinitely many pairwise distinct tuples $(x_1, \dots, x_n) \in \mathrm{PS}(\alpha)^n$ satisfying this equation. In this article, we investigate the solvability in $\mathrm{PS}(\alpha)$ of linear Diophantine equations

$$ax + by = cz \tag{1.1}$$

for all fixed $a, b, c \in \mathbb{N}$. For example, the solvability of the equation $y = \theta x + \eta$ for $\theta, \eta \in \mathbb{R}$ with $\theta \notin \{0, 1\}$ has been studied by Glasscock [Gla17, Gla20]. He asserts that if the equation $y = \theta x + \eta$ has infinitely many solutions $(x, y) \in \mathbb{N}^2$, then for Lebesgue-a.e. $\alpha > 1$ it is solvable or not in $\mathrm{PS}(\alpha)$ according as $\alpha < 2$ or $\alpha > 2$. As a direct consequence, for Lebesgue-a.e. $1 < \alpha < 2$, the equation $z = (a/c)x + (b/c)$ is solvable in $\mathrm{PS}(\alpha)$ for all $a, b, c \in \mathbb{N}$ with $\gcd(a, c) \mid b$. In other words, the equation (1.1) with $\gcd(a, c) \mid b$ is solvable in $\mathrm{PS}(\alpha)$. On the other hand, for $\alpha > 2$, we did not know at all whether the equation (1.1) is solvable in $\mathrm{PS}(\alpha)$ or not.

Our main result provides an answer to this question. We consider the set of α in a short interval $[s, t] \subset (2, \infty)$ such that (1.1) is solvable. The following theorem asserts that the Hausdorff dimension of this set is positive. To state the theorem, let $\{x\}$ be the fractional part of $x \in \mathbb{R}$, and $\dim_{\mathrm{H}}(X)$ the Hausdorff dimension of $X \subseteq \mathbb{R}$ (the definition will be recalled in Section 2).

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Theorem 1.1. *Let $a, b, c \in \mathbb{N}$. For all positive real numbers $2 < s < t$,*

$$\dim_{\mathrm{H}}(\{\alpha \in [s,t] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\})$$
$$\geq \begin{cases} \left(s + \dfrac{s^3}{(2+\{s\}-2^{1-\lfloor s \rfloor})(2-\{s\})}\right)^{-1} & \text{if } a = b = c, \\ 2\left(s + \dfrac{s^3}{(2+\{s\}-2^{1-\lfloor s \rfloor})(2-\{s\})}\right)^{-1} & \text{otherwise.} \end{cases}$$

Note that the lower bound in either case is greater than $1/s^3$ for all $2 < s < t$. The positivity of the Hausdorff dimension implies that this set is uncountable for any closed interval $[s,t] \subset (2,\infty)$. Moreover, we can easily prove the following:

Corollary 1.2. *For any closed interval $I \subset (2,\infty)$, the set of $\alpha \in I$ such that $ax + by = cz$ is solvable in $\mathrm{PS}(\alpha)$ is uncountable and dense in I.*

In particular, for $a = b = 1$, $c = 2$, a pairwise distinct tuple (x,z,y) satisfying (1.1) forms an arithmetic progression of length 3. Therefore Corollary 1.2 implies

Corollary 1.3. *For any closed interval $I \subset (2,\infty)$, the set of $\alpha \in I$ such that $\mathrm{PS}(\alpha)$ contains infinitely many arithmetic progressions of length 3 is uncountable and dense in I.*

There are some related works on arithmetic progressions and Piatetski-Shapiro sequences. It is an exercise to show that for all $1 < \alpha < 2$, the set $\mathrm{PS}(\alpha)$ contains arbitrarily long arithmetic progressions (consisting of consecutive elements). Frantzikinakis and Wierdl [FW09] proved that any set of positive integers with positive upper density contains arbitrarily long arithmetic progressions whose common difference belongs to $\mathrm{PS}(\alpha)$ for all non-integral $\alpha > 1$ (here we say that $A \subseteq \mathbb{N}$ has *positive upper density* if $\overline{\lim}_{N\to\infty} |A \cap \{1,\dots,N\}|/N > 0$). This result is an extension of Szemerédi's theorem [Sze75]. Furthermore, the second author and Yoshida [SY19] gave another extension of Szemerédi's theorem to Piatetski-Shapiro sequences by showing that for any $A \subseteq \mathbb{N}$ with positive upper density, the set $\{\lfloor n^\alpha \rfloor : n \in A\}$ with $1 < \alpha < 2$ contains arbitrarily long arithmetic progressions. They also posed a question:

Question 1.4 ([SY19, Question 13]). *Is it true that*

$$\sup\{\alpha \geq 1 \colon \mathrm{PS}(\alpha) \text{ contains arbitrarily long arithmetic progressions}\} = 2?$$

We do not get any answer to this question here, but surprisingly, by Corollary 1.3, the supremum of α such that $\mathrm{PS}(\alpha)$ contains infinitely many arithmetic progressions of length 3 is positive infinity. Glasscock also posed a related question for the equation (1.1) with $a = b = c = 1$.

QUESTION 1.5 ([Gla17, Question 6]). *Does there exist an $\alpha_S > 1$ with the property that for Lebesgue-a.e. or all $\alpha > 1$, the equation $x + y = z$ is solvable or not in* PS(α) *according as $\alpha < \alpha_S$ or $\alpha > \alpha_S$?*

By Corollary 1.2, the case with "all $\alpha > 1$" in Question 1.5 is false since the supremum of $\alpha > 0$ such that (1.1) is solvable in PS(α) is positive infinity. However, the case with "Lebesgue-a.e." in Question 1.5 is still open.

The rest of the article is organized as follows. First in Section 2 we define the discrepancy of the sequences and the Hausdorff dimension, and describe some known useful results. In Sections 3 and 4, we prove a series of lemmas. Finally we provide a proof of Theorem 1.1.

NOTATION. Let $\mathbb{N} = \{1, 2, \dots\}$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the integer part of x, $\{x\}$ denote the fractional part of x, and $\lceil x \rceil$ denote the minimum integer n such that $x \leq n$. A tuple $(x_1, \dots, x_k) \in \mathbb{R}^k$ is called *pairwise distinct* if $\#\{x_1, \dots, x_k\} = k$. Let $\sqrt{-1}$ denote the imaginary unit, and define $e(x)$ by $e^{2\pi\sqrt{-1}x}$ for all $x \in \mathbb{R}$.

2. Preparations.

For all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, define

$$\{\mathbf{x}\} = (\{x_1\}, \dots, \{x_d\}).$$

Let $(\mathbf{x}_n)_{1 \leq n \leq N}$ be a sequence composed of $\mathbf{x}_n \in \mathbb{R}^d$ for all $1 \leq n \leq N$. We define the *discrepancy* $D(\mathbf{x}_1, \dots, \mathbf{x}_N)$ of $(\mathbf{x}_n)_{n=1}^N$ by

$$\sup_{\substack{0 \leq a_i < b_i \leq 1 \\ 1 \leq i \leq d}} \left| \frac{\#\{n \in \mathbb{N} \cap [1, N] \colon \{\mathbf{x}_n\} \in \prod_{i=1}^d [a_i, b_i)\}}{N} - \prod_{i=1}^d (b_i - a_i) \right|.$$

In order to evaluate an upper bound for the discrepancy, we use the following inequality which was shown by Koksma [Kok50] and Szüsz [Szü52] independently: there exists $C_d > 0$ which depends only on d such that for all $K \in \mathbb{N}$, we have

$$D(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq C_d \left(\frac{1}{K} + \sum_{\substack{0 < \|\mathbf{k}\|_\infty \leq K \\ \mathbf{k} \in \mathbb{Z}^d}} \frac{1}{\nu(\mathbf{k})} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi\sqrt{-1}\,\langle \mathbf{k}, \mathbf{x}_n \rangle} \right| \right), \tag{2.1}$$

where we let $\langle \cdot, \cdot \rangle$ denote the standard inner product and define

$$\|\mathbf{k}\|_\infty = \max(|k_1|, \dots, |k_d|), \qquad \nu(\mathbf{k}) = \prod_{i=1}^d \max(1, |k_i|).$$

This inequality is sometimes reffered as the Erdős–Turán–Koksma inequality. We refer the readers to [DT97, Theorem 1.21] for more details on discrepancies and a proof of (2.1). This inequality reduces the estimate of the discrepancy to that of exponential sums. Furthermore, the exponential sum is evaluated by the following lemma.

LEMMA 2.1 (van der Corput's kth derivative test). *Let $f(x)$ be real and have continuous derivatives up to kth order, where $k \geq 4$. Let $\lambda_k \leq f^{(k)}(x) \leq h\lambda_k$ (or the same for $-f^{(k)}(x)$). Let $b-a \geq 1$. Then there exists $C(h,k) > 0$ such that*

$$\Big|\sum_{a<n\leq b} e^{2\pi\sqrt{-1}\,f(n)}\Big| \leq C(h,k)\big((b-a)\lambda_k^{1/(2^k-2)} + (b-a)^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}\big).$$

Proof. See Titchmarsh's book [Tit86, Theorem 5.13]. ∎

We next introduce the Hausdorff dimension. For every $U \subseteq \mathbb{R}$, write the diameter of U as $\operatorname{diam}(U) = \sup_{x,y\in U}|x-y|$. Fix $\delta > 0$. For all $F \subseteq \mathbb{R}$ and $s \in [0,1]$, we define

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum_{j=1}^{\infty}\operatorname{diam}(U_j)^s \colon F \subseteq \bigcup_{j=1}^{\infty} U_j,\ (\forall j\in\mathbb{N})\ \operatorname{diam}(U_j) \leq \delta\Big\},$$

and $\mathcal{H}^s(F) = \lim_{\delta\to+0}\mathcal{H}^s_\delta(F)$ is called the *s-dimensional Hausdorff measure* of F. Further,

$$\dim_{\mathrm{H}}(F) = \inf\{s\in[0,1] \colon \mathcal{H}^s(F) = 0\}$$

is called the *Hausdorff dimension* of F. Note that the Hausdorff dimension can be defined on all metric spaces, but we use only $\mathbb{R}$ in this article. By the definition, the following basic properties hold:

- (Monotonicity) for all $F \subseteq E \subseteq \mathbb{R}$, we have $\dim_{\mathrm{H}}(F) \leq \dim_{\mathrm{H}}(E)$;
- (Countable stability) if $F_1, F_2, \ldots \subseteq \mathbb{R}$ is a countable sequence of sets, then $\dim_{\mathrm{H}}(\bigcup_{n=1}^{\infty} F_n) = \sup_{n\in\mathbb{N}}\dim_{\mathrm{H}}(F_n)$.

We refer the readers to Falconer's book [Fal14] for more details on fractal dimensions. In order to prove Theorem 1.1, we construct a general Cantor set which is a subset of the set of all α such that (1.1) is solvable in $\mathrm{PS}(\alpha)$. In [Fal14, (4.3)], we can see a general construction of Cantor sets and a technique to evaluate their Hausdorff dimension as follows: Let $[0,1] = E_0 \supseteq E_1 \supseteq \cdots$ be a decreasing sequence of sets, with each E_k a union of a finite number of disjoint closed intervals called *kth level basic intervals*, with each interval of E_k containing at least two intervals of E_{k+1}, and with the maximum length of kth level intervals tending to 0 as $k\to\infty$. Then let

$$F = \bigcap_{k=0}^{\infty} E_k. \tag{2.2}$$

LEMMA 2.2 ([Fal14, Example 4.6(a)]). *Suppose in the general construction (2.2) each $(k-1)$st level interval contains at least $m_k \geq 2$ kth level* *intervals* ($k = 1, 2, \ldots$) *which are separated by gaps of at least* δ_k, *where* $0 < \delta_{k+1} < \delta_k$ *for each* k. *Then*

$$\dim_{\mathrm{H}}(F) \geq \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k\delta_k)}.$$

Since the Hausdorff dimension is stable under similarity transformations, the initial interval E_0 may be taken to be an arbitrary closed interval. Moreover, let E_k° be the set of interior points of E_k for all $k \in \mathbb{N}$. Then the Hausdorff dimension of $\bigcap_{k=0}^\infty E_k^\circ$ is equal to that of $\bigcap_{k=0}^\infty E_k$. To see why, let $\mathcal{N}_k$ be the boundary of E_k, that is, the set of all end points of kth level intervals. We easily see that

$$\mathcal{N} := F \setminus \bigcap_{k=0}^\infty E_k^\circ \subset \bigcup_{k=0}^\infty \mathcal{N}_k =: \mathcal{N}_\infty.$$

Since each $\mathcal{N}_k$ is a finite set, $\mathcal{N}_\infty$ is countable. By monotonicity, and the fact that the Hausdorff dimension of a countable set is 0, we get

$$0 \leq \dim_{\mathrm{H}}(\mathcal{N}) \leq \dim_{\mathrm{H}}(\mathcal{N}_\infty) = 0,$$

that is, $\dim_{\mathrm{H}}(\mathcal{N}) = 0$. Therefore by countable stability,

$$\dim_{\mathrm{H}}(F) = \max\Big\{\dim_{\mathrm{H}}\Big(\bigcap_{k=0}^\infty E_k^\circ\Big), \dim_{\mathrm{H}}(\mathcal{N})\Big\} = \dim_{\mathrm{H}}\Big(\bigcap_{k=0}^\infty E_k^\circ\Big).$$

To summarize this discussion, we have the following:

LEMMA 2.3. *Let* E_0 *be any open interval, and let* $E_0 \supseteq E_1 \supseteq \cdots$ *be a decreasing sequence of sets, with each* E_k *a union of a finite number of disjoint open intervals, and with the maximum length of* k*th level intervals tending to* 0 *as* $k \to \infty$. *Suppose each* $(k-1)$*st level interval contains at least* $m_k \geq 2$ k*th level intervals* ($k = 1, 2, \ldots$) *which are separated by gaps of at least* δ_k, *where* $0 < \delta_{k+1} < \delta_k$ *for each* k. *Then*

$$\dim_{\mathrm{H}}\Big(\bigcap_{k=0}^\infty E_k\Big) \geq \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k\delta_k)}.$$

3. Lemmas I. We write $O(1)$ for a bounded quantity. If this bound depends only on some parameters $a_1, \ldots, a_n$, then for instance we write $O_{a_1,\ldots,a_n}(1)$. As is customary, we often abbreviate $O(1)X$ and $O_{a_1,\ldots,a_n}(1)X$ to $O(X)$ and $O_{a_1,\ldots,a_n}(X)$ respectively for a non-negative quantity X. We also write $f(X) \ll g(X)$ and $f(X) \ll_{a_1,\ldots,a_n} g(X)$ if $f(X) = O(g(X))$ and $f(X) = O_{a_1,\ldots,a_n}(g(X))$ respectively, where $g(X)$ is non-negative.

Let us consider the solvability of the equation (1.1). In this and subsequent sections, we fix $a, b, c, d \in \mathbb{N}$ with $d \geq 2$ and $\beta, \gamma \in \mathbb{R}$ with $d < \beta < \gamma < d + 1$. Unless it causes confusion, we do not indicate their dependence hereafter. Take a large parameter $x_0 = x_0(a, b, c, d, \beta, \gamma) > 0$. For all integers $x \geq x_0$, we define

$$J_{a,b,c}(x) = \begin{cases} \left(\left(\dfrac{b}{cx^2 \log x} + \dfrac{a}{c}\right)^{1/\gamma} x,\ \left(\dfrac{a}{c}\right)^{1/\beta} x\right)_{\mathbb{N}} \setminus x\mathbb{N} & \text{if } c < a, \\ \left(\left(\dfrac{a}{c - b(x^2 \log x)^{-1}}\right)^{1/\beta} x,\ \left(\dfrac{a}{c}\right)^{1/\gamma} x\right)_{\mathbb{N}} & \text{if } a < c, \\ \left(2^{1/\gamma}\left(x + \dfrac{1}{x\lceil \log x \rceil}\right),\ 2^{1/\beta} x\right)_{\mathbb{N}} & \text{if } a = b = c, \end{cases}$$

where we let $(s,t)_{\mathbb{N}}$ denote $(s,t) \cap \mathbb{N}$, and set $x\mathbb{N} = \{xn \colon n \in \mathbb{N}\}$. Note that $J_{a,b,c}(x)$ is non-empty if x_0 is sufficiently large. When $a = c$ and $b \neq c$, $J_{a,b,c}(x)$ is not defined above, but this case comes down to the case when $a \neq c$ by switching the roles of (a, x) and (b, y). Thus the three cases in the definition of $J_{a,b,c}(x)$ are sufficient.

LEMMA 3.1. *Assume that $a \neq c$. Then there exists $C > 0$ such that for all integers $x \geq x_0$ and for all $z \in J_{a,b,c}(x)$, we can find $\alpha = \alpha(x, z) \in (\beta, \gamma)$ such that $ax^\alpha + b = cz^\alpha$, and*

$$\left| \alpha - \frac{\log(a/c)}{\log(z/x)} \right| \leq \frac{C}{x^2 \log x}. \tag{3.1}$$

Proof. Fix any $x \geq x_0$ and $z \in J_{a,b,c}(x)$. For all $u \in \mathbb{R}$, consider the continuous function $f(u) = ax^u + b - cz^u$. We consider two cases.

CASE $a > c$. Let

$$\alpha_0 = \frac{\log(a/c)}{\log(z/x)}, \qquad \alpha_1 = \frac{\log(a/c + b/(cx^2 \log x))}{\log(z/x)}.$$

Then $z \in J_{a,b,c}(x)$ implies $\beta < \alpha_0 < \alpha_1 < \gamma$. It follows that $f(\alpha_0) = b > 0$. By taking a larger x_0 if necessary, we have

$$f(\alpha_1) = x^{\alpha_1}(a + bx^{-\alpha_1} - c(z/x)^{\alpha_1}) \leq x^{\alpha_1}(a + b/(x^2 \log x) - c(z/x)^{\alpha_1}) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of f such that $\beta < \alpha_0 \leq \alpha \leq \alpha_1 < \gamma$. Since $\log(1+u) \leq u$ for all $u \in (-1, \infty)$, we have

$$|\alpha_1 - \alpha_0| = \frac{\log(1 + b/(ax^2 \log x))}{\log(z/x)} \leq \frac{b}{ax^2 \log x} \cdot \frac{1}{\log(z/x)}.$$

From this inequality and $1/\log(z/x) \ll_{a,c,\gamma} 1$, we obtain (3.1).

CASE $c > a$. Let

$$\alpha_0 = \frac{\log(c/a)}{\log(x/z)}, \qquad \alpha_1' = \frac{\log(c/a - b/(ax^2 \log x))}{\log(x/z)}.$$

Since $z \in J_{a,b,c}(x)$, we have $\beta < \alpha_1' < \alpha_0 < \gamma$ and $x \ll_{a,b,c,\beta,\gamma} z$. Then by the calculation in Case $a > c$, $f(\alpha_0) = b > 0$. Further, $x \ll z$ implies $z^{-\alpha_1'} \le z^{-\beta} \ll x^{-\beta}$. Thus if x_0 is sufficiently large, we have $z^{-\alpha_1'} \le 1/(x^2 \log x)$, which yields

$$f(\alpha_1') = z^{\alpha_1'}(a(x/z)^{\alpha_1'} + bz^{-\alpha_1'} - c) \le z^{\alpha_1'}(a(x/z)^{\alpha_1'} + b/(x^2 \log x) - c) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of f such that $\beta < \alpha_1' \le \alpha \le \alpha_0 < \gamma$. Since $|\log(1 - u)| \le 2u$ for all $u \in (0, 1/2)$, we have

$$|\alpha_0 - \alpha_1'| = \frac{|\log(1 - b/(cx^2 \log x))|}{\log(x/z)} \le \frac{2b}{cx^2 \log x} \cdot \frac{1}{\log(x/z)}$$

provided x_0 is sufficiently large. From this inequality and $1/\log(x/z) \ll_{a,c,\gamma} 1$, we obtain (3.1). ∎

Lemma 3.2. *There exists $C > 0$ such that for all integers $x \ge x_0$ and $z \in J_{1,1,1}(x)$, we can find $\alpha = \alpha(x, z) \in (\beta, \gamma)$ such that $x^\alpha + (x + (x\lceil \log x\rceil)^{-1})^\alpha = z^\alpha$, and*

$$\left|\alpha - \frac{\log 2}{\log(z/x)}\right| \le \frac{C}{x^2 \log x}. \tag{3.2}$$

Proof. Take any $x \ge x_0$ and $z \in J_{1,1,1}(x)$. For all $u \in \mathbb{R}$, consider the continuous function $f(u) = x^u + (x + (x\lceil \log x\rceil)^{-1})^u - z^u$, and set

$$\alpha_0 = \frac{\log 2}{\log(z/x)}, \quad \alpha_1 = \frac{\log 2}{\log\left(\dfrac{z}{x + (x\lceil \log x\rceil)^{-1}}\right)}.$$

By $z \in J_{1,1,1}(x)$, we get $\beta < \alpha_0 < \alpha_1 < \gamma$. By the definitions of α_0 and α_1, we have

$$f(\alpha_0) > z^{\alpha_0}\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0, \quad f(\alpha_1) < z^{\alpha_1}\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of f such that $\alpha_0 \le \alpha \le \alpha_1$. Further, we deduce (3.2) since

$$|\alpha_1 - \alpha_0| \le \frac{\gamma^2}{\log 2} \log\left(1 + \frac{1}{x^2 \log x}\right) \le \frac{\gamma^2}{\log 2} \cdot \frac{1}{x^2 \log x}. \ ∎$$

Lemma 3.3. *Let $\varepsilon > 0$ be an arbitrarily small real number. For all $X, Y, Z \in \mathbb{N}$, and $\alpha \in \mathbb{R}$ with $\beta < \alpha < \gamma$, if*

$$aX^\alpha + bY^\alpha = cZ^\alpha, \tag{3.3}$$

then there exists $n_0 \in \mathbb{N}$ such that

$$a\lfloor (n_0X)^\alpha\rfloor + b\lfloor (n_0Y)^\alpha\rfloor = c\lfloor (n_0Z)^\alpha\rfloor, \tag{3.4}$$

$$\max(\{(n_0X)^\alpha\}, \{(n_0Y)^\alpha\}, \{(n_0Z)^\alpha\}) < 1/2, \tag{3.5}$$

$$n_0 \ll_\varepsilon (X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon}. \tag{3.6}$$

Proof. Choose $X,Y,Z \in \mathbb{N}$ and α with $\beta<\alpha<\gamma$ satisfying (3.3). For all $n\in\mathbb{N}$,

$$c\lfloor (nZ)^\alpha\rfloor = c(nZ)^\alpha - c\{(nZ)^\alpha\} = a\lfloor (nX)^\alpha\rfloor + b\lfloor (nY)^\alpha\rfloor + \delta(n),$$

where we define $\delta(n) = a\{(nX)^\alpha\} + b\{(nY)^\alpha\} - c\{(nZ)^\alpha\}$. Let

$$A = \{n\in\mathbb{N}\colon |\delta(n)|<1,\ \max(\{(nX)^\alpha\},\{(nY)^\alpha\},\{(nZ)^\alpha\})<1/2\},$$

and note that any $n\in A$ satisfies (3.4) and (3.5). Let us show the existence of $n\in A$ satisfying (3.6). Take a small $\xi=\xi(d,\beta,\gamma,\varepsilon)>0$ and take a sufficiently large parameter $R=R(a,b,c,d,\beta,\gamma,\varepsilon)$. Set

$$N = \lceil R(X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon}\rceil, \tag{3.7}$$

and set $\psi = \{\beta\}-2+(2^{d+2}-2)(1/2^d-2\xi)$. Since

$$\psi = 2+\{\beta\}-2^{1-\lfloor\beta\rfloor}+O(\xi), \tag{3.8}$$

we have $0<\psi<\beta<\alpha$ for ξ small enough. Moreover, we let $L(h_1,h_2) = (h_1X^\alpha + h_2Y^\alpha)/c$.

Case 1. We first discuss the case when

$$|L(h_1,h_2)| \ge N^{-\psi} \tag{3.9}$$

for all $h_1,h_2\in\mathbb{Z}$ with $0<\max(|h_1|,|h_2|)\le N^\xi$. In this case, define

$$A_1 = \left\{n\in\mathbb{N}\colon 0\le \{(nX)^\alpha/c\} < \frac{1}{4ac},\ 0\le\{(nY)^\alpha/c\}<\frac{1}{4bc}\right\}. \tag{3.10}$$

Then we have $A_1\subseteq A$. Indeed, take any $n\in A_1$. We see that

$$(nX)^\alpha = c\lfloor (nX)^\alpha/c\rfloor + c\{(nX)^\alpha/c\}. \tag{3.11}$$

Since the first term on the right-hand side of (3.11) is an integer and the second term belongs to $[0,1)$ by $n\in A_1$, we get $\{(nX)^\alpha\} = c\{(nX)^\alpha/c\}$. This yields $\{(nX)^\alpha\}<1/(4a)$. Similarly, $\{(nY)^\alpha\}<1/(4b)$. Further,

$$\{(nZ)^\alpha\} = \{a(nX)^\alpha/c + b(nY)^\alpha/c\} \le a\{(nX)^\alpha/c\} + b\{(nY)^\alpha/c\} < \frac{1}{2c}.$$

Hence

$$|\delta(n)| \le a\{(nX)^\alpha\} + b\{(nY)^\alpha\} + c\{(nZ)^\alpha\} < \frac14+\frac14+\frac12 = 1.$$

Therefore $A_1\subseteq A$.

We now evaluate the distribution of A_1. Let $D_1(N)$ be the discrepancy of the sequence $((nX)^\alpha/c,\ (nY)^\alpha/c)_{N<n\le 2N}$. Then (2.1) with $K=\lfloor N^\xi\rfloor$ implies that

$$D_1(N) \ll N^{-\xi} + \sum_{0<\|(h_1,h_2)\|_\infty\le N^\xi} \frac{1}{\nu(h_1,h_2)}\left|\frac1N\sum_{N<n\le 2N} e(L(h_1,h_2)n^\alpha)\right|.$$

For all $u \in \mathbb{R}$, define $f(u) = L(h_1,h_2)u^{\alpha}$. For each $N < u \le 2N$,

$$|L(h_1,h_2)|N^{\alpha-(d+2)} \ll |f^{(d+2)}(u)| \ll |L(h_1,h_2)|N^{\alpha-(d+2)}.$$

Therefore Lemma 2.1 with $k = d+2$ yields

$$\begin{aligned}\frac{1}{N}&\sum_{N<n\le 2N} e(L(h_1,h_2)n^{\alpha})\\ &\ll (|L(h_1,h_2)|N^{\alpha-(d+2)})^{1/(2^{d+2}-2)} + \frac{(|L(h_1,h_2)|N^{\alpha-(d+2)})^{-1/(2^{d+2}-2)}}{N^{1/2^d}}\\ &\ll (L(N^{\xi},N^{\xi})N^{\{\gamma\}-2})^{1/(2^{d+2}-2)} + \frac{N^{(2-\{\beta\}+\psi)/(2^{d+2}-2)}}{N^{1/2^d}},\end{aligned}$$

where in the last inequality we used $\alpha - d < \{\gamma\}$ and $d+2-\alpha < 2-\{\beta\}$. By the definition of ψ, it follows that $(2-\{\beta\}+\psi)/(2^{d+2}-2) - 1/2^d = -2\xi$. Then

$$\frac{1}{N}\sum_{N<n\le 2N} e(L(h_1,h_2)n^{\alpha}) \ll ((X+Y)^{\gamma}N^{\{\gamma\}-2+\xi})^{1/(2^{d+2}-2)} + N^{-2\xi}.$$

Therefore, since

$$\sum_{0<\|(h_1,h_2)\|_\infty \le N^{\xi}} \frac{1}{\nu(h_1,h_2)} \ll (\log N^{\xi})^2 \ll_{\xi} N^{\xi/(2^{d+2}-2)},$$

we have

$$D_1(N) \ll_{\xi} N^{-\xi} + ((X+Y)^{\gamma}N^{\{\gamma\}-2+2\xi})^{1/(2^{d+2}-2)}. \tag{3.12}$$

Let $E_1(N)$ be the right-hand side of (3.12). By the definition of discrepancy,

$$\frac{\#(A_1 \cap (N,2N])}{N} = \frac{1}{16abc^2} + O_{\xi}(E_1(N)).$$

By (3.7), we have

$$(X+Y)^{\gamma}N^{\{\gamma\}-2+2\xi} \ll R^{\{\gamma\}-2+2\xi}(X+Y)^{e}. \tag{3.13}$$

Here

$$\begin{aligned} e &= \gamma + (\{\gamma\}-2+2\xi)\biggl(\frac{\gamma^2}{(2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\})} + \varepsilon\biggr)\\ &= \gamma\biggl(1 - \frac{\gamma}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}}\biggr) - \varepsilon(2-\{\gamma\}) + O(\xi)\\ &\le \gamma\cdot\frac{2+\{\beta\}-\gamma}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}} - \varepsilon(2-\{\gamma\}) + O(\xi) < 0\end{aligned}$$

for ξ small enough. This yields

$$E_1(N) \ll_{\xi} R^{-\xi} + R^{(\{\gamma\}-2+2\xi)/(2^{d+2}-2)}.$$

Therefore if ξ is sufficiently small and R is sufficiently large, then

$$\frac{1}{16abc^2} + O_\xi(E_1(N)) \geq \frac{1}{32abc^2}.$$

Hence, in this case, $\#(A \cap (N, 2N]) \geq \#(A_1 \cap (N, 2N]) \geq N/(32abc^2) > 0$, which implies that there exists $n_0 \in A$ satisfying (3.6).

Case 2. We next discuss the case when (3.9) is false, that is, there exist $h_1, h_2 \in \mathbb{Z}$ with $0 < \max(|h_1|, |h_2|) \leq N^\xi$ such that

$$|L(h_1, h_2)| < N^{-\psi}. \tag{3.14}$$

We observe that h_1 and h_2 are non-zero and have opposite signs, since if not, then $1/c \leq |L(h_1, h_2)| < N^{-\psi}$, which causes a contradiction when R is sufficiently large. Thus we may assume that $h_1 < 0 < h_2$ by multiplying both sides of (3.14) by $|(-1)|$ if necessary. Let $h_1' = -h_1$ and $\theta = L(h_1, h_2)/h_2$.

In the case $\theta \geq 0$, let

$$A_2 = \left\{ n \in [1, N^{\psi/\alpha}/(8bc)^{1/\alpha}] \cap \mathbb{N} \colon 0 \leq \{(nX)^\alpha/(ch_2)\} < \frac{1}{8abcN^\xi} \right\}; \tag{3.15}$$

then $A_2 \subseteq A$. To see why, suppose $n \in A_2$. Then

$$(nX)^\alpha/c = h_2 \lfloor (nX)^\alpha/(ch_2) \rfloor + h_2 \{(nX)^\alpha/(ch_2)\},$$

where the first term is an integer and the second term belongs to $[0, 1)$. This yields $\{(nX)^\alpha/c\} = h_2\{(nX)^\alpha/(ch_2)\}$. Thus we obtain $0 \leq \{(nX)^\alpha/c\} < 1/(4ac)$. Further, since

$$(nY)^\alpha/c = \frac{h_1'}{ch_2}(nX)^\alpha + n^\alpha\theta = h_1' \lfloor (nX)^\alpha/(ch_2) \rfloor + h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta,$$

$$h_1' \lfloor (nX)^\alpha/(ch_2) \rfloor \in \mathbb{Z}, \qquad 0 \leq h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta < \frac{1}{8bc} + \frac{1}{8bc} = \frac{1}{4bc},$$

we have $\{(nY)^\alpha/c\} = h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta$ and $0 \leq \{(nY)^\alpha/c\} < 1/(4bc)$. Hence, $A_2 \subseteq A_1 \subseteq A$.

We next evaluate the distribution of A_2. Let $V = N^{\psi/\alpha}/(2(8bc)^{1/\alpha})$, and $D_2(N)$ be the discrepancy of the sequence $((nX)^\alpha/(ch_2))_{V<n\leq 2V}$. Then by (2.1) with $K = \lfloor N^{2\xi} \rfloor$,

$$D_2(N) \ll \frac{1}{N^{2\xi}} + \sum_{0<|h|\leq N^{2\xi}} \frac{1}{|h|} \left| \frac{1}{V} \sum_{V<n\leq 2V} e\big((h/(ch_2))X^\alpha n^\alpha\big) \right|.$$

From Lemma 2.1 with $k = d + 2$ we deduce

$$D_2(N) \ll \frac{1}{N^{2\xi}} + \sum_{0<|h|\leq N^{2\xi}} \frac{1}{|h|} \left(\left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{1/(2^{d+2}-2)} + \frac{\left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2}\right)^{-1/(2^{d+2}-2)}}{V^{1/2^d}} \right).$$

We see that

$$\sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|}\left(\frac{|h|X^\alpha}{ch_2}V^{\alpha-d-2}\right)^{1/(2^{d+2}-2)}$$
$$\le (X^\gamma V^{\{\gamma\}-2})^{1/(2^{d+2}-2)}\cdot 2\sum_{1\le h\le N^{2\xi}} h^{-1+1/(2^{d+2}-2)}$$
$$\ll (X^\gamma V^{\{\gamma\}-2})^{1/(2^{d+2}-2)}\cdot N^{2\xi/(2^{d+2}-2)}.$$

In addition, since $d-\alpha<0$ and $h_2\le N^\xi$, we see that

$$\sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|}\cdot\frac{\left(\frac{|h|X^\alpha}{ch_2}V^{\alpha-d-2}\right)^{-1/(2^{d+2}-2)}}{V^{1/2^d}}$$
$$\le\left(\frac{ch_2}{X^\alpha}\right)^{1/(2^{d+2}-2)} V^{(2+d-\alpha)/(2^{d+2}-2)-1/2^d}\cdot 2\sum_{h=1}^\infty h^{-1-1/(2^{d+2}-2)}$$
$$\ll N^\xi\cdot V^{1/(2^{d+1}-1)-1/2^d} = N^\xi V^{(-1+2^{-d})/(2^{d+1}-1)}.$$

Hence

$$D_2(N)\ll \frac{1}{N^{2\xi}}+(X^\gamma N^{2\xi}V^{\{\gamma\}-2})^{1/(2^{d+2}-2)}+N^\xi V^{(-1+2^{-d})/(2^{d+1}-1)}$$
$$\ll \frac{1}{N^{2\xi}}+(X^\gamma N^{2\xi+\psi(\{\gamma\}-2)/\gamma})^{1/(2^{d+2}-2)}+N^{\xi+\psi(-1+2^{-d})/(\gamma(2^{d+1}-1))}.$$

Let $E_2(N)$ be the right-hand side. Now by (3.7), we have

$$X^\gamma N^{2\xi+\psi(\{\gamma\}-2)/\gamma}\ll R^{2\xi+\psi(\{\gamma\}-2)/\gamma}(X+Y)^{e'}. \tag{3.16}$$

Here

$$e'=\gamma+\left(2\xi+\frac{\psi}{\gamma}(\{\gamma\}-2)\right)\left(\frac{\gamma^2}{(2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\})}+\varepsilon\right)$$
$$=\gamma-\gamma\cdot\frac{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}+O(\xi)}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}}-\varepsilon\cdot\frac{\psi}{\gamma}(2-\{\gamma\})+O(\xi)$$
$$=-\varepsilon\cdot\frac{\psi}{\gamma}(2-\{\gamma\})+O(\xi),$$

where we have used (3.8). This implies that for ξ small enough,

$$E_2(N)\ll N^{-2\xi}+(R^{2\xi+\psi(\{\gamma\}-2)/\gamma}(X+Y)^{e'})^{1/(2^{d+2}-2)}$$
$$+N^{\xi+\psi(-1+2^{-d})/(\gamma(2^{d+1}-1))}$$
$$\ll N^{-2\xi}.$$

Therefore, by making ξ smaller and R larger if necessary, we get

$$\frac{\#(A_2\cap(V,2V])}{V}=\frac{1}{8abcN^\xi}+O(E_2(N))\ge\frac{1}{16abcN^\xi}>0.$$

Hence, there exists $n_0 \in A$ such that

$$n_0 \ll_\varepsilon ((X+Y)^{\psi/\alpha})^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon},$$

which implies the inequality (3.6) since $\psi < \alpha$. In the case $\theta < 0$, let $\theta' = L(h_1,h_2)/h_1 > 0$. By switching the roles of (θ, X^α) and (θ', Y^α), and by a similar argument to the case $\theta \geq 0$, we also find $n_0 \in A$ satisfying (3.6). ∎

LEMMA 3.4. *For all $\alpha > 0$ and $X, Y, Z \in \mathbb{N}$, define*

$$\eta(\alpha, X, Y, Z) = \min\left\{\frac{\log((\lfloor W^\alpha\rfloor + 1)W^{-\alpha})}{\log W} : W = X, Y, Z\right\}.$$

For all $\alpha > 0$ and $X, Y, Z \in \mathbb{N}$, if $a\lfloor X^\alpha\rfloor + b\lfloor Y^\alpha\rfloor = c\lfloor Z^\alpha\rfloor$, then for all $\tau \in (\alpha, \alpha + \eta(\alpha, X, Y, Z))$, we have

$$a\lfloor X^\tau\rfloor + b\lfloor Y^\tau\rfloor = c\lfloor Z^\tau\rfloor.$$

Proof. The claim is clear since we observe that

$$\lfloor X^\alpha\rfloor = \lfloor X^\tau\rfloor, \quad \lfloor Y^\alpha\rfloor = \lfloor Y^\tau\rfloor, \quad \lfloor Z^\alpha\rfloor = \lfloor Z^\tau\rfloor$$

for all $\tau \in (\alpha, \alpha + \eta(\alpha, X, Y, Z))$. ∎

4. Lemmas II. Let $2 \leq \beta < \gamma$, and let $a, b, c \in \mathbb{N}$ as in the previous section. Let $x_0 > 0$ be a large parameter. For each $x \geq x_0$, let $K(x) \subseteq \mathbb{N}$ be a non-empty finite set. For each $x \geq x_0$ and $z \in K(x)$, let $\theta(x,z)$ and $\ell(x,z)$ be positive real numbers, and define an interval $I(x,z) = (\theta(x,z), \theta(x,z) + \ell(x,z))$. For each $x \geq x_0$, define

$$G_x = \bigcup_{z \in K(x)} I(x,z).$$

Let us consider the following conditions:

(C1) for all integers $x \geq x_0$, $K(x)$ does not contain any multiples of x;
(C2) for all integers $x \geq x_0$ and $z \in K(x)$, if $z \neq \max K(x)$, then $z+1 \in K(x)$ or $z+2 \in K(x)$;
(C3) there exists $Q_1 > 0$ such that for all $x \geq x_0$,

$$\max(\inf\{|\beta - \alpha| : \alpha \in G_x\}, \inf\{|\gamma + x^{-2} - \alpha| : \alpha \in G_x\}) \leq Q_1 x^{-1};$$

(C4) there exists a real number $\kappa \in (0,\infty) \setminus \{1\}$ such that for all $x \geq x_0$ and $z \in K(x)$,

$$\theta(x,z) = \frac{\log \kappa}{\log(z/x)} + O\left(\frac{1}{x^2 \log x}\right);$$

(C5) there exist $Q_2, Q_3 > 0$ and $q > 2$ such that for all $x \geq x_0$ and $z \in K(x)$,

$$Q_2 x^{-q} \leq \ell(x,z) \leq Q_3 x^{-\beta};$$

(C6) for all integers $x \geq x_0$, $G_x \subseteq (\beta, \gamma + x^{-2})$;

(C7) for all integers $x \geq x_0$ and $z \in K(x)$, there exists a pairwise distinct tuple $(X(x,z), Y(x,z), Z(x,z)) \in \mathbb{N}^3$ such that for all $\tau \in I(x,z)$,

$$a\lfloor X(x,z)^\tau \rfloor + b\lfloor Y(x,z)^\tau \rfloor = c\lfloor Z(x,z)^\tau \rfloor, \qquad X(x,z) \geq x.$$

PROPOSITION 4.1. *Suppose that there exist x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$ satisfying* (C1) *to* (C7)*. Let q be as in* (C5)*. Then*

$$\dim_{\mathrm{H}}(\{\alpha \in [\beta,\gamma] : ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \geq 2/q.$$

REMARK 4.2. The idea of the proof of Proposition 4.1 comes from the proof of Jarník's theorem in Falconer's book [Fal14, Theorem 10.3]. Jarník's theorem states that for every $q > 2$ the set of $\alpha \in [0,1]$ such that there exist infinitely many $x, z \in \mathbb{N}$ with $|\alpha - z/x| \leq x^{-q}$ has Hausdorff dimension $2/q$.

The goal of this section is to prove Proposition 4.1. Suppose that there exist x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$ satisfying (C1) to (C7), and choose such x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$. Let Q_1, Q_2, Q_3, κ, q be as in (C3) to (C5). Let $x_1 > 0$ and $U_1 > 0$ be large parameters depending on a, b, c, d, β, γ, Q_1, Q_2, Q_3, κ, x_0, q. Below we do not indicate the dependence of those parameters. Let p denote a variable running over prime numbers.

LEMMA 4.3. *There exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, the intervals $I(p,z)$ and $I(p,z')$ are separated by a gap of at least $B_1 p^{-1}$ if x_1 is sufficiently large.*

Proof. By (C4) and (C6), for all $p \geq x_1$ and $z \in K(p)$, we have

$$\frac{\beta}{2} \leq \frac{\log \kappa}{\log(z/p)} \leq 2\gamma \tag{4.1}$$

if x_1 is sufficiently large. This implies that

$$p \ll z \ll p. \tag{4.2}$$

By (C4) and the inequalities (4.1) and (4.2), there exists $B_0 > 0$ such that

$$\begin{aligned}
|\theta(p,z) - \theta(p,z')| &= \left| \frac{\log \kappa}{\log \frac{z}{p}} - \frac{\log \kappa}{\log \frac{z'}{p}} + O\left(\frac{1}{p^2 \log p}\right) \right| \\
&\geq \frac{|\log \kappa|\, \left|\log \frac{z'}{z}\right|}{\left|\log \frac{z}{p}\right|\, \left|\log \frac{z'}{p}\right|} + O\left(\frac{1}{p^2 \log p}\right) \\
&\geq \frac{\beta^2}{4|\log \kappa|} \log\left(\frac{z+1}{z}\right) + O\left(\frac{1}{p^2 \log p}\right) \geq B_0 p^{-1}
\end{aligned}$$

for all $p \geq x_1$ and all $z, z' \in K(p)$ with $z < z'$. Further, since $\ell(p,z) \leq Q_3 p^{-2}$ by (C5), there exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, the intervals $I(p,z)$ and $I(p,z')$ are separated by a gap of at least

$$B_0 p^{-1} - Q_3 p^{-2} \geq B_1 p^{-1} \tag{4.3}$$

if x_1 is sufficiently large. ∎

Now we call the open interval $I(p,z)$ $(z \in K(p))$ a *basic interval* of G_p for all $p \geq x_1$. For each $U \geq U_1$, define

$$H_U = \bigcup_{\substack{U<p\leq 2U \\ p\,\mathrm{prime}}} G_p.$$

For all $U < p \leq 2U$, we also call a basic interval of G_p a basic interval of H_U.

LEMMA 4.4. *There exist $B_2, B_3 > 0$ such that for any $U \geq U_1$, all distinct basic intervals of H_U are separated by gaps of at least B_2U^{-2}, and the length of each basic interval of H_U is at least B_3U^{-q} if U_1 is sufficiently large.*

Proof. We take distinct prime numbers p and p' with $U < p, p' \leq 2U$. Then, for all $z \in K(p)$ and $z' \in K(p')$, the condition (C4), the inequality (4.1), and the mean value theorem imply that

$$\begin{aligned} |\theta(p,z) - \theta(p',z')| &\geq \left|\frac{\log\kappa}{\log(z/p)} - \frac{\log\kappa}{\log(z'/p')}\right| + O\left(\frac{1}{U^2\log U}\right) \\ &\geq \frac{\beta^2}{4|\log\kappa|}\left|\frac{z}{p} - \frac{z'}{p'}\right| \min\left(\frac{p}{z}, \frac{p'}{z'}\right) + O\left(\frac{1}{U^2\log U}\right). \end{aligned}$$

We may assume that $p'/z' > p/z$. By (C1), z and p are coprime, which yields $|zp' - z'p| \geq 1$. Therefore

$$\left|\frac{z}{p} - \frac{z'}{p'}\right| \min\left(\frac{p}{z}, \frac{p'}{z'}\right) = \left|\frac{z}{p} - \frac{z'}{p'}\right|\frac{p}{z} \geq \frac{1}{p'z} \gg U^{-2}$$

by the inequalities (4.2) and $U < p, p' \leq 2U$. Therefore for all $U \geq U_1$,

$$|\theta(p,z) - \theta(p',z')| \gg U^{-2} \tag{4.4}$$

if U_1 is sufficiently large. Further, for all $U < p \leq 2U$ and $z \in K(p)$, we deduce by (C5) that $\ell(p,z) \ll U^{-\beta}$, where $\beta \geq 2$. Hence there exists $D_1 > 0$ such that for all distinct prime numbers $U < p, p' \leq 2U$, and all $z \in K(p)$ and $z' \in K(p')$, the intervals $I(p,z)$ and $I(p',z')$ are separated by gaps of at least D_1U^{-2}. By combining this with Lemma 4.3, there exists $D_2 > 0$ such that distinct basic intervals of H_U are separated by gaps of at least D_2U^{-2}. Furthermore by (C5), for all $U < p \leq 2U$ and $z \in K(p)$, we have $Q_2 \cdot 2^{-q}U^{-q} \leq \ell(p,z)$. In conclusion, we find that all distinct basic intervals of H_U are separated by gaps of at least B_2U^{-2}, and have length of at least B_3U^{-q}, where we let $B_2 = D_2$ and $B_3 = Q_2 \cdot 2^{-q}$. ∎

LEMMA 4.5. *There exists $B_4 > 0$ such that the following statement holds: for every $U \geq U_1$, if an open interval $I \subset (\beta, \gamma + p^{-2})$ satisfies*

$$3B_4/\mathrm{diam}(I) < U < p \leq 2U, \tag{4.5}$$

then the open interval I completely includes at least

$$\frac{U^2}{6B_4\log U} \cdot \mathrm{diam}(I) \quad \textit{basic intervals of } H_U. \tag{4.6}$$

Proof. By (C4), (4.1), and (4.2), there exists $D_3 > 0$ such that for every $z \in K(p)$ and the minimum $z' \in K(p)$ with $z' > z$,

$$
\begin{aligned}
(4.7) \qquad |\theta(p,z) - \theta(p,z')| &= \left| \frac{\log \kappa}{\log(z/p)} - \frac{\log \kappa}{\log(z'/p)} + O\left(\frac{1}{p^2 \log p}\right) \right| \\
&\le \frac{4\gamma^2}{|\log \kappa|} \cdot \frac{1}{z} \cdot |z - z'| + O\left(\frac{1}{p^2 \log p}\right) \le D_3 p^{-1}.
\end{aligned}
$$

Here we apply (C2) when we deduce the last inequality. From (C3), (C6) and (4.7), there exists $B_4 > 0$ such that

$$
\begin{aligned}
(\beta, \gamma + p^{-2}) \subseteq (\beta, \beta + B_4 p^{-1}) \cup \bigcup_{z \in K(p)} (\theta(p,z), \theta(p,z) + B_4 p^{-1}) \\
\cup (\gamma + p^{-2} - B_4 p^{-1}, \gamma + p^{-2}).
\end{aligned}
$$

Therefore for all $U \ge U_1$ and $U < p \le 2U$, any open interval $I \subset (\beta, \gamma + p^{-2})$ satisfying (4.5) completely includes at least $B_4^{-1} p \cdot \mathrm{diam}(I) - 2 \ge (3B_4)^{-1} U \cdot \mathrm{diam}(I)$ basic intervals of G_p. Hence, by the prime number theorem, the open interval I completely includes at least (4.6) basic intervals of H_U for a large enough U_1. ∎

Proof of Proposition 4.1. Let B_3 and B_4 be constants as in Lemma 4.4 and Lemma 4.5, respectively. Let $u_1 = \max(U_1, 2)$. For every $k = 2, 3, \ldots,$ we put

$$
u_k = \max(u_{k-1}^k, \lceil 3(B_4/B_3) u_{k-1}^q \rceil),
$$

and $B_5 = B_3/(6B_4)$. Let E_1 be the open interval $(\beta, 2\gamma)$. For every $k = 2, 3, \ldots,$ let E_k be the union of basic intervals of H_{u_k} which are completely included by E_{k-1}. Let F be the intersection of all E_k's. Define $m_1 = 1$, and for $k \ge 2$, define

$$
m_k = \frac{u_k^2}{6B_4 \log u_k} B_3 u_{k-1}^{-q} = B_5 \frac{u_k^2 u_{k-1}^{-q}}{\log u_k}.
$$

Lemma 4.4 implies that each $(k-1)$st level interval of F has length at least $B_3 u_{k-1}^{-q}$. Then, by Lemma 4.5, each $(k-1)$st level interval of F contains at least m_k kth level intervals. In addition, by Lemma 4.4, disjoint kth level intervals of F are separated by gaps of at least $\delta_k = B_2 u_k^{-2}$. Therefore, Lemma 2.3 implies that

$$
\begin{aligned}
\dim_{\mathrm{H}}(F) &\ge \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(\delta_k m_k)} \\
&= \varliminf_{k\to\infty} \frac{2\log u_{k-1} + \log(B_5^{k-2} u_1^{-q} (u_2 \cdots u_{k-2})^{2-q} (\log u_2)^{-1} \cdots (\log u_{k-1})^{-1})}{q \log u_{k-1} + \log\log u_k - \log(B_2 B_5)}.
\end{aligned}
$$

Since $u_k \ge u_{k-1}^k$ for all $k \ge 2$, we have $\log u_k \ge k! \log u_1$ and $u_k \ge u_{k-1}$. Further, for $k \ge 1$ large enough, we have $u_k = u_{k-1}^k$. Thus for $k \ge 1$ large enough, we see that

$$2\log u_{k-1} = 2k^{-1}\log u_k, \qquad q\log u_{k-1} = qk^{-1}\log u_k,$$
$$|\log(B_5^{k-2}u_1^{-q}(u_2\cdots u_{k-2})^{2-q}(\log u_2)^{-1}\cdots(\log u_{k-1})^{-1})| \ll \log u_{k-2}.$$

Therefore, since $\log u_{k-2}/\log u_k = 1/(k(k-1)) \to 0$ as $k\to\infty$, we get

$$\dim_{\mathrm{H}}\Big(\bigcap_{k=1}^{\infty} E_k\Big) \geq \frac{2}{q}.$$

We finally show that for any $\tau \in F$, the equation $ax+by=cz$ is solvable in $\mathrm{PS}(\tau)$ and $\tau\in[\beta,\gamma]$. If this claim is true, we get the conclusion of Proposition 4.1 by the monotonicity of the Hausdorff dimension.

Take any $\tau\in F$. It is clear that $\tau\in[\beta,\gamma]$ since the condition (C6) yields $H_{u_k}\subseteq(\beta,\gamma+u_k^{-2})$, which implies $F\subseteq[\beta,\gamma]$. Further, by (C7), for all $k>1$, there exist a prime number $u_k<p_k\leq 2u_k$ and $z_k\in K(p_k)$ such that we find a pairwise distinct tuple $(X(p_k,z_k),Y(p_k,z_k),Z(p_k,z_k))\in\mathbb{N}^3$ such that

$$a\lfloor X(p_k,z_k)^\tau\rfloor + b\lfloor Y(p_k,z_k)^\tau\rfloor = c\lfloor Z(p_k,z_k)^\tau\rfloor, \qquad X(p_k,z_k)\geq p_k.$$

Since $X(p_k,z_k)\geq p_k\geq u_k\to\infty$ as $k\to\infty$, the equation $ax+by=cz$ is solvable in $\mathrm{PS}(\tau)$. ∎

5. Proof of Theorem 1.1. Fix any $a,b,c\in\mathbb{N}$. Without loss of generality, we may assume that either $a\neq c$ or $a=b=c=1$. Let $\varepsilon>0$ be arbitrarily small. Let $d=\lfloor s\rfloor$ and choose real numbers β,γ with $d\leq s<\beta<\gamma<\min(t,d+1)$. Let $x_0=x_0(a,b,c,d,\beta,\gamma)$ be as in Section 3. By the monotonicity of the Hausdorff dimension, we have

$$\begin{aligned}\text{(5.1)}\qquad &\dim_{\mathrm{H}}(\{\alpha\in[s,t]\colon ax+by=cz \text{ is solvable in } \mathrm{PS}(\alpha)\})\\ &\qquad\geq \dim_{\mathrm{H}}(\{\alpha\in[\beta,\gamma]\colon ax+by=cz \text{ is solvable in } \mathrm{PS}(\alpha)\}).\end{aligned}$$

Take $\alpha(x,z)$ as in Lemmas 3.1 and 3.2. Let $K(x)=J_{a,b,c}(x)$, $\theta(x,z)=\alpha(x,z)$. We give $\ell(x,z)$ later. Let us check the conditions (C1) to (C7), and apply Proposition 4.1.

Case $a>c$. By Lemma 3.1, for all $x\geq x_0$ and $z\in J_{a,b,c}(x)$ we have

$$ax^{\alpha(x,z)}+b=cz^{\alpha(x,z)}.$$

Thus by Lemma 3.3, there exists $n_0=n_0(x,z)\in\mathbb{N}$ such that

$$a\lfloor(n_0x)^\alpha\rfloor + b\lfloor n_0^\alpha\rfloor = c\lfloor(n_0z)^\alpha\rfloor, \tag{5.2}$$
$$\max(\{(n_0x)^\alpha\},\{(n_0)^\alpha\},\{(n_0z)^\alpha\})<1/2, \tag{5.3}$$
$$n_0 \ll_\varepsilon x^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon}. \tag{5.4}$$

Define η as in Lemma 3.4. Let $\ell(x,z)=\eta(\alpha(x,z),n_0x,n_0,n_0z)$. The condition (C1) is clear from the definition of $J_{a,b,c}(x)$. The condition (C2) is also clear since we find at most one multiple of x among any three consecutive integers if $x_0 \geq 3$. Lemma 3.1 implies (C4). By Lemma 3.4, for each $x \geq x_0$ and $z \in J_{a,b,c}(x)$, each $\tau \in (\alpha(x,z), \alpha(x,z) + \ell(x,z))$ satisfies

$$a\lfloor (n_0 x)^\tau \rfloor + b\lfloor n_0^\tau \rfloor = c\lfloor (n_0 z)^\tau \rfloor, \qquad n_0 x \geq x.$$

Therefore we have (C7). Let us prove (C3), (C5), (C6).

We show (C3). Let x be an integer with $x \geq x_0$. For each $i \in \{1,2\}$, let

$$z_{1,i} = \left\lfloor \left(\frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} x \right\rfloor + i, \qquad z_{2,i} = \lfloor (a/c)^{1/\beta} x \rfloor - i.$$

Note that $J_{a,b,c}(x)$ does not contain multiples of x. Thus we do not know whether $z_{1,i}, z_{2,i} \in J_{a,b,c}(x)$ for each $i \in \{1,2\}$. However, by (C2), there exist $i_1, i_2 \in \{1,2\}$ such that $z_{1,i_1}, z_{2,i_2} \in J_{a,b,c}(x)$. Lemma 3.1 implies that

$$\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\log(z_{1,i_1}/x)} + O\left(\frac{1}{x^2 \log x} \right).$$

Here we have

$$\begin{aligned}
\log(z_{1,i_1}/x) &= \log\left(\left(\frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} + O(x^{-1}) \right) \\
&= \frac{1}{\gamma} \log(a/c) + \log\left(1 + O\left(\frac{b}{a\gamma x^2 \log x} \right) + O(x^{-1}) \right) \\
&= \frac{1}{\gamma} \log(a/c) + O(x^{-1}).
\end{aligned}$$

Therefore

$$\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\frac{1}{\gamma}\log(a/c) + O(x^{-1})} + O\left(\frac{1}{x^2 \log x} \right) = \gamma + O(x^{-1}).$$

Similarly, we have $\alpha(x, z_{2,i_2}) = \beta + O(x^{-1})$. Hence we obtain (C3).

We next show (C5). For all $x \geq x_0$ and $z \in J_{a,b,c}(x)$, we have $x < z$ by the definition of $J_{a,b,c}(x)$. Recall that

$$\ell(x,z) = \eta(\alpha(x,z), n_0 x, n_0, n_0 z) = \frac{\log((\lfloor W^\alpha \rfloor + 1) W^{-\alpha})}{\log W},$$

where W is one of $n_0 x$, n_0, or $n_0 z$. From $\beta < \alpha(x,z)$, we have $\ell(x,z) \leq \log(1 + (n_0 x)^{-\beta}) \leq x^{-\beta}$. Further, by the facts (5.3), (5.4), $1 < x < z \ll x$, and $\alpha < \gamma$, we have

$$\ell(x,z) \geq \frac{\log(1 + 2^{-1} W^{-\alpha})}{\log W} \gg \frac{1}{(n_0 z)^\gamma \log(n_0 z)} \gg_\varepsilon x^{-q},$$

where

$$q = q(\beta, \gamma, \varepsilon) = (\gamma + \varepsilon)\left(\frac{\gamma^2}{(2 + \{\beta\} - 2^{1 - \lfloor \beta \rfloor})(2 - \{\gamma\})} + 1 + \varepsilon \right).$$

Therefore (C5) holds (with $Q_3 = 1$). The remaining condition (C6) is clear since $\beta < \alpha(x,z) < \gamma$ and $\alpha(x,z) + \ell(x,z) < \gamma + x^{-2}$ by (C5) (with $Q_3 = 1$).

Case $c > a$. Define $n_0 = n_0(x,z)$ and $\ell(x,z)$, $q(\beta,\gamma,\varepsilon)$ the same way as in Case $a > c$. The condition (C1) is clear since $z < x$ by the definition of $J_{a,b,c}(x)$. The condition (C2) is also clear since $J_{a,b,c}(x)$ forms a set of consecutive integers. Lemma 3.1 implies (C4). Similarly to the discussion in Case $a > c$, we have (C5)–(C7). To show (C3), let x be an integer with $x \geq x_0$. Let

$$z_1 = \left\lfloor \left(\frac{a}{c - b(x^2 \log x)^{-1}} \right)^{1/\beta} x \right\rfloor + 1, \qquad z_2 = \lfloor (a/c)^{1/\gamma} x \rfloor - 1.$$

We observe that $z_1, z_2 \in J_{a,b,c}(x)$ if x_0 is sufficiently large. Lemma 3.1 implies that $\alpha(x,z_1) = \beta + O(x^{-1})$ and $\alpha(x,z_2) = \gamma + O(x^{-1})$. This gives (C3).

Case $a = b = c = 1$. By Lemma 3.2, for all $x \geq x_0$ and $z \in J_{1,1,1}(x)$, by letting $X = X(x,z) = x^2 \lceil \log x \rceil$, $Y = Y(x,z) = x^2 \lceil \log x \rceil + 1$, $Z = Z(x,z) = zx\lceil \log x \rceil$, we have

$$X^{\alpha(x,z)} + Y^{\alpha(x,z)} = Z^{\alpha(x,z)}.$$

Therefore, from Lemma 3.3, there exists $n_0 = n_0(x,z) \in \mathbb{N}$ such that

$$\begin{aligned} &\lfloor (n_0 X)^\alpha \rfloor + \lfloor (n_0 Y)^\alpha \rfloor = \lfloor (n_0 Z)^\alpha \rfloor, \\ &\max(\{(n_0 X)^\alpha\}, \{(n_0 Y)^\alpha\}, \{(n_0 Z)^\alpha\}) < 1/2, \\ &n_0 \ll_\varepsilon (X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor \beta \rfloor})(2-\{\gamma\}))+\varepsilon}. \end{aligned} \tag{5.5}$$

Defining $r = r(\gamma,\beta,\varepsilon) = \gamma^2/((2+\{\beta\}-2^{1-\lfloor \beta \rfloor})(2-\{\gamma\})) + \varepsilon$, we obtain

$$n_0 \ll_\varepsilon x^{(2+\varepsilon)r}. \tag{5.6}$$

Let $\ell(x,z) = \eta(\alpha(x,z), n_0X, n_0Y, n_0Z)$ be as in Lemma 3.4.

The condition (C1) is clear since $x < z < 2x$ by the definition of $J_{1,1,1}(x)$. The condition (C2) is also clear since $J_{1,1,1}(x)$ forms a set of consecutive integers. Lemma 3.2 implies (C4). By Lemma 3.4, for all $x \geq x_0$ and $z \in J_{1,1,1}(x)$, each $\tau \in (\alpha(x,z), \alpha(x,z) + \ell(x,z))$ satisfies

$$\lfloor (n_0 X)^\tau \rfloor + \lfloor (n_0 Y)^\tau \rfloor = \lfloor (n_0 Z)^\tau \rfloor, \qquad n_0 X \geq x.$$

Therefore (C7) holds. It remains to prove (C3), (C5), and (C6).

Let us show (C3). Take any integer $x \geq x_0$. Let

$$z_1 = \lfloor 2^{1/\gamma}(x + (x\lceil \log x \rceil)^{-1}) \rfloor + 1, \qquad z_2 = \lfloor 2^{1/\beta} x \rfloor - 1.$$

It follows that $z_1, z_2 \in J_{1,1,1}(x)$ if x_0 is sufficiently large. Then Lemma 3.2 implies that $\alpha(x,z_1) = \gamma + O(x^{-1})$ and $\alpha(x,z_2) = \beta + O(x^{-1})$. Therefore we have (C3).

We next show (C5). Let x be an integer with $x \geq x_0$ and $z \in J_{1,1,1}(x)$. It is clear that $x < z$ and $X(x,z) < Y(x,z) < Z(x,z)$. Recall that

$$\ell(x,z) = \eta(\alpha(x,z), n_0X, n_0Y, n_0Z) = \frac{\log\left((\lfloor W^\alpha \rfloor + 1)W^{-\alpha}\right)}{\log W},$$

where W is one of n_0X, n_0Y, or n_0Z. Therefore, as $\beta < \alpha$, we have $\ell(x,z) \leq \log(1 + (n_0Z)^{-\beta}) \leq Z^{-\beta} \leq x^{-\beta}$. Further, from (5.5), (5.6) and $\alpha < \gamma$, we obtain

$$\ell(x,z) \geq \frac{\log(1 + 2^{-1}W^{-\alpha})}{\log W} \gg \frac{1}{(n_0Z)^\gamma \log(n_0Z)} \gg_\varepsilon x^{-(2+\varepsilon)(\gamma+\varepsilon)(r+1)}.$$

Hence, (C5) holds. The condition (C6) is clear since $\beta < \alpha(x,z) < \gamma$ and $\alpha(x,z) + \ell(x,z) < \gamma + x^{-2}$ by (C5).

To summarize the above discussion, define

$$D_{a,b,c}(\beta,\gamma,\varepsilon) = \begin{cases} \dfrac{2}{(2+\varepsilon)(\gamma+\varepsilon)(r(\beta,\gamma,\varepsilon)+1)} & \text{if } a = b = c, \\ \dfrac{2}{q(\beta,\gamma,\varepsilon)} & \text{otherwise.} \end{cases}$$

Cases $a > c$, $c > a$, $a = b = c = 1$ and Proposition 4.1 imply that

$$\dim_{\mathrm{H}}(\{\alpha \in [\beta,\gamma] : ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \geq D_{a,b,c}(\beta,\gamma,\varepsilon).$$

Therefore, by (5.1) and by letting $\varepsilon \to +0$, $\gamma \to \beta$, $\beta \to s$, we have

$$\dim_{\mathrm{H}}(\{\alpha \in [s,t] : ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \geq D_{a,b,c}(s,s,0).$$

By the definitions of q and r, we get the conclusion of Theorem 1.1.

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References

[DT97] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Math. 1651, Springer, Berlin, 1997.

[Fal14] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, 3rd ed., Wiley, Chichester, 2014.

[FW09] N. Frantzikinakis and M. Wierdl, *A Hardy field extension of Szemerédi's theorem*, Adv. Math. 222 (2009), 1–43.

[Gla17] D. Glasscock, *Solutions to certain linear equations in Piatetski-Shapiro sequences*, Acta Arith. 177 (2017), 39–52.

[Gla20] D. Glasscock, *A perturbed Khinchin-type theorem and solutions to linear equations in Piatetski-Shapiro sequences*, Acta Arith. 192 (2020), 267–288.

[Kok50] J. F. Koksma, *Some theorems on Diophantine inequalities*, Scriptum no. 5, Math. Centrum Amsterdam, 1950.

[SY19] K. Saito and Y. Yoshida, *Arithmetic progressions in the graphs of slightly curved sequences*, J. Integer Seq. 22 (2019), no. 2, art. 19.2.1, 25 pp.

[Sze75] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. 27 (1975), 199–245.

[Szü52] P. Szüsz, *Über ein Problem der Gleichverteilung*, in: Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août – 2 Septembre 1950, Akadémiai Kiadó, Budapest, 1952, 461–472.

[Tit86] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, New York, 1986.

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