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A SIMPLE PROOF OF THE REAL RIESZ–THORIN INTERPOLATION THEOREM IN THE LOWER TRIANGLE

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Abstract. A simple proof of the real Riesz–Thorin interpolation theorem for operators of strong types (p_0, q_0) and (∞, ∞) in the lower triangle, that is, if $0 < p_0 \le q_0 < \infty$ with the best estimate of the norms is presented.

1. Introduction. The classical Riesz–Thorin interpolation theorem is proved for L^p spaces of complex-valued functions, which is based on the three-line theorem. It states the following (see [BS88, BL76, BK91, Fo99]):

Suppose that (Ω_1, μ) and (Ω_2, ν) are σ-finite measure spaces. Let $0 < p_0, p_1, q_0, q_1 \le \infty$ with $p_0 \neq p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ for any $0 < \theta < 1$. If a linear operator $T\colon L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ is bounded from $L^{p_0}(\mu)$ to $L^{q_0}(\nu)$ and from $L^{p_1}(\mu)$ to $L^{q_1}(\nu)$, then it is also bounded from $L^p(\mu)$ to $L^q(\nu)$, and

$$\|T\|_{L^p(\mu)\to L^q(\nu)} \le \|T\|^{1-\theta}_{L^{p_0}(\mu)\to L^{q_0}(\nu)} \|T\|^{\theta}_{L^{p_1}(\mu)\to L^{q_1}(\nu)}. \tag{1.1}$$

We can briefly say that the pair $(L^p(\mu), L^q(\nu))$ is an *exact interpolation pair* with respect to the pairs $(L^{p_0}(\mu), L^{p_1}(\nu))$ and $(L^{q_0}(\mu), L^{q_1}(\nu))$, where "exact" refers to the fact that estimate (1.1) holds without an additional constant C as in (1.2) below.

Recall that if $0 < p_0 < p_1 \le \infty$, then $L^{p_0}(\mu) + L^{p_1}(\mu)$ is a quasi-Banach space with the quasi-norm

$$\|f\|_{L^{p_0}(\mu)+L^{p_1}(\mu)} = \inf\{\|f_0\|_{p_0} + \|f_1\|_{p_1} : f = f_0 + f_1, f_0 \in L^{p_0}(\mu), f_1 \in L^{p_1}(\mu)\}.$$

If $1 \le p_0 < p_1 \le \infty$, then $L^{p_0}(\mu) + L^{p_1}(\mu)$ is a Banach space. In a similar way the space $L^{q_0}(\nu) + L^{q_1}(\nu)$ is defined for $0 < q_0 < q_1 \le \infty$.

The theorem for L^p spaces of real-valued functions holds only if the exponents lie in the lower triangle, that is, if $p_0 \le q_0$ and $p_1 \le q_1$ (see, for example, [GM94, MS11]).

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If at least one exponent lies in the upper triangle, estimate (1.1) is replaced by

$$\|T\|_{L^p(\mu)\to L^q(\nu)} \le C\|T\|^{1-\theta}_{L^{p_0}(\mu)\to L^{q_0}(\nu)}\|T\|^{\theta}_{L^{p_1}(\mu)\to L^{q_1}(\nu)} \tag{1.2}$$

with some constant $C > 1$. This follows from estimates of the norm of the operator and its natural complexification, which was fully investigated in [MS11].

So we naturally come to the following question: is it possible to give a direct, elementary proof of the Riesz–Thorin theorem for real-valued functions without using complex variables?

Kruglyak [Kr07] gave a real proof with the constant $C = 3+2\sqrt{2} \approx 5.82$ (see also [BK91]). In [KM01] we presented a real proof for $1 < p_0 = q_0 < p_1 = q_1 < \infty$ with the constant $C = 2^{(1/p_0)(1-1/p_1)+\min(1/p_0,1-1/p_1)} < 4$.

Now, we present an elementary proof of the Riesz–Thorin interpolation theorem using only real variable techniques, with the exact estimate of norms (1.1) on any line segment going through the origin in the lower triangle, that is, if $0 < p_0 \le q_0 < \infty$, $p_1 = q_1 = \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0}$, $\frac{1}{q} = \frac{1-\theta}{q_0}$, $0 < \theta < 1$. The idea of the proof comes from [LS71, Ma89].

2. Real proof of the Riesz–Thorin interpolation theorem for $0 < p_0 \le q_0 < \infty$ and $p_1 = q_1 = \infty$. The proof is an extension of an idea coming from [LS71, Ma89] (see also [Mal89]). In [Ma89, Mal89] it was proved even more that the pair of Orlicz spaces $(L^{\varphi}(\mu), L^{\varphi}(\nu))$ is an exact interpolation pair with respect to $(L^1(\mu), L^1(\nu))$ and $(L^{\infty}(\mu), L^{\infty}(\nu))$ and not only for linear operators, but even for Lipschitz operators.

THEOREM 2.1. *If $0 < p_0 \le q_0 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0}$, $\frac{1}{q} = \frac{1-\theta}{q_0}$ with $0 < \theta < 1$, then pair $(L^p(\mu), L^q(\nu))$ is an exact interpolation pair with respect to $(L^{p_0}(\mu), L^{q_0}(\nu))$ and $(L^{\infty}(\mu), L^{\infty}(\nu))$. Moreover,*

$$\|T\|_{L^p(\mu)\to L^q(\nu)} \le \|T\|^{1-\theta}_{L^{p_0}(\mu)\to L^{q_0}(\nu)}\,\|T\|^{\theta}_{L^{\infty}(\mu)\to L^{\infty}(\nu)}. \tag{2.1}$$

Proof. The first step is the following equality: if $0 < p_0 < p < \infty$, then for all $u > 0$ we have

$$u^p\, C_{p_0,p} = \int_0^u (u-s)^{p_0} s^{p-p_0-1}\, ds = \int_0^{\infty} (u-s)_+^{p_0} s^{p-p_0-1}\, ds, \tag{2.2}$$

with

$$C_{p_0,p} = \frac{\Gamma(p_0+1)\Gamma(p-p_0)}{\Gamma(p+1)} = B(p_0+1, p-p_0),$$

where Γ, B stand for Gamma and Beta functions and $v_+ = \max(v,0)$ for $v \in \mathbb{R}$. It is enough to see that

$$\int_0^u (u-s)^{p_0} s^{p-p_0-1}\, ds = u^p \int_0^1 (1-t)^{p_0} t^{p-p_0-1}\, dt.$$

The second step is an estimate for truncation: for each $x \in L^{p_0}(\mu) + L^{\infty}(\mu)$ we have

$$|Tx(t) - (Tx)^{(\alpha)}(t)| \le |Tx(t) - T(x^{(\alpha/M_\infty)})(t)| \quad \nu\text{-a.e.}, \tag{2.3}$$

where for $\alpha > 0$ the truncation is defined by $x^{(\alpha)}(t) := \min(|x(t)|, \alpha) \operatorname{sgn} x(t)$ and $M_\infty = \|T\|_{L^\infty(\mu)\to L^\infty(\nu)}$.

Indeed, if $|Tx(t)| \le \alpha$ then (2.3) is obvious. On the other hand, if $|Tx(t)| > \alpha$ then since

$$\|T(x^{(\alpha/M_\infty)})\|_{L^\infty(\nu)} \le M_\infty \|x^{(\alpha/M_\infty)}\|_{L^\infty(\mu)} \le M_\infty \frac{\alpha}{M_\infty} = \alpha$$

it follows that $|T(x^{(\alpha/M_\infty)})(t)| \le \alpha$ ν-a.e. Hence,

$$\begin{aligned} |Tx(t) - (Tx)^{(\alpha)}(t)| &= |Tx(t) - \alpha \operatorname{sgn} Tx(t)| \\ &= |Tx(t)| - \alpha \le |Tx(t)| - |T(x^{(\alpha/M_\infty)})(t)| \\ &\le |Tx(t) - T(x^{(\alpha/M_\infty)})(t)| \quad \nu\text{-a.e.} \end{aligned}$$

and estimate (2.3) is proved.

Now, if $x \in L^p(\mu) \cap L^{p_0}(\mu)$, then using (2.2) twice and Fubini's theorem twice, with

$$M_0 := \|T\|_{L^{p_0}(\mu)\to L^{q_0}(\nu)}$$

we get

$$\begin{aligned} C_{q_0,q} \int_{\Omega_2} |Tx(t)|^q \, d\nu &= \int_{\Omega_2} \Big[\int_0^\infty (|Tx(t)| - s)_+^{q_0} \, s^{q-q_0-1} \, ds \Big] d\nu \\ &= \int_0^\infty \Big[\int_{\Omega_2} (|Tx(t)| - s)_+^{q_0} \, d\nu \Big] s^{q-q_0-1} \, ds \\ &= \int_0^\infty \Big[\int_{\Omega_2} |Tx(t) - (Tx)^{(s)}(t)|^{q_0} \, d\nu \Big] s^{q-q_0-1} \, ds \quad \text{[by (2.3)]} \\ &\le \int_0^\infty \Big[\int_{\Omega_2} |Tx(t) - T(x^{(s/M_\infty)})(t)|^{q_0} \, d\nu \Big] s^{q-q_0-1} \, ds \\ &= \int_0^\infty \|Tx - T(x^{(s/M_\infty)})\|_{L^{q_0}(\nu)}^{q_0} \, s^{q-q_0-1} \, ds \\ &\le M_0^{q_0} \int_0^\infty \|x - x^{(s/M_\infty)}\|_{L^{p_0}(\mu)}^{q_0} \, s^{q-q_0-1} \, ds \\ &= M_0^{q_0} \int_0^\infty \Big[\int_{\Omega_1} (|x(t)| - s/M_\infty)_+^{p_0} \, d\mu \Big]^{q_0/p_0} s^{q-q_0-1} \, ds \end{aligned}$$

$$= M_0^{q_0} \int_{\Omega_1} \Big[\int_0^\infty (|x(t)| - s/M_\infty)_+^{p_0} \, d\mu \Big]^{q_0/p_0} s^{q-q_0-1} ds$$

$$= M_0^{q_0} \Big\| \int_{\Omega_1} (|x(t)| - s/M_\infty)_+^{p_0} \, d\mu \Big\|_{L^{q_0/p_0}(s^{q-q_0-1}ds)}^{q_0/p_0} =: A.$$

Because we have $q_0/p_0 \geq 1$, using Minkowski's inequality and substituting $s = M_\infty |x(t)| u$ we obtain

$$A \leq M_0^{q_0} \Big\{ \int_{\Omega_1} \|(|x(t)| - s/M_\infty)_+^{p_0}\|_{L^{q_0/p_0}(s^{q-q_0-1}ds)} \, d\mu \Big\}^{q_0/p_0}$$

$$= M_0^{q_0} \Big\{ \int_{\Omega_1} \Big[\int_0^{M_\infty |x(t)|} (|x(t)| - s/M_\infty)^{q_0} s^{q-q_0-1} \, ds \Big]^{p_0/q_0} d\mu \Big\}^{q_0/p_0}$$

$$= M_0^{q_0} \Big\{ \int_{\Omega_1} \Big[\int_0^1 |x(t)|^{q_0} (1-u)^{q_0} (M_\infty |x(t)| u)^{q-q_0} \, du/u \Big]^{p_0/q_0} d\mu \Big\}^{q_0/p_0}$$

$$= M_0^{q_0} M_\infty^{q-q_0} \Big\{ \int_{\Omega_1} |x(t)|^{qp_0/q_0} \, d\mu \Big\}^{q_0/p_0} C_{q_0,q} = M_0^{(1-\theta)q} M_\infty^{\theta q} C_{q_0,q} \|x\|_{L^p(\mu)}^q,$$

since $\frac{1}{p} = \frac{1-\theta}{p_0}$, $\frac{1}{q} = \frac{1-\theta}{q_0}$ and so $p = q\frac{p_0}{q_0}$. Hence,

$$C_{q_0,q} \int_{\Omega_2} |Tx(t)|^q \, d\nu \leq M_0^{(1-\theta)q} M_\infty^{\theta q} C_{q_0,q} \|x\|_{L^p(\mu)}^q,$$

that is,

$$\|Tx\|_{L^q(\nu)} \leq M_0^{1-\theta} M_\infty^{\theta} \|x\|_{L^p(\mu)} \quad \text{for any } x \in L^p(\mu) \cap L^{p_0}(\mu).$$

We want to prove that the last estimate is true for functions from the entire space $L^p(\mu)$, so let $x \in L^p(\mu)$ and $A_n := \{t \in \Omega_1 : |x(t)| > 1/n\}$ for $n \in \mathbb{N}$. Then

$$\mu(A_n) \leq n^p \int_{A_n} |x(t)|^p \, d\mu \leq n^p \|x\|_{L^p(\mu)}^p < \infty$$

for any $n \in \mathbb{N}$. We have $x_n := x\chi_{A_n} \in L^p(\mu) \cap L^{p_0}(\mu)$ because by the Hölder–Rogers inequality with $p/p_0 > 1$,

$$\int_{A_n} |x(t)|^{p_0} \, d\mu \leq \Big(\int_{A_n} |x(t)|^p \, d\mu \Big)^{p_0/p} \mu(A_n)^{1-p_0/p} \leq \|x\|_{L^p(\mu)}^{p_0} \mu(A_n)^{1-p_0/p} < \infty.$$

Hence,

$$\|x - x_n\|_{L^{p_0}(\mu)+L^\infty(\mu)} \leq \|x\chi_{\Omega_1 \setminus A_n}\|_{L^\infty(\mu)} \leq 1/n \to 0 \quad \text{as } n \to \infty.$$

By the boundedness of T from $L^{p_0}(\mu) + L^\infty(\mu)$ to $L^{q_0}(\nu) + L^\infty(\nu)$ we obtain $\|Tx - Tx_n\|_{L^{q_0}(\nu)+L^\infty(\nu)} \to 0$ as $n \to \infty$ and hence there is a subsequence

(n_k) such that $Tx_{n_k} \to Tx$ ν-a.e. as $k \to \infty$. By the Fatou lemma,

$$\begin{aligned}\|Tx\|_{L^q(\nu)} &\le \liminf_{k\to\infty} \|Tx_{n_k}\|_{L^q(\nu)} \le M_0^{p_0/p} M_\infty^{1-p_0/p} \liminf_{k\to\infty} \|x_{n_k}\|_{L^p(\mu)} \\ &\le M_0^{p_0/p} M_\infty^{1-p_0/p} \|x\|_{L^p(\mu)},\end{aligned}$$

and we are done. ∎

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REFERENCES

[BS88] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[BL76] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer, Berlin, 1976.

[BK91] Yu. A. Brudnyi and N. Kruglyak, *Interpolation Functors and Interpolation Spaces, Vol. I*, North-Holland, Amsterdam, 1991.

[Fo99] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley, New York, 1999.

[GM94] J. Gasch and L. Maligranda, *On vector-valued inequalities of the Marcinkiewicz–Zygmund, Herz and Krivine type*, Math. Nachr. 167 (1994), 95–129.

[KM01] A. Yu. Karlovich and L. Maligranda, *On the interpolation constant for Orlicz spaces*, Proc. Amer. Math. Soc. 129 (2001), 2727–2739.

[Kr07] N. Kruglyak, *An elementary proof of the real version of the Riesz–Thorin theorem*, in: Interpolation Theory and Applications, Contemp. Math. 445, Amer. Math. Soc., Providence, 2007, 179–182.

[LS71] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. 159 (1971), 207–221.

[Ma89] L. Maligranda, *Some remarks on Orlicz's interpolation theorem*, Studia Math. 95 (1989), 43–58.

[Mal89] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics 5, Univ. Estadual de Campinas, Campinas, 1989.

[MS11] L. Maligranda and N. Sabourova, *Real and complex operator norms between quasi-Banach* L^p-L^q *spaces*, Math. Inequal. Appl. 14 (2011), 247–270.

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