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On some hereditable shape properties

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Abstract. Let A be a compactum. A compactum X lying in a space $M \in \mathrm{AR}(\mathfrak{m})$ is said to be *A-movable* if for every neighborhood U of X in M there exists a neighborhood U_0 of X (in M) such that for every neighborhood V of X (in M) each map $\alpha: A \to U_0$ is homotopic in U to a map with all values in V. It is shown that the choice of the space $M \in \mathrm{AR}(\mathfrak{M})$ containing X is immaterial and that the A-movability is a hereditable shape property of X. Some relations between the A-movability and the formerly known concepts of the movability and of the n-movability are established.

Among shape properties (concerning the notions belonging to the shape theory see, for instance [2]) a special role is played by the property of movability (see [1], p. 137), because several important theorems (as theorems corresponding in the shape theory to theorems of Hurewicz [4] and of Whitehead [5]) are true for movable compacta, but they fail if one omits the hypothesis of movability.

The aim of this note is to introduce a family of properties analogous to the movability, and to prove that they are hereditable shape properties.

Let A and X be two compacta. We may assume that X is a subset of a space $M \in \mathrm{AR}(\mathfrak{M})$. Let us say that X is *A-movable* in M if for every neighborhood U of X in M there exists a neighborhood U_0 of X (in M) such that for every neighborhood $\hat{U}$ of X (in M) and for every map $\alpha: A \to U_0$ there is a homotopy

$$\varphi: A \times \langle 0, 1\rangle \to U$$

such that

$$(1) \qquad \varphi(p, 0) = \alpha(p) \quad \text{and} \quad \varphi(p, 1) \in \hat{U} \quad \text{for every point } p \in A.$$

Examples. (a) If A is a contractible in itself compactum, then every compactum $X \subset M \in \mathrm{AR}(\mathfrak{M})$ is A-movable in M. In fact, for every neighborhood U of X in M there exists a neighborhood $U_0 \subset U$ of X in M such that for every point $x \in U_0$ there is a point $\lambda(x) \in X$ such that x and $\lambda(x)$ can be joined by an arc lying in U. Since A is contractible in itself, each map $\alpha: A \to U_0$ is homotopic in U to a map α' mapping A onto a single point $x \in U_0$. Then α' is homotopic in U to the map α'' mapping A onto the point $\lambda(x) \in X$, hence α is homotopic in U to the map α'' with values lying in each neighborhood $\hat{U}$ of X.

(b) If X is a dyadic solenoid of Van Dantzig, lying in the euclidean 3-space $M = E^3$, and if A is the circle S^1, then X is not A-movable in M. In fact, there exists in E^3 a sequence of solid tori $T_1, T_2, \ldots$ such that T_{k+1} lies in the interior of T_k and its oriented core is homologous in T_k to the twice described core of T_k and that $X = \bigcap_{k=1}^{\infty} T_k$. One sees easily that any homeomorphism α mapping A onto the core of T_k with $k > 1$ is not homotopic in T_1 to any map with values in T_{k+1}.

Now we prove the following

(2) THEOREM. *Let A, X, Y be compacta and M, N be $\mathrm{AR}(\mathfrak{M})$-spaces such that $X \subset M$, $Y \subset N$. If X is A-movable in M and if $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$, then Y is A-movable in N.*

Proof. The hypothesis $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ means that there exist two fundamental sequences:

$$\underline{f} = \{f_k, X, Y\}_{M,N} \quad \text{and} \quad \underline{g} = \{g_k, Y, X\}_{N,M}$$

such that

$$\underline{f}\underline{g} \simeq \underline{i}_{Y,N}. \tag{3}$$

Consider a neighborhood V of Y in N. Then there exists a neighborhood U of X in M and an index k_1 such that

$$f_k/U \simeq f_{k_1}/U \quad \text{in } V \quad \text{for every } k \geqslant k_1. \tag{4}$$

Moreover, (3) implies that there exists a neighborhood $V_1 \subset V$ of Y in N and an index $k_2 \geqslant k_1$ such that

$$f_k g_k/V_1 \simeq i/V_1 \quad \text{in } V \quad \text{for every } k \geqslant k_2. \tag{5}$$

Since X is A-movable in M, there exists a neighborhood U_0 of X in M such that for every neighborhood $\hat{U}$ of X in M and for every map $\alpha\colon A \to U_0$ there is a homotopy

$$\varphi\colon A \times \langle 0, 1\rangle \to U$$

satisfying condition (1). Since $\underline{g}$ is a fundamental sequence, there exists a neighborhood $V_0 \subset V_1$ of Y in N and an index $k_3 \geqslant k_2$ such that

$$g_k(V_0) \subset U_0 \quad \text{for every } k \geqslant k_3. \tag{6}$$

Now let β be a map of A into V_0 and let $\hat{V}$ be an arbitrary neighborhood of Y in N. Since $\underline{f}$ is a fundamental sequence, there is a neighborhood $\hat{U}$ of X in M and an index $k_4 \geqslant k_3$ such that

$$f_k(\hat{U}) \subset \hat{V} \quad \text{for every } k \geqslant k_4. \tag{7}$$

Setting

$$\alpha(p) = g_{k_4}\beta(p) \quad \text{for every point } p \in A, \tag{8}$$

one gets (in view of (5) and the inequality $k_4 \geqslant k_3$) a map $\alpha\colon A \to U_0$. Consequently there is a homotopy $\varphi\colon A\times\langle 0,1\rangle \to U$ satisfying condition (1). It follows by (1), (3) and (4) that the formula

(9) $$\psi'(p,t) = f_{k_4}\varphi(p,t) \quad \text{for every } (p,t)\in A\times\langle 0,1\rangle,$$

defines a homotopy

$$\psi'\colon A\times\langle 0,1\rangle \to V.$$

Using (1), (9), (8) and (7), we infer that

(10) $$\psi'(p,0) = f_{k_4}g_{k_4}\beta(p) \quad \text{and} \quad \psi'(p,1)\in \hat{V} \quad \text{for every point } p\in A.$$

Since $\beta(A)\subset V_0\subset V_1$, we infer by (5) and (6) that $f_{k_4}g_{k_4}\beta \simeq \beta$ in V, i.e. there exists a homotopy

$$\psi''\colon A\times\langle 0,1\rangle \to V$$

such that

(11) $$\psi''(p,0) = \beta(p) \quad \text{and} \quad \psi''(p,1) = f_{k_4}g_{k_4}\beta(p)$$

for every point $p\in A$.

Setting

$$\psi(p,t) = \psi''(p,2t) \qquad \text{for } (p,t)\in A\times\langle 0,\tfrac{1}{2}\rangle,$$

$$\psi(p,t) = \psi'(p,2t-1) \qquad \text{for } (p,t)\in A\times\langle \tfrac{1}{2},1\rangle,$$

we get a homotopy

$$\psi\colon A\times\langle 0,1\rangle \to V.$$

Using (10) and (11) we infer that

$$\psi(p,0) = \beta(p) \quad \text{and} \quad \psi(p,1) = \psi'(p,1)\in \hat{V} \quad \text{for every point } p\in A.$$

Thus Y is A-movable in N and the proof of theorem (2) is finished.

(12) COROLLARY. *The A-movability of X in M does not depend on the choice of the space $M\in \mathrm{AR}(\mathfrak{M})$ containing X.*

It follows that the words "in M" are for the property of A-movability superfluous. Thus, instead to say "X is A-movable in M" we can say shortly that X is A-movable.

(13) COROLLARY. *A-movability is a hereditable shape-property, that is if X is A-movable and $\mathrm{Sh}(X)\geqslant \mathrm{Sh}(Y)$, then Y is A-movable.*

In [3], p. 859 a property called the *n-movability* has been introduced for every $n = 1, 2, \ldots$ Let us recall its definition:

A compactum X is said to be *n-movable* if it is homeomorphic to a subset X' of the Hilbert cube Q such that for every neighborhood U of X' in Q there exists a neighborhood U_0 of X' in Q such that each compactum

$B \subset U_0$ with $\dim B \leqslant n$ is homotopic in U to a subset of an arbitrarily given neighborhood $\hat{U}$ of X' in Q. One knows that:

(14) *n-movability is a hereditable shape-property.*

(15) *Every movable compactum is n-movable for $n = 1, 2, \ldots$*

(16) *Solenoids of Van Dantzig are not 1-movable.*

Now we prove the following

(17) THEOREM. *If X is an n-movable compactum, then X is A-movable for every compactum A with $\dim A \leqslant n$.*

Proof. Using theorem (2), we may assume that $X \subset Q$. Let U be a neighborhood of X in Q. Since X is n-movable, there exists an open neighborhood U_0 of X (in Q) such that every at most n-dimensional compactum lying in U_0 is homotopic in U to a subset of an arbitrarily given neighborhood $\hat{U}$ of X (in Q).

Consider a map $a\colon A \to U_0$. It is well known that there exists a homeomorphism $a'\colon A \to U_0$ homotopic to a in U. Then $\dim a'(A) \leqslant n$ and consequently there is a homotopy carrying the set $a'(A)$ in U onto a subset of $\hat{U}$. Hence a is homotopic in U to map with all values in $\hat{U}$ and the proof of theorem (17) is finished.

(18) COROLLARY. *If X is movable, then X is movable for every compactum A.*

(19) PROBLEM. *Let S^k denote the k-dimensional sphere. Does there exist a compactum X which is S^k-movable for $k = 1, 2, \ldots, n$, but is not n-movable?*

References

[1] K. Borsuk, *On movable compacta*, Fund. Math. 66 (1969), p. 137–146.

[2] — *On the concept of shape for metrizable spaces*, Bull. Acad. Polon. Sci. 18 (1970), p. 127–132.

[3] — *On the n-movability*, ibidem 20 (1972), p. 859–865.

[4] K. Kuperberg, *An isomorphism theorem of Hurewicz type in the Borsuk's theory of shape*, Fund. Math. 77 (1973), p. 21–32.

[5] M. Moszyńska, *The Whitehead theorem in the theory of shape*, ibidem (to appear).

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