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Holomorphic continuation of harmonic functions

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Abstract. Put

$$B = \{x \in R^n: |x| < r\}, \quad \tilde{B} = \{x+iy \in C^n: |x|^2+|y|^2+2\sqrt{|x|^2|y|^2-\langle x, y\rangle^2} < r^2\},$$

where $|x|^2 = \sum_{j=1}^{n} x_j^2$, $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$. The set $\tilde{B}$ is called a *Lie ball* and it is an E. Cartan's classical domain of the fourth type.

Main result: For every function u of n real variables harmonic in B there exists a function $\tilde{u}$ of n complex variables holomorphic in $\tilde{B}$ such that $\tilde{u} = u$ in B (where B is identified with the set $\{x+iy \in C^n: |x| < r, y = 0\}$. Moreover, there exists a function u^* holomorphic in $\tilde{B}$ such that u^* cannot be continued analytically beyond $\tilde{B}$ and u^* restricted to B is harmonic.

1. Introduction. Let h be a function of n real variables harmonic in the ball $B_r = \{x \in R^n: |x| < r\}$, where $|x| = [\sum x_j^2]^{1/2}$. It is well known that harmonic functions are real analytic. Therefore in a neighbourhood of $0 \in R^n$

$$h(x) = \sum C_\alpha x^\alpha, \quad \text{where } C_\alpha = \frac{1}{\alpha!} D^\alpha h(0), \ \alpha = (\alpha_1, \ldots, \alpha_n) \in Z_+^n, \tag{1}$$

the series being convergent absolutely and uniformly. By grouping the terms of degree k one gets the following expansion

$$h(x) = \sum_0^\infty h_k(x), \quad \text{where } h_k(x) = \sum_{|\alpha|=k} C_\alpha x^\alpha. \tag{2}$$

One may ask to what extent the grouping of the terms in (1) is essential for the convergence. The following theorems are known:

THEOREM A. [3]. *If h is harmonic in the ball B_r, then the terms h_k of the series* (2) *are harmonic polynomials and the series converges uniformly and absolutely in B_ϱ, whenever $0 < \varrho < r$.*

THEOREM B. [5]. *If h is harmonic in B_r, then its multiple Taylor series expansion* (1) *converges uniformly and absolutely in B_ϱ, when $\varrho < r/\sqrt{2}$, but the series may diverge at some points of the sphere $\{|x| = r/\sqrt{2}\}$.*

THEOREM C. [5]. *If $n = 2$ and $h(x_1, x_2)$ is harmonic in the disk $\{x_1^2+x_2^2 < r^2\}$, then the Taylor series expansion* (1) *converges absolutely and* *uniformly on every compact subset of the square* $C = \{|x_1|+|x_2| < r\}$. *If* h *is not harmonic in any open disk of larger radius centered at the origin then* (1) *diverges at all points exterior to* C *for which* $x_1x_2 \neq 0$.

In the sequel we identify R^n with $\{z = x+iy \in C^n : y = 0\}$. Put

$$t(z) = \Big[\sum_1^n |z_j|^2 + \Big[\Big(\sum_1^n |z_j|^2\Big)^2 - \Big|\sum_1^n z_j^2\Big|^2\Big]^{1/2}\Big]^{1/2}, \quad z \in C^n, \tag{3}$$

$$\tilde{B}_r = \{z \in C^n : t(z) < r\}. \tag{4}$$

The domain $\tilde{B}_r$ is identical with the so called *Lie ball* — the classical domain of the fourth type ([6]). It is obvious that $t(x) = |x|$ for $x \in R^n$. Thus $B_r \subset \tilde{B}_r$.

The purpose of this note is to prove the following

THEOREM D. *If* h *is harmonic in* B_r, *then:*

1° *There exists a function* $\tilde{h}$ *holomorphic in* $\tilde{B}_r$ *such that* $\tilde{h} = h$ *in* B_r;

2° *The multiple Taylor series* (1) *converges uniformly and absolutely on every compact subset of the domain*

$$G_r = \Big\{z \in C^n : \sum_1^n |z_j|^2 + 2\Big(\sum_{j<k} |z_j|^2 |z_k|^2\Big)^{1/2} < r^2\Big\}. \tag{5}$$

In particular, it converges uniformly and absolutely on every compact subset of the domain

$$H_r = \Big\{z \in C^n : \sum_1^n |z_j| < r\Big\}; \tag{6}$$

3° *There exists a function* h^* *harmonic in* B_r *such that* h^* *can be continued to a holomorphic function in* $\tilde{B}_r$, *but it cannot be continued holomorphically to any larger domain* $D \supset \tilde{B}_r$.

Observe that if $n = 2$, then $H_r = G_r = \{z \in C^2 : |z_1|+|z_2| < r\}$. If $n > 2$, $H_r \subset G_r$ and $H_r \neq G_r$. One may easily check that $L_r \overset{\text{df}}{=} \{z \in C^n : |z| < r/\sqrt{2}\} \subset H_r$ and $L_r \neq H_r$. Thus we get an improvement of Hayman's Theorem B.

COROLLARY. *If* D *is an open connected set in* R^n, *then there exists an open connected set* $\tilde{D} \subset C^n$ *such that every function* h *harmonic in* D *may be continued to a holomorphic function* $\tilde{h}$ *in* $\tilde{D}$.

Indeed, it suffices to put $\tilde{D} = \bigcup_{a \in D} \{z \in C^n : t(z-a) < r_a\}$, where r_a denotes the distance of a to the boundary of D.

By property 3° the Lie ball $\tilde{B}_r$ may be considered as an "envelope of holomorphy" of B_r with respect to the family of harmonic functions in B_r.

The problem of determining the envelope of holomorphy of an arbitrary domain $D \subset R^n$ with respect to the family of harmonic functions in D was discussed by Lelong in [7].

It is rather a point of interest that the harmonic envelope of holomorphy of B_r is one of the classical E. Cartan domains.

2. Proof of Theorem D.

Ad 1°. We shall need the following known properties of the Lie ball $\tilde{B}_r$ (see [6]):

(i) $\tilde{B}_r$ is a domain of holomorphy;

(ii) $\tilde{B}_r$ is balanced, i.e. if $z \in \tilde{B}_r$, $\lambda \in C$, $|\lambda| \leqslant 1$, then $\lambda z \in \tilde{B}_r$;

(iii) $S_r = \{e^{i\theta} x : x \in R^n,\ |x| = r,\ \theta \in R\} \subset \partial \tilde{B}_r$ and for every function f holomorphic in $\tilde{B}_r$, continuous in the closure of $\tilde{B}_r$ we have

$$\sup\{|f(z)| : z \in S_r\} = \sup\{|f(z)| : z \in \tilde{B}_r\},$$

i.e. S_r is the Bergman–Šilov boundary of $\tilde{B}_r$.

First we shall prove the following

LEMMA 1. *If* $\sum_0^\infty f_k$ *is any series of homogeneous polynomials of n complex variables* ($\deg f_k = k$) *convergent at every point* $x \in B_r$, *then it converges uniformly and absolutely on every compact subset of the Lie ball* $\tilde{B}_r$. *Moreover,* $\tilde{B}_r$ *is the maximal domain in* C^n *with this property.*

Proof. It is known ([2], p. 89) that, given ϱ, $0 < \varrho < r$, one may find $M > 0$ and $0 < \theta < 1$ such that

$$|f_k(x)| \leqslant M\theta^k, \qquad x \in B_\varrho,\ k \geqslant 0.$$

By (iii)

$$\sup\{|f_k(z)| : z \in \tilde{B}_\varrho\} = \sup\{|f_k(z)| : z \in S_\varrho\} = \sup\{|f_k(x)| : x \in B_\varrho\}.$$

Thus

$$|f_k(z)| \leqslant M\theta^k, \qquad z \in \tilde{B}_\varrho,\ k \geqslant 0.$$

So the series $\sum_0^\infty f_k$ converges uniformly and absulutely in $\tilde{B}_\varrho$, $0 < \varrho < r$.

Since $\tilde{B}_r$ is a domain of holomorphy, one may find a function f holomorphic in $\tilde{B}_r$ which cannot be continued analytically to any larger domain. Since $\tilde{B}_r$ is balanced, f can be expanded into a series of homogeneous polynomials, $f = \sum_0^\infty f_k$, which converges uniformly and absolutely on every compact subset of $\tilde{B}_r$ ([1]). However, this series cannot converge in any larger domain containing $\tilde{B}_r$, because otherwise f would be continuable beyond $\tilde{B}_r$. The proof of Lemma 1 is concluded. (For a different proof of Lemma 1 see [4].)

By Lemma 1 the series (2) converges uniformly and absolutely on every compact subset of $\tilde{B}_r$. Its sum $\tilde{h}$ gives the required continuation of h.

Ad 2°. The domain G_r given by (5) has the following properties:

(P_1) $G_r \subset \tilde{B}_r$.

(P_2) If $a \in G_r$, then the set $\{z \in C^n: |z_j| \leqslant |a_j|,\ j = 1, \ldots, n\}$ is contained in G_r.

The second property is obvious. To prove (P_1), observe that

$$\Big(\sum |z_j|^2\Big)^2 - \Big|\sum z_j^2\Big|^2 = \sum |z_j|^2 \sum |z_k|^2 - \sum z_j^2 \sum \bar{z}_k^2$$
$$= 2\sum_{j<k} |z_j|^2 |z_k|^2 - \sum_{j<k} (z_j^2 \bar{z}_k^2 + \bar{z}_j^2 z_k^2) \leqslant 4\sum_{j<k} |z_j|^2 |z_k|^2.$$

Hence

$$t(z)^2 \leqslant \sum |z_j|^2 + 2\Big[\sum_{j<k} |z_j|^2 |z_k|^2\Big]^{1/2}, \qquad z \in C^n.$$

Therefore $G_r \subset \tilde{B}_r$.

Since by (P_2) the set G_r is a complete n-circular domain, then every function f holomorphic in G_r may be developped into a multiple Taylor series and the series converges absolutely and uniformly on every compact subset of G_r ([1]). In particular, the series (1) converges absolutely and uniformly on every compact subset of G_r.

Ad 3°. Put $H(\tilde{B}_r) = \{f \in \mathcal{O}(\tilde{B}_r): f|B_r \text{ is harmonic in } B_r\}$, where $\mathcal{O}(\tilde{B}_r)$ denotes the space of all holomorphic functions in $\tilde{B}_r$. By the Harnack theorem $H(\tilde{B}_r)$ is a real Frechet space, if it is endowed with the topology of uniform convergence on compact subsets of $\tilde{B}_r$.

Let $\{a_k\}$ be a denumerable dense subset of $\tilde{B}_r$. Let ϱ_k denotes the distance of a_k to the boundary of $\tilde{B}_r$. For every $k, l \geqslant 1$, consider the domain

$$D_{kl} = \tilde{B}_r \cup \left\{z \in C^n:\ |z - a_k| < \varrho_k + \frac{1}{l}\right\}.$$

A function $f \in H(\tilde{B}_r)$ may be continued holomorphically beyond $\tilde{B}_r$ if and only if there exist positive integers k, l and a function $\tilde{f} \in H(D_{kl})$ such that $\tilde{f} = f$ in $\tilde{B}_r$. Thus our aim is to show that

$$H(\tilde{B}_r) \setminus \bigcup_{k,l=1}^{\infty} r_{kl}(D_{kl}) \neq \emptyset,$$

where r_{kl} is the continuous linear mapping defined by

$$r_{kl}\colon\ H(D_{kl}) \ni f \to f|\tilde{B}_r \in H(\tilde{B}_r).$$

By a theorem of Banach $r_{kl}(D_{kl})$ is either identical with $H(\tilde{B}_r)$ or it is a subset of the first category of $H(\tilde{B}_r)$. We shall show that the second possibility holds true. Namely, we shall show that for every point $w \in \partial\tilde{B}_r$ there exists $f \in H(\tilde{B}_r)$ such that f cannot be continued holomorphically through w.

Case 1. $n = 2$. One may check that in this case $t(z) = \max\{|z_1+iz_2|, |z_1-iz_2|\}$. Therefore $\tilde{B}_r = \{z \in C^2\colon |z_1+iz_2| < r,\ |z_1-iz_2| < r\}$. Let $w \in \partial\tilde{B}_r$. Then either $|w_1+iw_2| = r$ or $|w_1-iw_2| = r$. Define

$$f(z) = \log\{[\mathrm{re}^{i\theta}-(z_1+iz_2)][re^{-i\theta}-(z_1-iz_2)]\}, \qquad z \in \tilde{B}_r,$$

where

$$re^{i\theta} = w_1+iw_2 \quad \text{ if } |w_1+iw_2| = r,$$

or

$$re^{-i\theta} = w_1-iw_2 \quad \text{ if } |w_1-iw_2| = r.$$

We take the branch of log such that $f(x) = \log|re^{i\theta}-(x_1+ix_2)|^2$ is positive for $x \in B_r$. The function f belongs to $H(\tilde{B}_r)$ and f is not continuable through w.

Case 2. $n \geqslant 3$. It is known that $t(z)^2 = |x|^2+|y|^2+2[|x|^2|y|^2-\langle x, y\rangle^2]^{1/2}$, where $z = x+iy$, $x, y \in R^n$, and $\langle x, y\rangle = \sum_1^n x_jy_j$ (see [6]). Since for every $a \in R^n$, $|a| = r$, the function $h(x) = |a-x|^{-n+2} = [\sum_1^n (a_j-x_j)^2]^{1-n/2}$ is harmonic in B_r, it may be continued holomorphically into $\tilde{B}_r$. Its continuation $\tilde{h}$ is of the form $\tilde{h}(z) = [\sum_1^n (a_j-z_j)^2]^{1-n/2}$, where the branch of the power is uniquely determined by the condition $\tilde{h} = h$ in B_r.

Now the existence of the required function f will follow from

LEMMA 2. *Given $w = u+iv \in \partial\tilde{B}_r$ one may find $a \in \partial B_r$ such that $\sum_1^n (a_j-w_j)^2 = 0$, or equivalently*

$$|a-u| = |v| \quad \text{and} \quad \langle a-u, v\rangle = 0. \tag{7}$$

In order to prove Lemma 2 we shall first show that $\partial\tilde{B}_r = \Gamma$, where

$$\Gamma = \{e^{i\theta}(x+iy)\colon \theta \in R,\ x, y \in R^n,\ |x|+|y| = r,\ \langle x, y\rangle = 0\}.$$

Indeed, let $w = e^{i\theta}z \in \Gamma$. Then

$$t(e^{i\theta}z)^2 = t(z)^2 = |x|^2+|y|^2+2|x||y| = (|x|+|y|)^2 = r^2.$$

Thus $w \in \partial\tilde{B}_r$.

Now let $w \in \partial \tilde{B}_r$. We want to show that $w \in \Gamma$, i.e. we are about to find $z = x+iy \in C^n$ and $\theta \in R$ such that $w = e^{i\theta}z$, $|x|+|y| = r$ and $\langle x, y\rangle = 0$. It is obvious that $w \in \Gamma$ if $\langle u, v\rangle = 0$. So assume $\langle u, v\rangle \neq 0$ and put $e^{i\theta} = a+i\beta$ and $z = (a-i\beta)(u+iv) = (au+\beta v)+i(av-\beta u)$. Thus

$$\langle x, y\rangle = (a^2-\beta^2)\langle u, v\rangle + a\beta(|v|^2-|u|^2)$$
$$= \langle u, v\rangle \cos 2\theta - \tfrac{1}{2}\sin 2\theta(|u|^2-|v|^2).$$

Therefore if θ satisfies

$$\cot 2\theta = \frac{|u|^2-|v|^2}{2\langle u, v\rangle}, \tag{8}$$

then $w = e^{i\theta}z$ and $\langle x, y\rangle = 0$. Since $r = t(w) = t(e^{i\theta}z) = |x|+|y|$, we see that $w \in \Gamma$. Thus $\partial \tilde{B}_r = \Gamma$.

Let w be a fixed point of $\partial \tilde{B}_r$. Take $z = x+iy \in C^n$ and $\theta \in R$ such that $w = e^{i\theta}z$, $\langle x, y\rangle = 0$ and $|x|+|y| = r$. Put $a = \cos\theta$, $\beta = \sin\theta$. Then $u = ax-\beta y$, $v = \beta x+ay$. We need find $a \in R^n$ such that $|a| = r$, $\langle a-u, v\rangle = 0$ and $|a-u|^2 = |v|^2$. The last two equations may be written in the following equivalent form:

$$\beta\langle a, x\rangle + a\langle a, y\rangle = a\beta r(|x|-|y|),$$
$$-2a\langle a, x\rangle + 2\beta\langle a, y\rangle = r[(|x|-|y|)(\beta^2-a^2)-r].$$

Therefore $\langle a, x\rangle = ar|x|$, $\langle a, y\rangle = -\beta r|y|$. Hence

(i) if $x = 0$ (resp. $y = 0$), $\beta = 0$ (resp. $a = 0$), we can take for a any vector orthogonal to y (resp. x), $|a| = r$;

(ii) if $x = 0$ (resp. $y = 0$), $\beta \neq 0$ (resp. $a \neq 0$) we may take for a any vector such that the angle between a and y is equal to $\theta+\frac{\pi}{2}$ (resp. the angle between a and x is equal θ), $|a| = r$;

(iii) if $|x|\,|y| \neq 0$, we may put $a = r\left(\frac{a}{|x|}x - \frac{\beta}{|y|}y\right)$.

Added in proof. The author has recently found that the points 1° and 3° of Theorem D were earlier obtained by a different method and in a more general context by C. O. Kiselman (*Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France 97 (4) (1969), p. 329–356).

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