ANNALES
POLONICI MATHEMATICI
XXIX (1974)

Existence of solutions of orientor fields with non-convex right-hand side

by H. Kaczyński and C. Olech (Warszawa)

Abstract. The existence of solutions of an orientor field

$$\dot{x} \in F(t, x), \quad x \in \mathbf{R}^n, \ 0 < t < 1,$$

is proved under the assumption: $F(t, x) \subset \mathbf{R}^n$ is compact (but not necessarily convex) continuous in the Hausdorff sense in x for each fixed t, measurable in t for each fixed x and there is an integrable function $\varphi: [0, 1] \to R$ such that $|v| \leqslant \varphi(t)$ if $v \in F(t, x)$. This is an extension of a recent result of Filippov, where existence was proved under the assumption that F is continuous with respect to both variables t and x.

Consider an orientor field, called also a *generalized differential equation* or a *differential equation* with multivalued right-hand side,

$$\dot{x} \in F(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$ and $F(t, x)$ is a subset of $\mathbf{R}^n$ for each $t \in I = [0, 1]$ and $x \in \mathbf{R}^n$. We shall consider solutions of (1) in the Carathéodory sense; that is $x(t)$ defined on I is a solution of (1) if it is absolutely continuous and satisfies (1) almost everywhere in I.

The usual way to show existence of solutions of the initial problem of (1) is to construct an equi-absolutely continuous sequence $x_n(t)$ of approximate solutions and to prove that there is a subsequence convergent to a solution of (1). To be more specific one constructs $x_n(t)$ with two properties:

$$d\big(\dot{x}_n(t), F(t, x_n(t))\big) \to 0 \quad \text{as } n \to \infty \text{ a.e. in } I, \tag{2}$$

where d stands for the distance of a point from a set and $x_n(t)$ contains a subsequence convergent uniformly to an absolutely continuous function $x_0(t)$. Then one tries to show that $x_0(t)$ is a solution of (1).

From (2) the only information one can have about the derivative $\dot{x}_0(t)$ is that (cf. [5])

$$\dot{x}_0(t) \in \bigcap_{\varepsilon > 0} clco \bigcup_{|x - x_0(t)| < \varepsilon} F(t, x) = \overline{F}\big(t, x_0(t)\big)$$

Thus the limit function would be a solution of (1) if $\overline{F}(t, x) = F(t, x)$, which is the case if F is compact convex and upper semicontinuous in x for each fixed t. In fact the later property was usually assumed in the known existence theorems (cf. [1], [7]).

If convexity of $F(t, x)$ is not assumed then the upper semi-continuity is not enough for existence of solutions of (1).

Hermes [3] posed a question: does there exist a solution of (1) if F is not assumed to be convex but is compact and continuous in x? Filippov [2] solved this question in the case when $F(t, x)$ is assumed to be compact and continuous in both variables. The purpose of this note is to show the existence of solutions of (1) in the case of Carathéodory assumptions. Namely we will prove the following theorem.

THEOREM 1. *Assume that $F(t, x)$ is compact for $(t, x) \in I \times \boldsymbol{R}^n$, continuous in x in the sense of Hausdorff metric for each fixed t, measurable in t for each fixed x and that there is an integrable function $m\colon I \to R$ such that*

$$|v| \leqslant m(t) \quad \text{if } v \in F(t, x) \quad \text{for each } x \in \boldsymbol{R}^n. \tag{3}$$

Under these assumptions there exist a solution of (1) *satisfying the initial condition $x(0) = x_0$.*

(For the definition and basic properties of measurable set-valued functions we refer the reader to [6].)

The difficulties which one has to face when one wants to construct a solution of (1) if $F(t, x)$ is not convex is that approximate solution have to be defined in a more accurate way to assure not only convergence of $x_n(t)$ but also of $\dot{x}_n(t)$ in some stronger sense. It is evident that if $x_n(t)$ converges uniformly to absolutely continuous $x_0(t)$ and $\dot{x}_n(t)$ converges point-wise to $\dot{x}_0(t)$ a.e. in t, then (2) implies that $x_0(t)$ is a solution of (1).

This is what Filippov [2] did and the construction of approximate solutions we are presenting in this note is very much like the one in Filippov's paper except that in our case $x_n(t)$ cannot be piece-wise linear because F is not continuous in t.

In Section 1 we introduce some notations and an auxiliary construction while in Sention 2 the approximate solutions are defined and Theorem 1 is proved.

This note is a part of a paper of the first of the authors presented as his thesis for master (magister) degree at Warsaw University in June, 1972. Theorem 1 was obtained independently by Hermes and Van Vleck [4] in a paper to be published in Journal of Differential Equations.

1. In this section we shall define an L_1 - conditionaly compact sequence of integrable functions which we will use in the next section to construct approximate solutions of (1).

Without any loss of generality we restrict ourselves to the initial condition $x(0) = 0$.

Denote by K the closed ball in $\boldsymbol{R}^n$ centered at zero and of radius $M = \int_0^1 m(t)\,dt$. By (3), any possible solution of (1) assumes values in K for $t \in I$.

Denote by $\eta(t,r)$ the modulus of continuity of F; that is

$$\eta(t,r) = \max\{h(F(t,\bar{x}),F(t,x))\,|\,x,\bar{x}\in K,\ |x-\bar{x}|\leqslant r\},$$

where h stands for the Hausdorff distance between two sets. One can check that assumptions of Theorem 1 imply that η is integrable in t for each fixed r, is non-increasing and continuous in r for each fixed t.

Let r_i be a decreasing sequence of reals tending to zero as $i\to\infty$, let A_i be a finite subset of K such that

$$K\subset\bigcup_{a\in A_i}\{x|\,|x-a|<r_i/4\}. \tag{4}$$

We define now inductively sets $B_i\subset A_1\times A_2\times\ldots\times A_i$. For $i=1$ we put $B_1=A_1$. If B_{i-1} is defined, then

$$B_i=\{b\,|\,b=(a^1,\ldots,a^{i-1},a^i),\quad (a^1,\ldots,a^{i-1})\in B_{i-1},\ a^i\in A_i \text{ and } |a^{i-1}-a^i|\leqslant r_{i-1}\}.$$

Notice that for each $a^i\in A$ there are $a^j\in A_j$, $j=1,\ldots,i-1$ such that $(a^1,\ldots,a^{i-1},a^i)\in B_i$. To each $b\in B_i$ we assign, in an inductive way, an integrable function $u_b\colon I\to\boldsymbol{R}^n$ satisfying the following conditions:

$$u_b(t)=u_{(a^1,\ldots,a^i)}(t)\in F(t,a^i)\quad \text{a.e. in } I, \tag{5}$$

and if $i>1$, then

$$|u_b(t)-u_{(a^1,\ldots,a^{i-1})}(t)|\leqslant\eta(t,r_{i-1})\quad \text{a.e. in } I. \tag{6}$$

Such u_b exists since $|a^{i-1}-a^i|\leqslant r_{i-1}$ and therefore the intersection

$$F(t,a^i)\cap\{x|\,|x-u_{(a^1,\ldots,a^{i-1})}(t)|\leqslant\eta(t,r_{i-1})\}$$

is not empty for each $t\in I$ and measurable as function of t. Thus any measurable selection of the above intersection satisfies (5) and (6).

Let h_i be a decreasing to zero sequence such that both $1/h_i$ and h_i/h_{i+1} are integers.

By C_i we denote the set of functions $c\colon I\to B_i$ such that if $c(t)=(a^1(t),\ldots,a^i(t))$, then for $1\leqslant j\leqslant i$, $a^j(t)=\text{const}$ on intervals $[sh_j,(s+1)h_j)$ for each integer $s=0,\ldots,1/h_j-1$. It is clear from (3) and (5) that for each $c\in C_i$

$$|u_{c(t)}(t)|\leqslant m(t)\quad \text{if } t\in I \text{ and } i=1,2,\ldots$$

Consider now the sets

$$W_i=\{u: I\to\boldsymbol{R}^n\,|\,u=u_c,\ c\in C_i\}.$$

It is clear that W_i is finite.

Denote by

$$U=\bigcup_{i=1}^{\infty}W_i$$

We will prove the following lemma.

LEMMA 1. *If*

$$\int_0^1 \sum \eta(t, r_i)\, dt < +\infty, \tag{7}$$

then U is conditionaly compact in L_1.

Proof. Take an arbitrary $\varepsilon > 0$. By (7) there is an integer m such that

$$\int_0^1 \sum_{i=m}^{\infty} \eta(t, r_i)\, dt < \varepsilon. \tag{8}$$

We will prove that for each $u \in W_{m+p}$, $p \geqslant 1$, there is $\overline{u} \in W_m$ such that $\|\overline{u} - u\|_{L_1} \leqslant \varepsilon$. Hence we will prove that there is a finite covering of U with balls of radius ε which means compactness of U.

If $u(t) = u_{(a^1(t), \ldots, a^m(t), \ldots, a^{m+p}(t))}(t)$, then we put $\overline{u}(t) = u_{(a^1(t), \ldots, a^m(t))}(t)$. It is clear that $\overline{u} \in W_m$. We have:

$$|u_{(a^1(t), \ldots, a^{m+p}(t))}(t) - u_{(a^1(t), \ldots, a^m(t))}(t)| \leqslant \sum_{i=1}^{p} |u_{(a^1(t), \ldots, a^{m+i}(t))}(t) - u_{(a^1(t), \ldots, a^{m+i-1}(t))}(t)|.$$

Using (6) and (8) we obtain

$$\|u - \overline{u}\|_{L_1} \leqslant \int_0^1 \sum_{i=1}^{p} \eta(t, r_{m+i-1})\, dt \leqslant \varepsilon,$$

what was to be proved.

2. In this section we construct the sequence $x_n(t)$ of approximate solutions of (1) and prove Theorem 1. For each n, x_n will be an absolutely continuous function from I into $\boldsymbol{R}^n$ such that, $x_n(0) = 0$ and

$$\dot{x}_n(t) = u_{c(t)}(t) \quad \text{for some } c \in C_n \text{ and each } t \in I \tag{9}$$

and if $c(t) = \big(a^1(t), \ldots, a^n(t)\big)$, then

$$|x_n(t) - a^n(t)| \leqslant r_n \quad \text{for } t \in I, \tag{10}$$

For this purpose assume that h_n is chosen so small that

$$\int_t^{t+h_n} m(t)\, dt < r_n/4, \quad n = 1, 2, \ldots \tag{11}$$

and r_i are such that (7) holds and

$$r_{i+1} < r_i/4. \tag{12}$$

To define x_n we have to define $c(t) = \big(a^1(t), \ldots, a^n(t)\big)$, where $c \in C_n$.

For $t \in [0, h_n]$ we put

$$a^i(t) = a^i = \text{const}, \tag{13}$$

where a_i is such that

$$|a^i - x(0)| = |a_i| \leqslant r_i/4, \quad i = 1, 2, \ldots, n. \tag{14}$$

For induction argument assume that $c(t)$ is defined for $t \in [0, sh_n]$, $1 \leqslant s < 1/h_n$.

If for a fixed $i < n$, $sh_n \neq ph_i$ for each integer p, then we put

$$a^i(t) = a^i\big((s-1)h_n\big) \quad \text{for } t \in [sh_n, (s+1)h_n].$$

If there is an integer p such that $sh_n = ph_i$, for a fixed i, then we put

$$a^i(t) = a^i = \text{const} \quad \text{for } t \in [sh_n, (s+1)h_n],$$

where $a^i \in A_i$ and $|a^i - x_n(sh_n)| \leqslant r^i/4$. Such a^i exists by (4).

It is easy to check that for each i and each integer $p < 1/h_i$, $a^i(t)$ is constant on $[ph_i, (p+1)h_i]$ and

$$|x_n(sh_n) - a^i(sh_n)| < r_i/2, \quad i = 1, 2, \ldots, n. \tag{15}$$

The latter inequality has to be checked in the first case and it follows from (11). From (12) and (15) we see that so defined c has values in B_n. Therefore $c \in C_n$. Inequality (10) follows easily from (11) and (15).

Proof of Theorem 1. By definition, x_n are absolutely continuous functions uniformly bounded by M and such that $\{\dot{x}_n\} \subset U$. Thus by Lemma 1 $\{\dot{x}_n\}$ is conditionally compact in L_1.

Therefore there is a subsequence, for simplicity still denoted by $\{\dot{x}_n\}$, and an integrable function $v: I \to \boldsymbol{R}^n$

$$\|\dot{x}_n - v\|_{L_1} \to 0 \quad \text{and} \quad \dot{x}_n(t) \to v(t) \quad \text{a.e. in } I. \tag{15}$$

This implies that

$$x_n(t) \rightrightarrows \int_0^t v(t)\,dt = x_0(t). \tag{16}$$

To prove that x_0 is a solution of (1) we note that

$$d\Big(\dot{x}_n(t), F\big(t, x_n(t)\big)\Big) = d\Big(u_{(a^1(t),\ldots,a^n(t))}(t), F\big(t, x_n(t)\big)\Big)$$
$$\leqslant d\Big(u_{(a^1(t),\ldots,a^n(t))}(t), F\big(t, a^n(t)\big)\Big) + h\Big(F\big(t, x_n(t)\big), F\big(t, a^n(t)\big)\Big).$$

The first component of the above sum is equal zero because of (5) and the second can be estimated by $\eta(t, r_n)$ because of (10). Hence we have the inequality $d\Big(\dot{x}_n(t), F\big(t, x_n(t)\big)\Big) \leqslant \eta(t, r_n)$, which together with (15), (16) and continuity of F with respect to x implies $d\Big(\dot{x}_0(t), F\big(t, x_0(t)\big)\Big)$

$= 0$. Hence $\dot{x}_0(t) \in F(t, x_0(t))$ a.e. in I, which completes the proof of Theorem 1.

Remark 1. Assumption (3) in Theorem 1 can be replaced by the inequality $|v| \leqslant m(t, r)$ for each $v \in F(t, x)$ if $|x| \leqslant r$, where m is assumed to be integrable in t for each fixed r. In this case the existence theorem would have local character; that is for each initial condition there is an interval such that there is a solution of (1) defined on this interval and satisfying given initial condition.

Remark 2. In the case considered by Filippov [2], when F is continuous in both variables, η can be taken as independent on t and u_c as piece-wise constant function. Therefore in this case the difference

$$u_{(a^1(t), \ldots, a^{m+p}(t))}(t) - u_{(a^1(t), \ldots, a^m(t))}(t)|$$

in the proof of Lemma 1 can be estimated uniformly (for each t) and consequently we can prove that the convergence of $\dot{x}_n(t)$ to $v(t)$, in (15), holds for each t and is uniform. Taking into account that $\dot{x}_n(t)$ is piece-wise constant the latter implies that $v(t)$ may be discontinuous only on a denumerable subset of I. In fact this is what Filippov proved in [2] about the derivative of the solution of (1).

References

[1] A. F. Filippov, *Differential equations with multivalued right-hand side* (in Russian), Dokl. Akad. Nauk SSSR 151 (1963), p. 65–68.

[2] — *On existence of solutions of multivalued differential equations* (in Russian), Mat. Zametki, 10 (1971), p. 307–313.

[3] H. Hermes, *The generalized differential equation $\dot{x} \in R(t, x)$*, Advances in Math. 4 (1970), p. 149–169.

[4] — and F. S. Van Vleck, *The existence of solutions of generalized differential equations satisfying Carathéodory conditions*, to appear in J. of Diff. Equations.

[5] A. Lasota and C. Olech, *On the closedness of the set of trajectories of a control system*, Bull. Acad. Polon. Sci., Ser. sci. math. astronom. et phys. 14 (1966), p. 615–621.

[6] R. T. Rockafellar, *Measurable dependence of convex sets and functions on parameters*, J. of Math. Anal. and Appl. 28 (1969), p. 4–25.

[7] T. Ważewski, *Systèmes de commande et équations an contingent*, Bull. Acad. Polon. Sci., Ser. sci. math., astronom. et phys. 9 (1961), p. 151–155.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 11. 1. 1973