A sharp integral inequality for compact Weingarten hypersurfaces under an Okumura type inequality

by

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Summary. Our aim in this paper is to provide a sharp integral inequality involving the norm of the traceless second fundamental form of a wide class of compact (without boundary) linear Weingarten hypersurfaces (including those with two distinct principal curvatures) immersed into a Riemannian space form. In particular, we generalize the results of Alías Meléndez (2020) when the ambient space form is the unit Euclidean sphere and give a new estimate when the space form is either the Euclidean space or the hyperbolic space. The sharpness of our integral inequality is realized by the totally umbilical spheres and, when the ambient space is the unit Euclidean sphere, by Clifford tori.

1. Introduction and statements of the main results. Let us denote by $\mathbb{M}^{n+1}_c$ the standard model of an $(n + 1)$-dimensional Riemannian space form with constant sectional curvature $c \in \{-1, 0, 1\}$, that is, $\mathbb{M}^{n+1}_c$ is either the hyperbolic space $\mathbb{H}^{n+1}$, when $c = -1$, the Euclidean space $\mathbb{R}^{n+1}$, when $c = 0$, or the unit Euclidean sphere $\mathbb{S}^{n+1}$, when $c = 1$. The study of the geometry of hypersurfaces in space forms has been of great interest in differential geometry. In particular, the investigation of hypersurfaces with constant scalar curvature has a long and interesting history, and still constitutes an active research field.

Since the ideas presented by Cheng and Yau in their seminal paper [10], much work has been done in this area. In [10] Cheng and Yau introduced a differential operator, the so-called Cheng–Yau operator, which has become
one of the most efficient tools to deal with rigidity of constant scalar curvature hypersurfaces in space forms. Among other results, Cheng and Yau proved that a compact (without boundary) hypersurface immersed in $\mathbb{M}^{n+1}_c$ having nonnegative sectional curvature and constant normalized scalar curvature $R \geq c$ must be either totally umbilical, a product of two totally umbilical constantly curved submanifolds or possibly a flat manifold which is different from the first two types. In the noncompact case, they extended their results to the ambient space $\mathbb{R}^{n+1}$.

There exists a vast literature related to the problem of establishing rigidity results in the same spirit as in [10] under various hypotheses about the geometry of such (not necessarily compact) hypersurfaces (see, for instance, [1, 2, 3, 6, 8, 9, 13, 14, 15, 19, 20] and the references therein). For instance, in a very recent work Alías and Meléndez [2] obtained a sharp integral inequality for the norm of the traceless second fundamental form of a compact (without boundary) hypersurface with constant scalar curvature and satisfying an Okumura type inequality, immersed in $\mathbb{S}^{n+1}$.

A natural and important generalization of hypersurfaces with constant scalar curvature are those in which scalar and mean curvatures are linearly related. In the recent literature, these hypersurfaces are known as linear Weingarten hypersurfaces. Regarding their geometry, in the last few years many authors have obtained rigidity results for these hypersurfaces in Riemannian space forms, extending previous results that pertained to constant scalar curvature hypersurfaces (see, for instance, [4, 5, 7, 11, 12, 16]).

Denoting by $A$ the second fundamental form and by $H$ the mean curvature function of a hypersurface $\Sigma^n$ immersed in a space form $\mathbb{M}^{n+1}_c$, let us recall that the traceless second fundamental form (or total umbilicity tensor) of $\Sigma^n$ is defined to be $\Phi = A - HI$, where $I$ stands for the identity operator on the tangent bundle of $\Sigma^n$. We will write $R$ to denote the normalized scalar curvature of $\Sigma^n$ (for more details, see Section 2).

Considering this context and assuming an Okumura type inequality for $\Phi$ originally introduced by Meléndez [17], our purpose is to extend the ideas of [2, 11] in order to prove a sharp integral inequality concerning compact (without boundary) linear Weingarten hypersurfaces immersed in $\mathbb{M}^{n+1}_c$, as stated below:

**Theorem 1.1.** Let $\Sigma^n$ be a compact linear Weingarten hypersurface immersed in a Riemannian space form $\mathbb{M}^{n+1}_c$, $c \in \{-1, 0, 1\}$, such that $R = aH + b$ for some constants $a \geq 0$ and $b \geq c$. In the case $b = c$, assume further that $H$ does not change sign on $\Sigma^n$. Suppose that the traceless second fundamental form $\Phi$ satisfies

\[
\text{tr}(\Phi^3) \geq -\frac{n - 2k}{\sqrt{nk(n - k)}}|\Phi|^3
\]
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for some \( k \) with \( 1 \leq k \leq (n - \sqrt{n})/2 \). Then, for all constants \( p > 2 \),

\[
\int_{\Sigma} |\Phi|^{p+2} Q_{a,b,n,k,c}(|\Phi|) \leq 0,
\]

where the real valued function \( Q_{a,b,n,k,c}(t) \) is given by

\[
Q_{a,b,n,k,c}(t) = -\frac{n-2}{n-1} t^2 \left( \frac{n(n-2k)}{\sqrt{nk(n-k)}} t - na \right) \sqrt{\frac{t^2}{n(n-1)} + \frac{a^2}{4} + b - c}
\]

\[
- \frac{n(n-2k)a}{2\sqrt{nk(n-k)}} t + n \left( \frac{a^2}{2} + b \right).
\]

Moreover, if in addition \( b > \max\{0,c\} \), then equality holds in (1.2) if and only if either

(i) \( \sup_{\Sigma} |\Phi| = 0 \) and \( \Sigma^n \) is a totally umbilical sphere, or

(ii) in the case \( c = 1 \),

\[
|\Phi| = \alpha_1(a,b,n,k) > 0,
\]

where \( \alpha_1(a,b,n,k) \) is a positive constant that depends only on \( a, b, n \) and \( k \) satisfying

\[
Q_{a,b,n,k,1}(\alpha_1(a,b,n,k)) = 0,
\]

and \( \Sigma^n \) is isometric to a Clifford torus \( S^k(\sqrt{1-r^2}) \times S^{n-k}(r) \subset S^{n+1} \), with

\[
r^2 = \frac{(n-1)(nR+(n-2k)) - \sqrt{\delta}}{2n(n-1)R},
\]

where \( \delta = [n(n-1)(r-1)+2k(n-k)]^2 - 4k(n-k)(k(n-k)-(n-1)) \).

In the case \( a = 0 \) and \( c = 1 \), we obtain as a consequence of Theorem 1.1 the results due to Alías and Meléndez [12, Theorems 1.2 and 4.1]. It is important to observe that in this case the function \( Q_{R,n,k,1}(t) \) in (1.3) coincides up to a negative multiplicative constant with the function \( Q_{R,k}(t) \) [2, (1.3) and (4.2)]. In particular, there is a change of sign in inequality (1.2) compared to that in [2, (1.2) and (4.1)]. When the ambient space is either the Euclidean space or the hyperbolic space, Theorem 1.1 provides a new sharp integral inequality, and in particular equality (1.2) gives a characterization of the totally umbilical spheres of these ambient spaces.

It is also worth pointing out that since \( \Phi \) is traceless, by the classical Okumura lemma [18, inequality (1.1) is automatically true when \( k = 1 \). Furthermore, when \( 1 < k < n/2 \), inequality (1.1) follows under the geometric assumption that the hypersurface has two distinct principal curvatures with multiplicities \( k \) and \( n-k \). Indeed, in the latter case \( \Phi \) also has two distinct
eigenvalues, say $\mu$ and $\nu$, with multiplicities $k$ and $n-k$, respectively. In particular, we get

$$\mu = -\frac{n-k}{k}\nu \quad \text{and} \quad |\Phi|^2 = k\mu^2 + (n-k)\nu^2,$$

which implies that

$$\text{tr}(\Phi^3) = k\mu^3 + (n-k)\nu^3 = \pm \frac{2k-n}{\sqrt{n(k-n)}} |\Phi|^3,$$

proving our claim.

Theorem 1.1 leads to new proofs of some results in [14,15]. More precisely, we get

**Corollary 1.2** ([14, Theorems 2 and 3] and [15, Corollary 2.2]). Let $\Sigma^n$ be a compact hypersurface immersed into a space form $M^{n+1}_c$, $c \in \{-1, 0, 1\}$, and having constant normalized scalar curvature $R > c$. Suppose that

$$n(R - 1) \leq |A|^2 \leq \frac{n[n(n-1)(R-c)^2 + 4(n-1)(R-c)c + nc^2]}{(n-2)(n(R-c) + 2c)}.$$

Then either $|A|^2 = n(R - c)$ and $\Sigma^n$ is a totally umbilical sphere, or

$$|A|^2 = \frac{n[n(n-1)(R-c)^2 + 4(n-1)(R-c)c + nc^2]}{(n-2)(n(R-c) + 2c)};$$

this formula holds if and only if $c = 1$ and $\Sigma^n$ is isometric to the Clifford torus

$$S^1(\sqrt{1-r^2}) \times S^{n-1}(r) \subset S^{n+1}, \quad \text{for some} \quad 0 < r < 1.$$

The proofs of Theorem 1.1 and Corollary 1.2 are given in Section 3.

**2. Set up.** Throughout this paper we will deal with a compact (without boundary), connected and oriented hypersurface $\Sigma^n$ isometrically immersed in a Riemannian space form $M^{n+1}_c$, $c \in \{-1, 0, 1\}$. Let $A : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ be its second fundamental form with respect to a globally defined unit normal vector field $N \in \mathfrak{X}^\perp(\Sigma)$ and $H = \frac{1}{n}\text{tr}(A)$ the corresponding mean curvature function of $\Sigma^n$. It is well known that, by the Gauss equation, the normalized scalar curvature $R$ of $\Sigma^n$ is given by

$$n(n-1)R = n(n-1)c + n^2H^2 - |A|^2.\quad (2.1)$$

The traceless second fundamental form of $\Sigma^n$, denoted by $\Phi$, is the tensor $\Phi : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ defined by

$$\Phi = A - HI, \quad (2.2)$$

where $I$ is the identity operator on $\mathfrak{X}(\Sigma)$. From (2.2) it is not difficult to verify that $\Phi$ is a traceless tensor, that is, $\text{tr}(\Phi) = 0$, and that

$$|\Phi|^2 = |A|^2 - nH^2. \quad (2.3)$$
From (2.1) and (2.3) we conclude that $|\Phi|$ vanishes identically on $\Sigma^n$ if and only if $\Sigma^n$ is a totally umbilical hypersurface of $\mathbb{M}_c^{n+1}$. For this reason, $\Phi$ is also called the total umbilicity tensor of $\Sigma^n$. We also note that, due to (2.1) and (2.3), the following relation is trivially satisfied:

$$(2.4) \quad n(n-1)R = n(n-1)(c + H^2) - |\Phi|^2.$$ 

Let us assume that $\Sigma^n$ is a linear Weingarten hypersurface, which means that there exist constants $a, b \in \mathbb{R}$ such that $R = aH + b$. Closely related to the geometry of this kind of hypersurfaces is a well known Cheng–Yau type operator. To be more precise, let us define the second order linear differential operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by

$$L = L - \frac{n-1}{2}a\Delta,$$

where $\Delta$ is the Laplacian on $\Sigma^n$ and $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the standard Cheng–Yau operator, which is given by

$$Lu = \text{div}_\Sigma(P\nabla u)$$

for every $u \in C^\infty(\Sigma)$. Here, $\text{div}_\Sigma$ stands for the divergence operator on $\Sigma^n$ and $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the first Newton tensor of $\Sigma^n$, that is, $P = nHHI - A$. Thus,

$$Lu = \text{div}_\Sigma(\mathcal{P}\nabla u),$$

where $\mathcal{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the tensor

$$\mathcal{P} = P - \frac{n-1}{2}aI.$$

In particular, when $b > c$ (respectively $b \geq c$) then $L$ is an elliptic (respectively semi-elliptic) operator or, equivalently, $\mathcal{P}$ is positive definite (respectively semi-definite). For the proof of this property see, for instance, [5, Proposition 3.3].

**3. Proofs of Theorem 1.1 and Corollary 1.2.** Let us observe that under our assumptions we can apply [11, Proposition 3.1] to obtain a lower bound for the operator $L$ acting on the squared norm of the traceless second fundamental form $\Phi$ as follows:

$$(n-1)|\Phi|^2 Q_{a,b,n,k,c}(\Phi) \sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + b - c} \leq \frac{1}{2}L(|\Phi|^2).$$

Here $Q_{a,b,n,k,c}(t)$ is just the real valued function given by (1.3). Equivalently,

$$(3.1) \quad |\Phi|^{p+2} Q_{a,b,n,k,c}(\Phi) \leq \frac{1}{2(n-1)}L(|\Phi|^2) \sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + b - c}$$
for every constant \( p > 2 \). Then, integrating both sides of (3.1) we obtain

\[
(3.2) \quad \int_{\Sigma} |\Phi|^{p+2} Q_{a,b,n,k,c}(|\Phi|) d\Sigma \leq \frac{1}{2} \sqrt{\frac{n}{n-1}} \int_{\Sigma} \mathcal{L}(|\Phi|^2)f(|\Phi|^2) d\Sigma,
\]

where \( f : [0, \infty) \rightarrow \mathbb{R} \) is given by

\[
f(t) = \frac{t^{p/2}}{\sqrt{t + n(n-1)(a^2/4 + b - c)}}.
\]

Let us observe that for every function \( u \in C^\infty(\Sigma) \) one has the relation

\[
f(u)\mathcal{L}(u) = \text{div}_{\Sigma}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle,
\]

because \( \mathcal{L}u = \text{div}_{\Sigma}(\mathcal{P}(\nabla u)). \) Hence, by the Stokes Theorem combined with (3.2) we find

\[
(3.3) \quad \int_{\Sigma} |\Phi|^{p+2} Q_{a,b,n,k,c}(|\Phi|) \leq -\frac{1}{2} \sqrt{\frac{n}{n-1}} \int_{\Sigma} f'(|\Phi|^2)\langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle.
\]

But, since we are assuming \( b \geq c \), \( \mathcal{P} \) is positive semi-definite, which gives

\[
\langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle \geq 0.
\]

On the other hand, since

\[
f'(t) = \frac{(p-1)t^{p/2} + n(n-1)(a^2/4 + b - c)t^{(p-2)/2}}{2(t + n(n-1)(a^2/4 + b - c))^{3/2}},
\]

it is easy to check that \( f'(t) \) is a nonnegative function for every \( p > 2 \). In particular, we prove the first part of the theorem by deducing from (3.3) that

\[
(3.5) \quad \int_{\Sigma} |\Phi|^{p+2} Q_{a,b,n,k,c}(|\Phi|) \leq 0.
\]

When equality occurs in (3.5), the same holds in (3.3), so that

\[
\int_{\Sigma} |\Phi|^{p+2} Q_{a,b,n,k,c}(|\Phi|) = \int_{\Sigma} f'(|\Phi|^2)\langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle = 0.
\]

This means that we must have

\[
(3.6) \quad f'(|\Phi|^2)\langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle = 0 \quad \text{on} \quad \Sigma^n.
\]

Supposing at this point that \( b > c \), due to positivity of \( \mathcal{P} \), we see that

\[
\langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle = 0 \text{ if and only if } \nabla|\Phi|^2 = 0.
\]

Concerning \( f'(|\Phi|^2) \), we also see from (3.4) that \( f'(|\Phi|^2) = 0 \) if and only if \( |\Phi| = 0 \), because \( p > 2 \) and \( b > c \).

Hence, (3.6) says that \( |\Phi|^2 \) must be constant, and so either \( |\Phi| = 0 \) or \( |\Phi| \) is a positive constant. Since \( \Sigma^n \) is compact, the first case means that \( \Sigma^n \) must be a totally umbilical sphere in \( \mathbb{M}^{n+1}_c \). The second case gives

\[
(3.7) \quad Q_{a,b,n,k,c}(|\Phi|) = 0.
\]
We claim that (3.7) holds only when $c = 1$. Indeed, let us reason as in [11, proof of Theorem 1.1]. It was shown there that $Q_{a, b, n, k, c}(0) > 0$ and $Q_{a, b, n, k, c}(t)$ is strictly decreasing for $t \geq 0$, because our assumption $2k \leq n - \sqrt{n}$ implies that $Q'_{a, b, n, k, c}(t) < 0$ for $t > 0$ (see Remark 3.1). In particular, there exists a unique $\alpha(a, b, n, k, c) > 0$, depending only on $a$, $b$, $n$, $p$ and $c$, such that $Q_{a, b, n, k, c}(\alpha(a, b, n, k, c)) = 0$. Thus, the validity of (3.7) ensures that

$$|\Phi| = \alpha(a, b, n, k, c) > 0.$$ But from [11, Theorem 1.1] we conclude that $\Sigma^n$ must be isometric to one of the following embedded standard products:

(i) $\mathbb{H}^k(\sqrt{1 + r^2}) \times S^{n-k}(r) \subset \mathbb{H}^{n+1}$ with $r > 0$, when $c = -1$;
(ii) $\mathbb{R}^k \times S^{n-k}(r) \subset \mathbb{R}^{n+1}$ with $r > 0$, when $c = 0$;
(iii) $S^k(\sqrt{1 - r^2}) \times S^{n-k}(r) \subset S^{n+1}$ with $0 < r < 1$, when $c = 1$.

Due to compactness of $\Sigma^n$, cases (i) and (ii) cannot occur and we conclude that (3.7) holds just when $c = 1$. Moreover, the constant $\alpha_1(a, b, n, k) = \alpha(a, b, n, k, 1) = \alpha(R, n, k, 1)$ depends only on $R$, $n$ and $k$ (see Remark 3.2).

Finally, as proved in [11, Theorem 1.1], we must have

$$r^2 = \frac{(n - 1)(nR + (n - 2k)) - \sqrt{\delta}}{2n(n - 1)R},$$

where $\delta = [n(n - 1)(R - 1) + 2k(n - k)]^2 - 4k(n - k)(k(n - k) - (n - 1))$. This proves the characterization of the equality case in (1.2), completing the proof of Theorem 1.1.

Remark 3.1 ([11, Remark 3.3]). As for the claim that $Q_{a, b, n, k, c}(x)$ is strictly decreasing on $[0, \infty)$, let us just observe the following: With a straightforward computation we can verify that

$$Q'_{a, b, n, k, c}(x) = -\frac{2(n - 2)}{n - 1}x - \frac{n(n - 2k)}{\sqrt{nk(n - k)}} \sqrt{\frac{x^2}{n(n - 1)}} + \frac{a^2}{4} + b - c$$

$$- \frac{(n - 2k)x^2}{(n - 1)\sqrt{nk(n - k)}} + \frac{a}{2}x + b - c$$

$$+ \frac{ax}{(n - 1)\sqrt{\frac{x^2}{n(n - 1)}} + \frac{a^2}{4} + b - c}$$

$$- \frac{n(n - 2k)a}{2\sqrt{nk(n - k)}}$$

$$= \tilde{Q}(x) + \frac{ax}{(n - 1)\sqrt{\frac{x^2}{n(n - 1)}} + \frac{a^2}{4} + b - c} - \frac{n(n - 2k)a}{2\sqrt{nk(n - k)}},$$
where, for every \( x > 0 \),
\[
\tilde{Q}(x) = -\frac{2(n-2)}{n-1}x - \frac{n(n-2k)}{\sqrt{nk(n-k)}}\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} + b - c} - \frac{(n-2k)x^2}{(n-1)\sqrt{nk(n-k)}}\sqrt{x^2_{n(n-1)}} + \frac{a^2}{4} + b - c < 0,
\]
and
\[
\frac{ax}{(n-1)\sqrt{x^2_{n(n-1)}} + \frac{a^2}{4} + b - c} - \frac{n(n-2k)a}{2\sqrt{nk(n-k)}} \leq \frac{ax}{(n-1)\sqrt{x^2_{n(n-1)}}} - \frac{n(n-2k)a}{2\sqrt{nk(n-k)}} = \frac{\sqrt{n}a}{\sqrt{n-1}} - \frac{n(n-2k)a}{2\sqrt{nk(n-k)}} \leq 0.
\]
But since we have \( p \leq (n - \sqrt{n})/2 \), the last inequality holds, and our claim follows.

**Remark 3.2.** According to Theorem 1.1 and \([3, (1.2)]\), the constant \( \alpha(R, n, k, c) \) is given by
\[
\alpha^2(R, n, 1, c) = \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)};
\]
and, for \( 2 \leq k \leq n-1 \),
\[
\alpha^2(R, n, k, c) = \frac{(n-1)[(n-1)(R-c)(n-2k)^2 + 2k(n-k)(n-2)R]}{2n(k-1)(n-k-1)} - \frac{(n-2k)\sqrt{\Delta}}{2n(k-1)(n-k-1)},
\]
where
\[
\Delta = (n-1)^4 \left[ (nR-c(n-2k))^2 - \frac{4ncRk(k-1)}{n-1} \right].
\]
We conclude the paper by proving Corollary 1.2.

**Proof of Corollary 1.2.** It follows immediately from (2.4) that
\[
|\Phi|^2 \leq \alpha^2(R, n, 1, c).
\]
Hence, as \( Q_{a,b,n,c,1} = Q_{R,n,c} \) is strictly decreasing for \( x \geq 0 \), we infer that \( Q_{R,n,c}(|\Phi|) \geq 0 \). Therefore, one has equality in (1.2), which proves the result.■
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