ACTA ARITHMETICA
204.4 (2022)

Coincidence of L-functions

by

YUTA KATAYAMA and MASANARI KIDA (Tokyo)

1. Introduction. Let K be a Galois extension over the rational number field $\mathbb{Q}$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Let χ be an irreducible character of G afforded by a complex linear representation $\rho : G \to \mathrm{GL}(V)$ of G. For the character χ (or the representation ρ), we can define the Artin L-function $L(s, \chi)$. By the *coincidence of L-functions*, we mean that there exist different subfields F_1, F_2 of K and ray class characters ψ_1, ψ_2 of F_1 and F_2, respectively, and the equality among Hecke L-functions and the Artin L-function

$$L(s, \psi_1) = L(s, \chi) = L(s, \psi_2)$$

holds up to a finite number of Euler factors. Such a coincidence was first observed by Hecke [6, 7] for the case where $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}] = 2$. If F_1 is an imaginary quadratic field and F_2 is a real quadratic field, then this causes an interaction of the arithmetic of abelian extensions of imaginary and real quadratic fields. In fact, Hecke proved the coincidence of indefinite binary theta functions and definite binary theta functions. As another example, the decomposition law in $K/\mathbb{Q}$ is described by three different quadratic forms with distinct discriminants (see [8] and [17, Section 5]). This also leads to an application in [9] on simultaneous representations of primes by different quadratic forms.

Furthermore, Shintani [22] proved his own conjecture on units of a certain abelian extension of a real quadratic field in the case where the coincidence of such L-functions occurs. He used elliptic units in a ray class field of an imaginary quadratic field to prove the existence of the so-called Shintani–Stark units in the corresponding ray class field of a real quadratic field. Here we again encounter an interaction between real and imaginary quadratic fields.

2020 *Mathematics Subject Classification*: Primary 11R42; Secondary 11R32, 11R21, 12F12.
Key words and phrases: Artin L-function, Hecke L-function, isoclinism, Heisenberg group, Shintani–Stark unit.
Received 12 October 2021; revised 6 June 2022.
Published online 8 August 2022.

DOI: 10.4064/aa211012-14-6

Ishii [11] further studied the coincidence in a more general setting and obtained a sufficient condition in terms of the projective image of the representation under the assumption that $F_i/\mathbb{Q}$ is a cyclic extension.

The aim of this paper is to give a more precise condition for the coincidence in terms of the Galois group under a slightly stronger condition than Ishii's: his condition is that $F_i/\mathbb{Q}$ is a cyclic extension of prime degree (see Remark 3.1). The main theorem of this paper is the following.

THEOREM 1.1. *Let p be a prime number. Let $K/\mathbb{Q}$ be a Galois extension with the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Assume that there is an abelian normal subgroup H of G such that $[G : H] = p$ and that there exists a linear character ψ of H whose induction ψ^G to G is an irreducible character of G. Then the following conditions are equivalent:*

(i) *There exist an abelian normal subgroup H_1 of G which is different from H and a character $\xi \in \mathrm{Irr}(H_1)$ such that $\xi^G = \psi^G$.*
(ii) *The Galois group G is isoclinic to the Heisenberg group He_p over the finite field $\mathbb{F}_p$ of p elements.*

For the definitions of isoclinism and the Heisenberg group He_p, see Section 2.

From the proof of Theorem 1.1 and the well-known properties of Artin L-functions on induced character, we have the following coincidence of L-functions.

COROLLARY 1.2. *Let p be a prime number. Let $K/\mathbb{Q}$ be a Galois extension with the Galois group G isoclinic to the Heisenberg group He_p. For each irreducible character ρ of G of degree p, there exist $p+1$ subfields F_i $(i = 1, \ldots, p+1)$ of K which are cyclic extensions of $\mathbb{Q}$ of degree p and a ray class character ξ_i of a certain ray class field of each F_i such that*

$$L(s, \xi_i) = L(s, \rho)$$

holds for $i = 1, \ldots, p+1$ up to a finite number of Euler factors.

The difference between Euler factors of $L(s, \rho)$ and $L(s, \xi_i)$ occurs at the ramifying primes in $F_i/\mathbb{Q}$ (see [21, VII, (10.6)]). To complement the missing factors, we require a careful computation of the image of the Artin map at ramifying primes (see [16, Lemma 6.10] for example).

Although we restrict ourselves to the case where the base field is $\mathbb{Q}$ for simplicity, Theorem 1.1 remains valid if we replace $\mathbb{Q}$ by any number fields. Conversely, as was pointed out in [11] (and proved in [18]), the coincidence of L-functions implies (i) in Theorem 1.1 if the base field is the field of rational numbers and if the characters are faithful.

For the case $p = 2$, one implication of Theorem 1.1 is essentially proved by Kida and Namura [17, Proposition 3.3] and the other direction is proved by Kani [13, Proposition 11]. Therefore our theorem generalizes both results.

As is stated above, Shintani used the coincidence of L-functions to prove the Shintani–Stark conjecture under certain conditions. This paper, in fact, grew out of the effort to understand his assumptions in [22]. We reformulate Shintani's theorem as follows by analyzing Shintani's assumptions.

THEOREM 1.3. *Let $K/\mathbb{Q}$ be a Galois extension with Galois group G. Assume that G is isoclinic to D_4 and that K is an imaginary non-CM field. Then there is an abelian normal subgroup H of G of index 2 containing the conjugacy class C of the complex conjugation.*

Let F be the fixed subfield of K by H and consider F as a subfield of $\mathbb{R}$. If we choose $s \in C$ so that $K^{\langle s\rangle}/F$ is unramified at the chosen infinite place, then the Shintani–Stark unit exists in $K^{\langle s\rangle}$.

For the definition of the Shintani–Stark unit, see Theorem 4.2.

We here note that the dihedral group D_4 of order 8 is isomorphic to the Heisenberg group He_2. Therefore, if the conditions in Theorem 1.3 are satisfied, then the assumptions in Theorem 1.1 indeed hold and the coincidence of L-functions follows.

Throughout this paper, we use the following notation. For a finite group G, we denote by $Z(G)$ the center of G and by G' the commutator subgroup of G. A *G-extension* is a Galois extension over $\mathbb{Q}$ whose Galois group is isomorphic to G. Standard finite group names such as C_n, D_n (of order $2n$) will be used. All representations appearing in this paper are complex linear representations. We denote by $\mathrm{Irr}(G)$ the set of the irreducible characters of G. We divide $\mathrm{Irr}(G)$ into the disjoint subsets

$$\mathrm{Irr}(G)_i = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = i\}.$$

Let H be a subgroup of G. For $\chi \in \mathrm{Irr}(G)$, we denote by χ_H the restriction of χ to H, and also for $\xi \in \mathrm{Irr}(H)$, we denote by ξ^G the induction of ξ to G.

The outline of the paper is as follows. In Section 2 we give some group-theoretic preliminaries. In Section 3 we give the proof of Theorem 1.1. We study Shintani's assumptions and prove Theorem 1.3 in Section 4. We also explain its connection to modular forms of weight 1.

2. Preliminaries

2.1. Isoclinism. The notion of isoclinism on finite groups was introduced by P. Hall [5]; it is a weaker equivalence than isomorphism. The definition is as follows.

DEFINITION 2.1. Two finite groups G_1 and G_2 are *isoclinic* if there exist isomorphisms $\varphi : G_1/Z(G_1) \xrightarrow{\sim} G_2/Z(G_2)$ and $\psi : G_1' \xrightarrow{\sim} G_2'$ such that the following diagram is commutative:

$$\begin{tikzcd} G_1/Z(G_1) \times G_1/Z(G_1) \arrow[r, "k_{G_1}"] \arrow[d, "\varphi\times\varphi"'] & G_1' \arrow[d, "\psi"] \\ G_2/Z(G_2) \times G_2/Z(G_2) \arrow[r, "k_{G_2}"] & G_2' \end{tikzcd}$$

where k_{G_1} and k_{G_2} are the commutator maps. If G_1 and G_2 are isoclinic, then we write $G_1 \sim G_2$ and we call the pair (φ, ψ) an *isoclinism.*

The orders of G' and $G/Z(G)$ are obviously invariants of the isoclinism class. Moreover, if G_1 or G_2 is nilpotent, then both of them are nilpotent and the nilpotency class is also an invariant of the class [5, Section 3].

Isoclinism is closely related to central extensions of finite groups. Let G be a central extension of Q by Z:

$$1 \to Z \to G \to Q \to 1 \quad \text{(exact)}. \tag{2.1}$$

Let $M(Q) = \mathrm{H}^2(Q, \mathbb{C}^\times)$ be the Schur multiplier of Q. By [23, (1.7)], two extension groups in (2.1) are isoclinic if and only if they correspond to a same subgroup of $M(Q)$. When the central extension G in (2.1) satisfies $|M(Q)| = |Z \cap G'|$, then it is called a *Schur cover* of Q, which is determined up to isoclinism.

For more information on isoclinism, we refer to [1].

2.2. Heisenberg groups. The Heisenberg group He_p over $\mathbb{F}_p$ is a unitriangular matrix group over $\mathbb{F}_p$ defined by

$$\mathrm{He}_p = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{F}_p \right\}.$$

The order of the group is p^3. We see that

$$Z(\mathrm{He}_p) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; z \in \mathbb{F}_p \right\}, \quad \text{and} \quad (\mathrm{He}_p)' = Z(\mathrm{He}_p).$$

Moreover, we have $\mathrm{He}_p / Z(\mathrm{He}_p) \cong C_p \times C_p$.

Non-abelian groups of order p^3 with odd prime p are isomorphic to He_p or $C_{p^2} \rtimes C_p$ [4, Section 5.5]. They both are Schur covers of $C_p \times C_p$ and hence are isoclinic. If $p = 2$, then the non-abelian groups of order 8 are D_4 and the quaternion group Q_8 and we can show $\mathrm{He}_2 \cong D_4$.

We also need some facts on the irreducible representations of He_p. Since there exists a normal abelian subgroup of index p in He_p, it follows from [10, Theorem (6.15)] that the character degrees are 1 and p. Since the maximal abelian quotient $\mathrm{He}_p/(\mathrm{He}_p)'$ is isomorphic to $C_p \times C_p$, we have $|\mathrm{Irr}(\mathrm{He}_p)_1| = p^2$ and thus $|\mathrm{Irr}(\mathrm{He}_p)_p| = p-1$. Also the group is nilpotent and hence is monomial by [10, Corollary (6.14)]. Therefore, any $\chi \in \mathrm{Irr}(\mathrm{He}_p)_p$ is induced from a linear character ψ of order p of a subgroup of index p. While we have $\chi(z) = p\psi_Z(z)$ for a central z, the character χ vanishes outside the center by the definition of the induced character.

These properties of the representations of He_p are inherited by those of isoclinic groups (see [1, III.5]).

3. Coincidence of L-functions. In this section, we shall prove Theorem 1.1.

We first recall the notation and the assumptions of Theorem 1.1. The Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ has an abelian normal subgroup H of index p a prime. We assume that there exists $\psi \in \mathrm{Irr}(H)_1$ such that the induction ψ^G is irreducible.

Proof of Theorem 1.1. First we assume (i). Namely, there exist a normal subgroup H_1 of G of index p different from H and $\xi \in \mathrm{Irr}(H_1)_1$ such that $\xi^G = \psi^G$. We have an exact sequence

$$1 \longrightarrow H \cap H_1 \longrightarrow G \longrightarrow G/(H \cap H_1) \longrightarrow 1. \tag{3.1}$$

By assumption, we have $HH_1 = G$ and thus $G/(H \cap H_1)$ is isomorphic to $C_p \times C_p$.

Since H and H_1 are normal in G, we see that $\psi^G(x) = 0$ for $x \in G - H$ and that $\xi^G(x) = 0$ for $x \in G - H_1$. If $x \in Z(G)$, then $\psi^G(x) \neq 0$ and $\xi^G(x) \neq 0$, hence $Z(G) \subset H \cap H_1$. Also, since H and H_1 are abelian and $G = HH_1$, if $x \in H \cap H_1$, then x commutes with every element in G, that is, $x \in Z(G)$. Therefore we conclude $Z(G) = H \cap H_1$. By (3.1), the group G is a central extension of $C_p \times C_p$. By [14, Corollary 2.2.12], the Schur multiplier of $C_p \times C_p$ is isomorphic to C_p. Hence the Schur cover is a non-abelian group of order p^3. Thus G is isoclinic to a Schur cover of $C_p \times C_p$. It is known that if $p = 2$, then the Schur cover is isomorphic to $D_4 \cong \mathrm{He}_2$ or Q_8, and if $p > 2$, then it is isomorphic to He_p or $C_{p^2} \rtimes C_p$. In either case, it is isoclinic to the Heisenberg group He_p. This shows the first half of Theorem 1.1.

Conversely, we assume (ii). From Section 2.2, we have an exact sequence

$$1 \to Z(G) \to G \to C_p \times C_p \to 1.$$

Let H_i $(i = 1, \dots, p+1)$ be the subgroups of G of index p containing $Z(G)$. Since $H_i/Z(G) \cong C_p$, if we denote a generator of the quotient group by $hZ(G)$ with $h \in H_i$, then every element of H_i can be written as $h^j z$ with $j \in \mathbb{Z}$, $z \in Z(G)$. From this it is easy to see that the commutator subgroup of H_i is trivial and hence the group H_i is abelian. Recall that the character degrees are 1 and p as in Section 2.2. Since He_p is monomial, for every

$\chi \in \mathrm{Irr}(G)_p$, there exist an abelian normal subgroup H of G of index p and $\psi \in \mathrm{Irr}(H)_1$ such that $\psi^G = \chi$. Since H is normal in G, we have $\psi^G(x) = 0$ for $x \in G - H$ and we obtain $Z(G) \subset H$ as before. This implies that $H = H_i$ for some $i \in \{1, \dots, p+1\}$.

Now we shall show a stronger claim: for every $j \neq i$, there exists $\psi_j \in \mathrm{Irr}(H_j)$ such that $\chi = \psi_j^G$. Let $K_{ij} = H_i \cap H_j$. We then obtain

$$\chi_{H_j} = (\psi^G)_{H_j} = (\psi_{K_{ij}})^{H_j}$$

by [10, Problem (5.2)]. Since the degree of ψ is 1, the restriction $\psi_{K_{ij}}$ is an irreducible character of K_{ij}. Since every irreducible character of H_j is of degree 1, we have a decomposition of $(\psi_{K_{ij}})^{H_j}$ into irreducible constituents ψ_j^τ:

$$(\psi_{K_{ij}})^{H_j} = \sum_{\tau \in G/T} \psi_j^\tau,$$

where ψ_j^τ is the conjugate of ψ_j defined by $\psi_j^\tau(h) = \psi_j(\tau h \tau^{-1})$ and $T = \{\sigma \in G \mid \psi_j^\sigma = \psi_j\}$ is the inertia group. Hence we conclude $T = H_j$. By [10, Theorem (6.11)], we conclude $(\psi_j^\tau)^G = \chi$ as claimed. This completes the proof of Theorem 1.1. ∎

If the conditions of Theorem 1.1 are satisfied, then $G/Z(G)$ is abelian and this implies $G' \subset Z(G)$. We thus see that G is a nilpotent group of nilpotency class 2. It is known that G is monomial, namely, every irreducible representation of G is an induced representation from a linear character.

As the proof above indicates, if the conditions of Theorem 1.1 hold, then $p+1$ L-functions of $p+1$ different C_p-extensions coincide as stated in Corollary 1.2.

Remark 3.1. In his paper [11], Ishii assumed that the induced character is faithful and that G/H is cyclic and then proved that if $G/Z(G)$ is abelian, then the coincidence occurs. Although our assumption on G/H is stronger, we can deduce that $G/Z(G)$ is abelian and can also prove the reverse implication.

By [10, Theorem (2.32)], if ψ^G is a faithful irreducible character, then $Z(G)$ is cyclic. Suppose that G is isoclinic to He_p. If $p = 2$, then it is shown in [16, Theorem 3.5] that the converse is also true. Also for the case of general p, it is plausible that the converse holds.

Remark 3.2. Although in both of our Theorem 1.1 and Ishii's theorem, $G/Z(G)$ is abelian in conclusion, there are several non-abelian groups $G/Z(G)$ for which the coincidence occurs. The smallest example is the case $[G : H] = 4$ and $G/Z(G) \cong C_2^2 \rtimes C_4$ where C_4 acts via C_4/C_2. An example of such a G is $C_2^3 \rtimes C_4$ where C_4 acts faithfully. In this case, an irreducible character of G of degree 4 is induced from linear characters of five abelian subgroups of index 4.

Here we give a list of finite groups of small order isoclinic to He_p. The groups are indicated by GAP ids for small groups.

p	Groups
2	$\underline{(8,3)}, (8,4), \underline{(16,3)}, (16,4), \underline{(16,6)}, \underline{(16,11)}, (16,12), \underline{(16,13)}, \underline{(24,10)},$ $(24,11), (32,2), (32,4), \underline{(32,5)}, (32,12), \underline{(32,17)}, \underline{(32,22)}, (32,23),$ $\underline{(32,24)}, \underline{(32,25)}, (32,26), \underline{(32,37)}, \underline{(32,38)}, \underline{(32,46)}, (32,47), \underline{(32,48)},$ $\underline{(40,10)}, (40,11), \underline{(48,21)}, (48,22), \underline{(48,24)}, \underline{(48,45)}, (48,46), \underline{(48,47)}$
3	$(27,3), (27,4), (54,10), (54,11), (81,3), (81,4), (81,6), (81,12), (81,13),$ $(81,14), (108,13), (108,14), (108,30), (108,31), (135,3), (135,4), (162,24),$ $(162,25), (162,27), (162,48), (162,49), (162,50), (189,10), (189,11)$
5	$(125,3), (125,4), (250,10), (250,11), (375,4), (375,5),$ $(500,13), (500,14), (500,35), (500,36)$

The entries for $p = 2$ with underline are groups related to the result in Section 4. See Lemma 4.4 below and the remark after its proof.

In the following, we shall give an explicit example of the coincidence.

Although the construction of He_p-extensions is discussed in [12, Section 6.6] in the context of generic polynomials, we give a simpler method over $\mathbb{Q}$. Our construction results in tamely ramified extensions and this helps us to compute the conductors of the ray class groups (see Example 3.4 below).

Proposition 3.3. *Let p be an odd prime and q, ℓ distinct primes congruent to* 1 *modulo p. If there exists a C_p-extension $F/\mathbb{Q}$ exactly ramified at q and ℓ whose class group is isomorphic to $C_p \times C_p$, then the Galois group of the Hilbert class field $\widetilde{F}$ of F over $\mathbb{Q}$ is isomorphic to* He_p*:*

$$\mathrm{Gal}(\widetilde{F}/\mathbb{Q}) \cong \mathrm{He}_p\,.$$

Proof. We start with an elementary general remark. If a Galois extension K/k of degree p^2 has an intermediate field k' such that K/k' is an unramified extension and k'/k is a ramified extension, then $\mathrm{Gal}(K/k)$ is isomorphic to $C_p \times C_p$. This follows from the fact that the inertia field of a prime ideal of K lying above a ramifying prime in k'/k does not coincide with k'.

Let F_q (resp. F_ℓ) be the unique C_p-extension over $\mathbb{Q}$ inside the qth (resp. ℓth) cyclotomic field. By the ramification property, the field F is contained in F_qF_ℓ. Also, since F_qF_ℓ is unramifed over F, the field $\widetilde{F}$ contains the composite field. By the above remark, the extensions $\widetilde{F}/F_q$ and $\widetilde{F}/F_\ell$ are $C_p \times C_p$-extensions.

Let M be an intermediate field of $\widetilde{F}$ and F. If it is an abelian extension over the rationals, then $M/\mathbb{Q}$ is unramified outside q and ℓ, and M coincides with F_qF_ℓ. Since there are $p+1$ intermediate fields between $\widetilde{F}$ and F, there must be such a field that is non-normal over $\mathbb{Q}$. This implies that $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$ is not an abelian group.

It remains to show that $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$ is not isomorphic to $C_{p^2} \rtimes C_p$, which has a presentation

$$\langle a, b \mid a^{p^2} = b^p = 1,\, bab^{-1} = a^{p+1}\rangle.$$

Suppose to the contrary that it is isomorphic to $C_{p^2} \rtimes C_p$. The field $\widetilde{F}^{\langle a^p, b\rangle}$ is the only C_p-extension over $\mathbb{Q}$ over which $\widetilde{F}$ is a $C_p \times C_p$-extension. On the other hand, our $\widetilde{F}/\mathbb{Q}$ contains at least two such subextensions $\widetilde{F}/F_q$ and $\widetilde{F}/F_\ell$, as we have seen above. This is a contradiction. ∎

We use Magma [2] to compute the class groups of cubic cyclic extensions and obtain the following explicit example of coincidence of L-functions.

Example 3.4. For $p = 3$, we have the following examples of pairs of (q, ℓ) satisfying the assumptions of Proposition 3.3:

$$(q, \ell) = (7, 181), (13, 103), (7, 223), (7, 337), (19, 151), (7, 421), (13, 229).$$

Let us consider the case $(q, \ell) = (7, 181)$. We have a cyclic cubic field F defined by $f(x) = x^3 - x^2 - 422x - 3191$. The discriminant of the polynomial is $7^2 \cdot 181^2$ and the class group Cl_F of F is isomorphic to $C_3 \times C_3$. Therefore the field F satisfies the assumptions of Proposition 3.3. The Hilbert class field $\widetilde{F}$ is given as the splitting field of

$$x^3 + (1044\alpha^2 - 12279\alpha - 309714)x + 19026\alpha^2 - 226583\alpha - 5561753$$

with $f(\alpha) = 0$ and we have $\mathrm{Gal}(\widetilde{F}/F) \cong \mathrm{He}_3$. The four cubic cyclic subfields k_i $(i = 1, 2, 3, 4)$ of $\widetilde{F}$ are $k_1 = F$ and those defined by

$$k_2 :\ x^3 + x^2 - 422x - 3144, \quad k_3 : x^3 - x^2 - 60x + 67, \quad k_4 :\ x^3 - x^2 - 2x + 1.$$

While the fields k_1 and k_2 are subfields of the cyclotomic field $Z_{7\cdot 181}$, we have $k_3 \subset Z_{181}$ and $k_4 \subset Z_7$. Let α_i $(i = 1, 2, 3, 4)$ be a root of each of the above defining polynomials, which is a primitive element of k_i. Since $\widetilde{F}$ is the Hilbert class field of k_1, the conductors of $\widetilde{F}/k_1$ and $\widetilde{F}/k_2$ are both trivial. The conductor of $\widetilde{F}/k_3$ is (7) and that of $\widetilde{F}/k_4$ is (181). We denote by $\mathrm{Cl}_k(\mathfrak{f})$ the ray class group modulo $\mathfrak{f}$ of k. When $\mathfrak{f} = 1$, we drop it from the notation. We compute

$$\begin{aligned} &\mathrm{Cl}_{k_1} \cong C_3 \times C_3, &&\mathrm{Cl}_{k_2} \cong C_6 \times C_6,\\ &\mathrm{Cl}_{k_3}(7) \cong C_3 \times C_3, &&\mathrm{Cl}_{k_4}(181) \cong C_2 \times C_6 \times C_{90}. \end{aligned}$$

There are the surjections induced from the Artin maps

$$\mathrm{Ar}_i : \mathrm{Cl}_{k_i}(\mathfrak{f}) \to \mathrm{Gal}(\widetilde{F}/k_i).$$

Let Ker_i be the kernel of Ar_i. By the results in Section 2.2, there are two irreducible characters of degree 3 of $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$, which are induced from every $\mathrm{Gal}(\widetilde{F}/k_i)$, and they are determined by the values on the center $Z(\mathrm{Gal}(\widetilde{F}/\mathbb{Q})) = \mathrm{Gal}(\widetilde{F}/k_1k_2)$. Hence the ideal classes corresponding to the center $\mathrm{Gal}(\widetilde{F}/k_1k_2)$ by Ar_i are important for our purpose and we call them the *central classes*. By computing $\mathrm{Ar}_1([(7,\alpha_1+2)])$, we find that it generates the center of $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$. Here we denote by $[\mathfrak{a}]$ the ray class containing an ideal $\mathfrak{a}$. We can also compute

$$\begin{aligned}\mathrm{Ar}_1([(7,\alpha_1+2)]) &= \mathrm{Ar}_2\left(\left[4, \frac{-\alpha_2^2+7\alpha_2-6}{6}\right]\right)\\ &= \mathrm{Ar}_3([(-22\alpha_3^2-145\alpha_3+177)])\\ &= \mathrm{Ar}_4([19\alpha_4^2-79\alpha_4+27]).\end{aligned}$$

If we denote by c_i $(i=1,\dots,4)$ the above ideal classes sent by the Artin map, then the characters ψ_i of the ray class groups $\mathrm{Cl}_{k_i}(\mathfrak{f}_i)$ such that the value $\psi_1(c_1)=\psi_2(c_2)=\psi_3(c_3)=\psi_4(c_4)$ is one of the primitive third roots of unity induce an irreducible representation of $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$ and we have a coincidence of L-functions of the cubic subfields

$$L(s,\psi_1)=L(s,\psi_2)=L(s,\psi_3)=L(s,\psi_4)$$

by Theorem 1.1. Let ω be a primitive third root of unity. The Artin L-functions of the irreducible 3-dimensional representations of $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$ coincide with $L(s,\psi_i^{\mathrm{Gal}(\widetilde{F}/\mathbb{Q})})$ or its complex conjugation and one is given by

$$\frac{1}{1^s}+\frac{\omega}{7^s}+\frac{1}{8^s}+\frac{1}{27^s}+\frac{3\omega^2}{29^s}+\frac{\omega^2}{49^s}+\frac{\omega}{56^s}+\frac{1}{64^s}+\frac{3}{71^s}+\cdots.$$

To end this section, we state without proof the following proposition concerning a construction of $C_p^2\rtimes C_p$-extensions over $\mathbb{Q}$. We recall that $C_p^2\rtimes C_p$ is another Schur cover of $C_p\times C_p$ causing the coincidence of L-functions.

PROPOSITION 3.5. *Let p be an odd prime and q,ℓ distinct primes satisfying $q\equiv 1 \pmod p$ and $\ell\equiv 1 \pmod{2p}$ and $\ell\not\equiv 1 \pmod{p^2}$. If there exists a C_p-extension $F/\mathbb{Q}$ exactly ramified at q whose ray class group modulo ℓ is a cyclic group of order divisible by p^2, then the subfield $\widetilde{F}$ of the ray class field of F modulo ℓ of degree p^3 satisfies*

$$\mathrm{Gal}(\widetilde{F}/\mathbb{Q})\cong C_{p^2}\rtimes C_p.$$

Several examples of pairs (q,ℓ) for $p=3$ satisfying the conditions of Proposition 3.5 are the following:

$$(q,\ell)=(7,13),(7,43),(7,97),(13,31),(13,73),(13,79).$$

When $(q,\ell)=(7,13)$, a defining polynomial of the $C_9\rtimes C_3$-extension is given by

$$x^9-6x^8+2x^7+28x^6-28x^5-28x^4+42x^3-10x^2-3x+1.$$

4. On Shintani's assumptions. In this section, we study Shintani's assumptions in his paper [22] and prove Theorem 1.3.

We use the following notation throughout this section. For a real quadratic field F, we fix an embedding into $\mathbb{R}$ and call the chosen infinite place ∞_1 and the other infinite place ∞_2. We denote by $\mathscr{O}_F$ the ring of integers of F and by $\mathscr{O}_F^\times$ its unit group. The subgroup of totally real units is denoted by $\mathscr{O}_{F,+}^\times$. For any element u of F, we write u' for the conjugate of u. For a modulus $\mathfrak{f}$ of F, we denote by $\mathfrak{f}_0$ the finite part of $\mathfrak{f}$ and by $\mathrm{Cl}_F(\mathfrak{f})$ the ray class group.

To state Shintani's theorem [22, Theorem 2] in an almost original form, we need the following lemma.

Lemma 4.1. *Let $\mathfrak{f} = \mathfrak{f}_0\infty_1\infty_2$ be a modulus of a real quadratic field F. We assume that $\mathfrak{f}_0$ is invariant under the action of* $\mathrm{Gal}(F/\mathbb{Q})$. *Let μ and ν be algebraic integers satisfying the following conditions:*

(4.1) $\mu \in \mathscr{O}_F, \quad \mu < 0,\ \mu' > 0, \quad \mu - 1 \in \mathfrak{f}_0,$

(4.2) $\nu \in \mathscr{O}_{F,+}, \quad \nu + 1 \in \mathfrak{f}_0.$

The assumptions (0-3) *and* (0-6) *in* [22] *can be reformulated as follows:*

(i) *The assumption* (0-6) *that*

$$\text{there is no unit } u \in \mathscr{O}_F^\times \text{ satisfying } u > 0,\ u' < 0, \text{ and } u - 1 \in \mathfrak{f}_0$$

is equivalent to the class of (μ') *in* $\mathrm{Cl}_F(\mathfrak{f})$ *being non-trivial.*

(ii) *The assumption* (0-3) *that*

$$\text{for all } u \in \mathscr{O}_{F,+}^\times, \text{ we have } u + 1 \notin \mathfrak{f}_0$$

is equivalent to the class of (ν) *in* $\mathrm{Cl}_F(\mathfrak{f})$ *being non-trivial.*

Proof. By [3, Proposition 3.2.3], we have the exact sequence

$$1 \to \mathscr{O}_{F,\mathfrak{f}}^\times \to \mathscr{O}_F^\times \xrightarrow{\rho} (\mathscr{O}_F/\mathfrak{f}_0)^\times \times \{\pm 1\}^2 \xrightarrow{\Psi} \mathrm{Cl}_F(\mathfrak{f}) \to \mathrm{Cl}_F \to 1,$$

where $\mathscr{O}_{F,\mathfrak{f}}^\times = \{u \in \mathscr{O}_{F,+}^\times \mid u \equiv 1 \pmod{\mathfrak{f}_0}\}$ and $\rho(u) = (u+\mathfrak{f}_0, \mathrm{sgn}(u), \mathrm{sgn}(u'))$ and Ψ maps $\rho(u)$ to the class of (u).

(i) A preimage of the class of (μ') by Ψ is $(1, 1, -1)$ by (4.1). On the other hand, the assumption (0-6) means that there is no unit in F sent to $(1, 1, -1)$ by ρ. Thus the equivalence follows.

(ii) A preimage of the class of (ν) by Ψ is $(-1, 1, 1)$ by (4.2). Since (0-3) means that no unit is sent to $(-1, 1, 1)$ by ρ, the equivalence holds. ∎

We can now state Shintani's theorem.

Theorem 4.2 (Shintani). *Let F be a real quadratic field. Let $\mathfrak{f} = \mathfrak{f}_0\infty_1\infty_2$ be a modulus of F. Assume that $\mathfrak{f}$ satisfies* (0-3) *and* (0-6) *in Lemma* 4.1 *and also that $\mathfrak{f}_0$ is invariant under the action of* $\mathrm{Gal}(F/\mathbb{Q})$. *Choose $\mu \in \mathscr{O}_F$ and $\nu \in \mathscr{O}_{F,+}$ satisfying* (4.1) *and* (4.2) *in Lemma* 4.1. *Let $\widetilde{F}(\mathfrak{f})$ be the ray class field modulo $\mathfrak{f}$, and $\widetilde{G}$ its Galois group over $\mathbb{Q}$. Let* $\mathrm{Ar} : \mathrm{Cl}_F(\mathfrak{f}) \to \widetilde{G}$ *be the*

Artin map. Let G_1 be a normal subgroup of $\widetilde{G}$ fixing F such that neither $\mathrm{Ar}(\mu)$ *nor* $\mathrm{Ar}(\nu)$ *are contained in G_1. Let $K = \widetilde{F}(\mathfrak{f})^{G_1}$ and $L = K^{\langle \mathrm{Ar}(\mu)\rangle}$. Assume further that*

(0-9) *the maximal abelian subfield L^* of L over $\mathbb{Q}$ satisfies $[L : L^*] = 2$ and L/F is unramified at ∞_1 and is ramified at ∞_2.*

If we define, for $\mathfrak{c} \in \mathrm{Cl}_F(\mathfrak{f})$,

$$X_{\mathfrak{f}}(\mathfrak{c}) = \exp\bigl({\zeta_F}'(0, \mathfrak{c}) - {\zeta_F}'(0, \mathfrak{c}(\nu))\bigr)$$

with the partial zeta function $\zeta_F(s, \mathfrak{c})$, then a power of

$$\overline{X_{\mathfrak{f}}(\mathfrak{c})} := \prod_{\tau \in \mathrm{Gal}(\widetilde{F}(\mathfrak{f})/L)} X_{\mathfrak{f}}(\mathrm{Ar}^{-1}(\tau)\mathfrak{c})$$

is a unit in L.

We believe that the statement of Theorem 1.3 is much easier to understand than that of the above theorem.

The unit $\overline{X_{\mathfrak{f}}(\mathfrak{c})}$ is the *Shintani–Stark unit* of Theorem 1.3. In [22], Shintani expressed the invariant $X_{\mathfrak{f}}(\mathfrak{c})$ in terms of a product of double sine functions.

We remark here that by the preceding lemma, neither $\mathrm{Ar}(\mu)$ nor $\mathrm{Ar}(\nu)$ is trivial.

To show the equivalence of the theorems we need the lemmas below.

Lemma 4.3. *Let G be a non-abelian finite group. Assume that there is an abelian normal subgroup H of index 2 in G. Then the following conditions are equivalent:*

(i) *The order of the commutator subgroup G' of G is 2.*
(ii) *The group G is isoclinic to D_4.*

Proof. Since the commutator subgroups of isoclinic groups are isomorphic (see Definition 2.1), if G is isoclinic to D_4, then G' is isomorphic to D_4', which is a cyclic group of order 2.

We shall show the converse. Since H is an abelian subgroup of G of index 2, it follows from [10, Theorem (6.15)] that $\chi(1)$ divides 2 for all $\chi \in \mathrm{Irr}(G)$. Since G is not an abelian group, we have $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\} = \{1, 2\}$. Hence G' is contained in $Z(G)$ by [10, Problem (5.14a)]. Then from [10, Problem (2.13)], we have $|G/Z(G)| = \chi(1)^2 = 4$ for any non-linear character $\chi \in \mathrm{Irr}(G)$. If $G/Z(G) \cong C_4$, then G is abelian, a contradiction. Thus $G/Z(G) \cong C_2 \times C_2$. Therefore it follows from [17, Lemma 2.6] that G is isoclinic to D_4. ∎

Lemma 4.4. *Let $K/\mathbb{Q}$ be an imaginary Galois extension with Galois group G isoclinic to D_4. Assume that there exists an abelian normal subgroup H of G such that $F = K^H$ is a real quadratic field. Let $\mathfrak{f}$ be the conductor of* *the abelian extension K/F. As before, denote by* $\mathrm{Ar} : \mathrm{Cl}_F(\mathfrak{f}) \to \mathrm{Gal}(K/F)$ *the Artin map. Then the following conditions are equivalent:*

(i) *There is a conjugacy class C of G of length* 2 *and the complex conjugation of $K/\mathbb{Q}$ lies in the class C.*
(ii) *K is not a CM field.*
(iii) *For some element $\mu \in \mathscr{O}_F$ satisfying* (4.1), *we have* $\mathrm{Ar}(\mu) \neq \mathrm{Ar}(\mu')$.

Proof. Since G is isoclinic to D_4, the length of any conjugacy class of G is less than or equal to 2 by [17, Lemma 2.6]. Therefore conditions (i) and (ii) are equivalent to the assertion that complex conjugation is not contained in $Z(G)$.

We shall show that (i) implies (iii). By assumption, complex conjugation lies in a conjugacy class C of length 2. Thus there is an element s of C such that $K^{\langle s\rangle}/F$ is unramified at ∞_1. Since the conductor of $K^{\langle s\rangle}/F$ is divisible by $\mathfrak{f}_0\infty_2$, the inverse image $\mathrm{Ar}^{-1}(s)$ is a non-trivial element of the kernel of the natural homomorphism from $\mathrm{Cl}_F(\mathfrak{f})$ onto $\mathrm{Cl}_F(\mathfrak{f}_0\infty_2)$. This kernel has order 2 and is generated by the class of the principal ideal (μ), where μ satisfies condition (4.1). Hence $\mathrm{Ar}(\mu) = s$ holds. Since the length of the conjugacy class of G containing s is 2, we have proved that $\mathrm{Ar}(\mu) \neq \mathrm{Ar}(\mu')$.

Conversely, if $\mathrm{Ar}(\mu) \neq \mathrm{Ar}(\mu')$, then by (4.1), $\mathrm{Ar}(\mu)$ is contained in the conjugacy class C of complex conjugation. This shows that (iii) implies (i). ∎

Among the groups of small order isoclinic to D_4 in the table in Section 3, the groups with underline have a conjugacy class of order 2 and length 2 and potentially satisfy the equivalent conditions of Lemma 4.4.

We shall prove the equivalence of Theorems 1.3 and 4.2.

Theorem 4.2 implies Theorem 1.3. Let $\mathfrak{f}$ be the conductor of the abelian extension K/F. Since K is a totally imaginary field, $\mathfrak{f}$ is divisible by $\infty_1\infty_2$. By noting that $K/\mathbb{Q}$ is a Galois extension, K is contained in both $\widetilde{F}(\mathfrak{f})$ and $\widetilde{F}(\mathfrak{f}')$, where $\mathfrak{f}'$ is the conjugate of $\mathfrak{f}$ by $\mathrm{Gal}(F/\mathbb{Q})$. Hence $\mathfrak{f}$ is invariant under the action of $\mathrm{Gal}(F/\mathbb{Q})$. Let μ, ν be integers satisfying the conditions (4.1), (4.2) of Lemma 4.1, respectively. Then from Lemma 4.4(iii), we have $\mathrm{Ar}(\mu) \neq \mathrm{Ar}(\mu')$. Since $-\mu\mu'$ maps to $(-1, 1, 1)$ via Ψ in the proof of Lemma 4.1, we have $[(\mu\mu')] = [(\nu)]$. In particular, the classes $[(\mu')]$ and $[(\nu)]$ are non-trivial in $\mathrm{Cl}_F(\mathfrak{f})$. From Lemma 4.1, the conductor $\mathfrak{f}$ satisfies the conditions (0-3) and (0-6). Since K is not a CM field, $\mathrm{Ar}(\mu) = s$ follows from Lemma 4.4. By the assumption that G is isoclinic to D_4, we have $|G'| = 2$ and thus the extension degree of K over the maximal abelian subfield $K^{G'}$ is 2. Let $L = K^{\langle s\rangle}$. By the assumption of Theorem 1.3, L/F is ramified only at ∞_2 among the infinite places of F. On the other hand, since $L/\mathbb{Q}$ is not a Galois extension by Lemma 4.4, the maximal abelian subfield $L \cap K^{G'}$ of $L/\mathbb{Q}$

satisfies the condition $[L : L \cap K^{G'}] = 2$. This shows the assumption (0-9) in Theorem 4.2. Thus, we have shown all the assumptions in Theorem 4.2 and the result follows. ∎

Theorem 1.3 implies Theorem 4.2. Let $G = \mathrm{Gal}(K/\mathbb{Q})$, $H = \mathrm{Gal}(K/F)$. If $Z(G) \not\subset H$, then there exists an element g such that $g \in Z(G)$ and $g \notin H$. Then $G = \langle g, H\rangle$ is abelian, a contradiction. Hence we have $Z(G) \subset H$. Let $Q = G/H$. Since H is an abelian group, we can consider it as a Q-module. Since $Z(G)$ is contained in H, the Q-invariant submodule H^Q coincides with $Z(G)$. Let γ be the non-trivial element of Q and $\tilde{\gamma} \in G$ be an element which maps to γ via the natural surjection $G \to Q$. We denote the centralizer of $\tilde{\gamma}$ in G by $Z_G(\tilde{\gamma})$. From $G = H \sqcup H\tilde{\gamma}$, it follows that $G' = \{h^\gamma h^{-1} \mid h \in H\}$ by direct computation. If $\mathrm{Ar}(\mu) \in G'$, then there exists $h \in H$ such that $\mathrm{Ar}(\mu) = h^\gamma h^{-1}$ and we have

$$\mathrm{Ar}(\mu)^\gamma = (h^\gamma h^{-1})^\gamma = (h^\gamma h^{-1})^{-1} = \mathrm{Ar}(\mu)^{-1} = \mathrm{Ar}(\mu).$$

This contradicts $L = K^{\langle \mathrm{Ar}(\mu)\rangle}$ not being a Galois extension over $\mathbb{Q}$. Hence we see $G' \cap \langle \mathrm{Ar}(\mu)\rangle = 1$ and conclude that $|G'| = [K\colon K^{G'}] = [L\colon L^*] = 2$ by the assumption (0-9). From Lemma 4.3, it follows that G is isoclinic to D_4. Since L/F is ramified at ∞_2, so is K/F. Because $K/\mathbb{Q}$ is a normal extension, K is a totally imaginary field. Since $\mathrm{Ar}(\mu) \notin G_1$ and $\mathrm{Ar}(\nu) \notin G_1$, they are not trivial in $\mathrm{Gal}(K/F)$. In particular, we have $\mathrm{Ar}(\mu) \neq \mathrm{Ar}(\mu')$. Lemma 4.4 implies that K is not a CM field. Thus, we have shown all the conditions in Theorem 1.3 and the result follows. ∎

Although several examples of explicit units in Theorem 4.2 are given in [22, Section 3], they almost all lie in a quartic field (i.e., $G \cong D_4$). We give an example in a larger degree.

EXAMPLE 4.5. Let M_1 be a number field adjoining a real root of $x^4 - x^3 - x^2 - x + 1$. Let K_1 be the normal closure of M_1 over $\mathbb{Q}$, which is a D_4-extension over $\mathbb{Q}$. The field K_1 contains three quadratic fields

$$F_1 = \mathbb{Q}(\sqrt{-3}), \quad F_2 = \mathbb{Q}(\sqrt{13}), \quad F_3 = \mathbb{Q}(\sqrt{-39}).$$

The extension K_1/F_3 is a C_4-extension and, in fact, K_1 is the Hilbert class field of F_3. Let $K_2/\mathbb{Q}$ be the C_4-extension defined by x^4+13x^2+13. The field K_2 contains F_2. If we set $K = K_1K_2$, then $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the group with GAP id $(16, 3)$. Since K is a composite field of a D_4-extension and an abelian extension, the Galois group G is clearly isoclinic to D_4. Also, the fact that $K_1/\mathbb{Q}$ is an imaginary non-CM field implies that so is $K/\mathbb{Q}$. Hence all the assumptions in Theorem 1.3 are satisfied. In fact, the group $(16, 3)$ has a presentation

$$G \cong \langle s_1, s_2, t \mid s_1^2 = s_2^2 = t^4 = 1,\ [s_1, s_2] = 1,\ t^{-1}s_1t = s_2,\ t^{-1}s_2t = s_1\rangle.$$

The set $\{s_1, s_2\}$ is a conjugacy class of G of length 2 and order 2. We compute the Shintani–Stark units of the subfield $M = \mathrm{Fix}(\langle s_1\rangle)$ of degree 8 defined by a real root of $x^8 + 3x^7 - 12x^5 - 25x^4 - 24x^3 + 24x + 16$. We have the following Galois correspondence:

$$\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[dr, dash] & & \\
M \arrow[d, dash] \arrow[dr, dash] & & K_1 \arrow[dl, dash] \arrow[d, dash] & \\
K_2 \arrow[dr, dash] & M_1 \arrow[d, dash] & F_1F_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
 & F_2 \arrow[dr, dash] & F_3 \arrow[d, dash] & F_1 \arrow[dl, dash] \\
 & & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, dash] \arrow[dr, dash] & & \\
\langle s_1\rangle \arrow[d, dash] \arrow[dr, dash] & & \langle t^2\rangle \arrow[dl, dash] \arrow[d, dash] & \\
\langle s_1, s_2\rangle \arrow[dr, dash] & \langle s_1, t^2\rangle \arrow[d, dash] & \langle t^2, s_1s_2\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
 & \langle s_1, s_2, t^2\rangle \arrow[dr, dash] & \langle ts_1, t^2\rangle \arrow[d, dash] & \langle t, s_1s_2\rangle \arrow[dl, dash] \\
 & & G &
\end{tikzcd}$$

The minimal polynomial of the Shintani unit in M_1 is

$$x^2 + \frac{1+\sqrt{13}}{2}x + 1$$

and that of the one in M is

$$f(x) = x^4 - \frac{97+27\sqrt{13}}{2}x^3 + \frac{873+243\sqrt{13}}{2}x^2 - \frac{97+27\sqrt{13}}{2}x + 1.$$

To solve f by radicals, we put $y = x + x^{-1}$. The minimal polynomial $g(y) = y^2 - \frac{97+27\sqrt{13}}{2}y + \frac{869+243\sqrt{13}}{2}$ of y satisfies the equation $f(x) = x^4 g(y)$. The two roots y_1, y_2 of $g(y)$ are

$$y_1 = \frac{\frac{97+27\sqrt{13}}{2} + \sqrt{\frac{5967+1647\sqrt{13}}{2}}}{2}, \qquad y_2 = \frac{\frac{97+27\sqrt{13}}{2} - \sqrt{\frac{5967+1647\sqrt{13}}{2}}}{2}.$$

To simplify the notation, for $\sigma \in \mathrm{Gal}(K/F_1) \le G$, we denote $\overline{X_{\mathfrak{f}}(\mathrm{Ar}^{-1}(\sigma))}$ by $\overline{X_{\mathfrak{f}}(\sigma)}$. Taking the numerical values of the roots into account, we have the equalities

$$\overline{X_{\mathfrak{f}}(1)} = \frac{y_1 + \sqrt{y_1^2 - 4}}{2}, \qquad \overline{X_{\mathfrak{f}}(s_2)} = \frac{y_1 - \sqrt{y_1^2 - 4}}{2},$$
$$\overline{X_{\mathfrak{f}}(t)} = \frac{y_2 + \sqrt{y_2^2 - 4}}{2}, \qquad \overline{X_{\mathfrak{f}}(s_2 t)} = \frac{y_2 - \sqrt{y_2^2 - 4}}{2}.$$

Shintani's assumptions are closely related to the modularity of Galois representation of G, which we now explain. Under the same assumption of Lemma 4.4, the field K is a non-CM imaginary field. In [16, Theorem 3.8 and

Remark 3.9(ii)], it is shown that this is also equivalent to G having an *odd* 2-dimensional irreducible representation, where "odd" means that the image of complex conjugation under the representation is $-E$. Moreover, if G has an odd faithful representation, then it is modular by the theorem of Khare and Wintenberger (see [15]). For the existence of a faithful representation, it is necessary that the center of G is cyclic, but that is not sufficient in general. If G is isoclinic to D_4, then it is also sufficient, as we noticed in Remark 3.1. Hence if G is a finite group isoclinic to D_4 with cyclic center, then the inverse Mellin transforms of these Hecke L-functions give rise to modular forms. The coincidence of L-functions in Theorem 1.1 yields corresponding equalities between three modular forms of weight 1. This observation implies the following proposition.

PROPOSITION 4.6. *If a Galois extension $K/\mathbb{Q}$ is an imaginary non-CM field whose Galois group G is isoclinic to D_4 with cyclic center, then G has an odd faithful* 2*-dimensional complex representation induced from three abelian subgroups of G, and the three corresponding modular forms of weight* 1 *coincide.*

The modular forms appearing in Proposition 4.6 are theta functions with congruence condition defined by

$$\theta([a,b,c],(x_1,y_1,d),f) = \sum_{\substack{x\equiv x_1 \pmod{d},\\ y\equiv y_1 \pmod{d}}} q^{(ax^2+bxy+cy^2)/f},$$

where the sum is taken over all integers x, y satisfying the congruence condition if the quadratic form $[a,b,c]$ is positive definite and over all positive integers satisfying the congruence condition if it is indefinite.

As we have seen in Section 1, such a coincidence of theta series was first studied by Hecke [6]. Furthermore, Köhler stated a conjecture on the coincidence of such theta series in [19, Conjecture 5.5] (see also [20]). The conjecture roughly states that if two theta functions of binary quadratic forms of distinct discriminants coincide, then there exists a binary quadratic form with a third discriminant such that the three theta series coincide. Thus our theorems also give the conditions for Köhler's conjecture to be valid.

If the conditions in Proposition 4.6 are satisfied, then K contains three quadratic fields fixed by $Z(G)$, two of which are imaginary and the third is real. Therefore we have a coincidence of an indefinite theta series and two positive definite theta series. We thus have an indefinite theta series which is an elliptic modular form of weight 1. Hecke developed the theory of indefinite theta functions in [7] in this way. Although until recently this seems to be the only resource for such indefinite theta series, it is shown in [16, Section 6] that in the isoclinism class of D_8 there are groups admitting an odd faithful representation induced only from a linear character corresponding to a real quadratic field. See Example 6.2 of that paper for an example.

Acknowledgements. This work was supported by JSPS KAKENHI Grant Number 20K03521.

References

[1] F. R. Beyl and J. Tappe, *Group Extensions, Representations, and the Schur Multiplicator*, Lecture Notes in Math. 958, Springer, Berlin, 1982.
[2] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), 235–265.
[3] H. Cohen, *Advanced Topics in Computational Number Theory*, Springer, New York, 2000.
[4] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed., Wiley, Hoboken, NJ, 2004.
[5] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. 182 (1940), 130–141.
[6] E. Hecke, *Über einen neuen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 1925, 35–44.
[7] E. Hecke, *Zur Theorie der elliptischen Modulfunktionen*, Math. Ann. 97 (1927), 210–242.
[8] T. Hiramatsu, N. Ishii, and Y. Mimura, *On indefinite modular forms of weight one*, J. Math. Soc. Japan 38 (1986), no. 1, 67–83.
[9] H. Horiba, M. Kida, and G. Koda, *Galois theoretic study on simultaneous representation of primes by binary quadratic forms*, J. Number Theory 213 (2020), 370–387.
[10] I. M. Isaacs, *Character Theory of Finite Groups*, Pure Appl. Math. 69, Academic Press, New York, 1976.
[11] H. Ishii, *On the coincidence of L-functions*, Japan. J. Math. (N.S.) 12 (1986), 37–44.
[12] C. U. Jensen, A. Ledet, and N. Yui, *Generic Polynomials*, Math. Sci. Res. Inst. Publ. 45, Cambridge Univ. Press, Cambridge, 2002.
[13] E. Kani, *Binary theta series and modular forms with complex multiplication*, Int. J. Number Theory 10 (2014), 1025–1042.
[14] G. Karpilovsky, *The Schur Multiplier*, London Math. Soc. Monogr. (N.S.) 2, Clarendon Press, New York, 1987.
[15] C. Khare, *Serre's conjecture and its consequences*, Jpn. J. Math. 5 (2010), 103–125.
[16] M. Kida and G. Koda, *Isoclinism classes of Galois groups of number fields*, Acta Arith. 191 (2019), 115–149.
[17] M. Kida and N. Namura, *On Artin L-functions of certain central extensions*, J. Number Theory 173 (2017), 147–169.
[18] J. Klüners and F. Nicolae, *Are number fields determined by Artin L-functions?*, J. Number Theory 167 (2016), 161–168.
[19] G. Köhler, *Eta Products and Theta Series Identities*, Springer Monogr. Math., Springer, Heidelberg, 2011.
[20] G. Köhler, *Eta products of weight one, and decomposition of primes in non-abelian extensions*, Math. Semesterber. 65 (2018), 183–193.

[21] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. 322, Springer, Berlin, 1999.
[22] T. Shintani, *On certain ray class invariants of real quadratic fields*, J. Math. Soc. Japan 30 (1978), no. 1, 139–167.
[23] J. Tappe, *On isoclinic groups*, Math. Z. 148 (1976), 147–153.

Yuta Katayama, Masanari Kida
Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka Shinjuku
Tokyo 162-8601, Japan
E-mail: 1120701@ed.tus.ac.jp
kida@rs.tus.ac.jp