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EXTENDED COMPARISON BETWEEN TWO NEWTON–JARRATT SIXTH ORDER SCHEMES FOR NONLINEAR MODELS UNDER THE SAME SET OF CONDITIONS

Abstract. Two sixth order convergence order schemes are compared and extended to solve Banach space valued models. Earlier studies have used derivatives and Taylor expansions up to order seven to show the convergence order in a finite-dimensional Euclidean space setting. We compute the order by finding computational convergence order or approximate computational convergence order, and condition only on the derivative that is present in the schemes. Moreover, a computable convergence radius, upper error bounds and uniqueness of the solution are provided. Numerical applications illustrate the theoretical results.

1. Introduction. Let X and Y denote Banach spaces, and $D \subseteq X$ be convex and open. Further, suppose that $L(X,Y)$ denotes the bounded linear operator mappings of X into Y. In applied mathematics many problems can be expressed in the form

$$H(x) = 0, \tag{1.1}$$

where $H : D \subset X \to Y$ is a differentiable mapping. Numerous schemes to compute a solution α of (1.1) are iterative, since solutions are rarely obtained in closed form (see [1–3, 6–10, 12]).

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Here, we investigate two sixth order schemes given in [4,11] for $X = Y = \mathbb{R}^m$ respectively by

$$
\begin{aligned}
y_n &= x_n - \tfrac{2}{3}H'(x_n)^{-1}H(x_n),\\
z_n &= x_n - \tfrac{1}{2}A_n^{-1}(3H'(y_n) + H'(x_n))H'(x_n)^{-1}H(x_n),\\
x_{n+1} &= z_n - 2A_n^{-1}H(z_n),
\end{aligned}
\tag{1.2}
$$

where $A_n = 3H'(y_n) - H'(x_n)$ and

(1.3)

$$
\begin{aligned}
y_n &= x_n - \tfrac{2}{3}H'(x_n)^{-1}H(x_n),\\
z_n &= x_n - \big(\tfrac{23}{8}I - \big(3I - \tfrac{9}{8}H'(x_n)^{-1}H'(y_n)\big)H'(x_n)^{-1}H'(y_n)\big)H'(x_n)^{-1}H(x_n),\\
x_{n+1} &= z_n - \tfrac{1}{2}\big(5I - 3H'(x_n)^{-1}H'(y_n)\big)H'(x_n)^{-1}H(z_n).
\end{aligned}
$$

The sixth order convergence of (1.2) and (1.3) was established in [4, 11] respectively by using Taylor series and conditions on $H^{(i)}$, $i = 1, \ldots, 7$. These conditions limit the applicability for these schemes. Notice that only the first derivatives are used in the schemes. Indeed, consider $H : D = [-5/2, 2] \to \mathbb{R}$ defined by

$$
H(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin \frac{1}{x}, & x \neq 0,\\ 0, & x = 0,\end{cases}
$$

leading to

$$
H'''(x) = \frac{1}{x}\left[(1-36x^2)\cos\left(\frac{1}{x}\right) + x\left(22 + 6\log(\pi^2 x^2) + (60x^2 - 9)\sin\left(\frac{1}{x}\right)\right)\right].
$$

Hence, $H'''(x)$ is not continuous on D, so the results of [4,11] are not applicable.

The goal of this paper is to extend the applicability of methods (1.2) and (1.3) in cases not covered in earlier studies, which required the use of derivatives up to order seven not appearing in the methods. The price we pay by using conditions on the first derivative which actually appears in the method is that we show only linear convergence. To find the convergence order is not however our intention, since this is already known in the m-dimensional Euclidean space. Notice also that the order is rediscovered by using ACOC or COC (see Remark 2.1), which requires only the first derivative. Moreover, in earlier studies using Taylor series no computable error distances are available based say on generalized Lipschitz conditions. So, we do not know for example in advance how many iterates are needed to achieve a predetermined error tolerance. Furthermore, no uniqueness of the solution results are available in the aforementioned studies; we also provide such results. Our technique can be used to extend the applicability of other methods in an analogous way, since it is fairly general. Finally, notice that local results of this type are important, since they demonstrate the difficulty in choosing initial points.

We summarize the contents of the paper. In Section 2, the local convergence analysis is presented including a convergence radius. Some numerical applications are developed in Section 3.

2. Analysis. Some parameters and functions will be used in the local convergence analysis of method (1.2). Consider $D = [0, \infty)$. Suppose the following:

(i) There exists a function $v_0 : D \to D$ continuous and nondecreasing and such that the equation

$$v_0(t) - 1 = 0 \tag{2.1}$$

has a minimal zero $r_0 \in D - \{0\}$.

(ii) Set $D_0 = [0, 2r_0)$. Let $v : D_0 \to D$ be a continuous nondecreasing function. Define $h_1 : D_0 \to D$ by

$$h_1(t) = \frac{\int_0^1 v((1-\theta)t)\, d\theta + \frac{1}{3}\int_0^1 v_1(\theta t)\, d\theta}{1 - v_0(t)}.$$

The equation

$$h_1(t) - 1 = 0 \tag{2.2}$$

has a minimal zero $\delta_1 \in D_0 - \{0\}$.

(iii) The equation

$$q(t) - 1 = 0, \quad \text{where} \quad q(t) = \tfrac{1}{2}(3v_0(h_1(t)t) + v_0(t)), \tag{2.3}$$

has a minimal zero $r_1 \in D_0 - \{0\}$.

(iv) Let $r_2 = \min\{r_0, r_1\}$ and $D_1 = [0, r_2)$. Let $v_1 : D_1 \to D$ be a continuous nondecreasing function. Define $h_2 : D_1 \to D$ by

$$h_2(t) = \frac{\int_0^1 v((1-\theta)t)\, d\theta}{1 - v_0(t)} + \frac{3(v_0(t) + v_0(h_1(t)t))\int_0^1 v_1(\theta t)\, d\theta}{4(1 - v_0(t))(1 - q(t))}.$$

The equation

$$h_2(t) - 1 = 0 \tag{2.4}$$

has a minimal zero $\delta_2 \in D_1 - \{0\}$.

(v) The equation

$$v_0(h_2(t)t) - 1 = 0$$

has a minimal zero $r_3 \in D_1 - \{0\}$.

(vi) Let $r = \min\{r_2, r_3\}$ and $D_2 = [0, r)$. Define $h_3 : D_2 \to D$ by

$$h_3(t) = \left[\frac{\int_0^1 v((1-\theta)h_2(t)t)\,d\theta}{1 - v_0(h_2(t)t)} + \frac{(v_0(t) + 3v_0(h_1(t)t) + 2v_0(h_2(t)t))\int_0^1 v_1(\theta h_2(t)t)\,d\theta}{2(1 - v_0(h_2(t)t))(1 - q(t))}\right] h_2(t).$$

The equation

$$h_3(t) - 1 = 0 \tag{2.5}$$

has a minimal zero $\delta_3 \in D_2 - \{0\}$.

We shall prove that

$$\delta = \min\{\delta_i : i = 1, 2, 3\}, \tag{2.6}$$

is a convergence radius for method (1.2). Let $D_3 = [0, \delta)$.

The definition of δ implies that for all $t \in D_3$,

$$0 \le v_0(t) < 1, \tag{2.7}$$

$$0 \le v_0(h_2(t)t) < 1, \tag{2.8}$$

$$0 \le h_i(t) < 1. \tag{2.9}$$

We denote by $\bar{B}(\alpha, \delta)$ the closure of the ball $B(\alpha, \delta)$ of radius $\delta > 0$ and center α. From now on the functions v_i are as above.

The local convergence analysis requires the following conditions (called *conditions* A). Assume:

(A_1) $H : D \to Y$ is continuously differentiable with α a simple solution of equation (1.1).

(A_2) $\|H'(\alpha)^{-1}(H'(x) - H'(\alpha))\| \le v_0(\|x - \alpha\|)$ for all $x \in D$.

(A_3) Set $D_0 = D \cap B(\alpha, r_0)$. Then $\|H'(\alpha)^{-1}(H'(x) - H'(y))\| \le v(\|x - y\|)$ and $\|H'(\alpha)^{-1}H'(x)\| \le v_1(\|x - \alpha\|)$ for all $x, y \in D_0$.

(A_4) $\bar{B}(\alpha, R) \subset D$ for some $R > 0$ to be determined later.

(A_5) There exists $x^* \ge \delta$ satisfying

$$\int_0^1 v_0(\theta x^*)\,d\theta < 1.$$

Set $D_1 = D \cap \bar{B}(\alpha, x^*)$.

Next, conditions (A) are used in the local convergence result for method (1.2).

THEOREM 2.1. *Suppose conditions* (A) *with* $R = \delta$ *hold. Then the following assertions hold for method* (1.2) *provided that* $x_0 \in B(\alpha, \delta) - \{\alpha\}$*:*

$$\{x_n\} \subset B(\alpha, \delta), \qquad \lim_{n\to\infty} x_n = \alpha, \tag{2.10}$$

$$\|y_n - \alpha\| \le h_1(d_n)d_n \le d_n < \delta, \tag{2.11}$$

$$\|z_n - \alpha\| \le h_2(d_n)d_n \le d_n, \tag{2.12}$$

$$\|x_{n+1} - \alpha\| \le h_3(d_n)d_n \le d_n, \tag{2.13}$$

with $d_n = \|x_n - \alpha\|$. *Moreover, the only solution of equation* (1.1) *in the set* D_1 *given in* (A_5) *is* α.

Proof. Assertions (2.10)–(2.13) will be proved by induction on n. Choose $w \in B(\alpha, \delta) - \{\alpha\}$. Using ($A_1$), ($A_2$), (2.6) and (2.7), we have

$$\|H'(\alpha)^{-1}\big(H'(w) - H'(\alpha)\big)\| \le v_0\|w - \alpha\| < v_0(\delta) < 1. \tag{2.14}$$

But then (2.14) and Banach's lemma [3] on inverses of linear operators imply $H'(w)^{-1} \in L(Y, X)$ and

$$\|H'(w)^{-1}H'(\alpha)\| \le \frac{1}{1 - v_0\|w - \alpha\|}. \tag{2.15}$$

Moreover y_0 and z_0 exist by the first and second substep of (1.2) for $n = 0$ (if $w = x_0$ in (2.15)), from which we can also write

(2.16)
$$\begin{aligned} y_0 - \alpha &= x_0 - \alpha - H'(x_0)^{-1}H(x_0) + \tfrac{1}{3}H'(x_0)^{-1}H(x_0) \\ &= [H'(x_0)^{-1}H'(\alpha)]\Big[\int_0^1 H'(\alpha)^{-1}(H'(\alpha+\theta(x_0-\alpha)) - H'(x_0))\, d\theta(x_0-\alpha)\Big] \\ &\quad + \tfrac{1}{3}H'(x_0)^{-1}H(x_0) \end{aligned}$$

and

$$\begin{aligned} z_0 - \alpha = x_0 - \alpha \\ - H'(x_0)^{-1}H(x_0) + \tfrac{3}{2}A_0^{-1}(H'(y_0) - H'(x_0))H'(x_0)^{-1}H(x_0). \end{aligned} \tag{2.17}$$

Then by (2.6), (2.9) (for $i = 1, 2$), (A_3), (2.15) (for $w = x_0$), (2.16) and (2.17), we have in turn

$$\begin{aligned} \|y_0 - \alpha\| &\le \frac{\int_0^1 v((1-\theta)d_0)\, d\theta\, d_0 + \frac{1}{3}\int_0^1 v_1(\theta d_0)\, d\theta\, d_0}{1 - v_0(d_0)} \\ &= h_1(d_0)d_0 \le d_0 < \delta, \end{aligned} \tag{2.18}$$

and

(2.19)
$$\begin{aligned} \|z_0 - \alpha\| &\le \left[\frac{\int_0^1 v((1-\theta)d_0)\, d\theta}{1 - v_0(d_0)} + \frac{3(v_0(d_0) + v_0(\|y_0 - \alpha\|))\int_0^1 v_1(\theta d_0)\, d\theta}{4(1 - v_0(d_0))(1 - q(d_0))}\right] d_0 \\ &\le h_2(d_0)d_0 \le d_0, \end{aligned}$$

which proves that $y_0, z_0 \in B(\alpha,\delta)$ and (2.10)–(2.12) hold with $n = 0$, where we have also used

$$\begin{aligned}\|(2H'(\alpha))^{-1}(A_0 - H'(\alpha))\| &\leq \tfrac{1}{2}(3\|H'(\alpha)^{-1}(H'(y_0) - H'(\alpha))\| \\ &\quad + \|H'(\alpha)^{-1}(H'(x_0) - H'(\alpha))\|) \\ &\leq \tfrac{1}{2}(3v_0(\|y_0 - \alpha\|) + v_0(d_0)) \\ &\leq q(d_0) \leq q(\delta) < 1,\end{aligned}$$

so

$$\|A_0^{-1}H'(\alpha)\| \leq \frac{1}{2(1 - q(d_0))}.$$

The iterate x_1 is well defined by the third substep of method (1.2), from which we can also write

$$\begin{aligned}x_1 - \alpha &= z_0 - \alpha \\ &\quad - H'(z_0)^{-1}H(z_0) + H'(z_0)^{-1}(A_0 - 2H'(z_0))A_0^{-1}H(z_0).\end{aligned} \tag{2.20}$$

It follows that in view of (2.6), (2.9) (for $i = 3$), (2.15) (for $w = z_0$) and (2.18)–(2.20),

$$\begin{aligned}\|x_1 - \alpha\| &\leq \bigg[\frac{\int_0^1 v((1-\theta)\|z_0 - \alpha\|)\,d\theta}{1 - v_0(\|z_0 - \alpha\|)} \\ &\quad + \frac{(3v_0(\|y_0-\alpha\|)+v_0(d_0)+2v_0(\|z_0-\alpha\|))\int_0^1 v_1(\theta\|z_0-\alpha\|)\,d\theta}{2(1-q(d_0))(1-v_0(\|z_0-\alpha\|))}\bigg] \\ &\quad \times \|z_0 - \alpha\| \\ &\leq h_3(d_0)d_0 \leq d_0,\end{aligned} \tag{2.21}$$

proving $x_1 \in B(\alpha,\delta)$ and (2.13) for $n = 0$. Replace x_0, y_0, z_0, x_1 by x_n, y_n, z_n, x_{n+1} in the preceding calculations to terminate the induction. Then, by the estimation

$$\|x_{n+1} - \alpha\| \leq \beta d_n < \delta, \tag{2.22}$$

where $\beta = h_3(d_0) \in [0,1)$, we get $x_{n+1} \in B(\alpha,\delta)$, and $\lim_{n\to\infty} x_n = \alpha$.

Suppose $u \in D_1$ with $H(u) = 0$, and set $T = \int_0^1 H'(u + \theta(\alpha - u))\,d\theta$. By ($A_2$) and ($A_5$), we get

$$\begin{aligned}\|H'(\alpha)^{-1}(T - H'(\alpha))\| &\leq \int_0^1 v_0(\theta\|\alpha - u\|)\,d\theta \\ &\leq \int_0^1 v_0(\theta x^*)\,d\theta < 1,\end{aligned} \tag{2.23}$$

so $u=\alpha$ follows, since $T^{-1}\in L(Y,X)$, and

$$0=H(\alpha)-H(u)=T(\alpha-u). \ \blacksquare$$

Next, we develop the local convergence analysis of method (1.3) in an analogous manner. Now the h functions are defined by

$$\begin{aligned}
\bar h_1(t)&=\frac{\int_0^1 v((1-\theta)t)\,d\theta+\frac13\int_0^1 v_1(\theta t)\,d\theta}{1-v_0(t)}=h_1(t),\\
\bar h_2(t)&=\frac{\int_0^1 v((1-\theta)t)\,d\theta}{1-v_0(t)}+\frac38\biggl(3\Bigl(\frac{v_0(t)+v_0(h_1(t)t)}{1-v_0(t)}\Bigr)^2\\
&\quad+2\frac{v_0(t)+v_0(h_1(t)t)}{1-v_0(t)}\biggr)\frac{\int_0^1 v_1(\theta t)\,d\theta}{1-v_0(t)},\\
\bar h_3(t)&=\biggl[\frac{\int_0^1 v((1-\theta)\bar h_2(t)t)\,d\theta}{1-v_0(\bar h_2(t)t)}+\frac{(v_0(\bar h_2(t)t)+v_0(t))\int_0^1 v_1(\theta\bar h_2(t)t)\,d\theta}{(1-v_0(t))(1-v_0(\bar h_2(t)t))}\\
&\quad+3\frac{(v_0(t)+v_0(\bar h_1(t)t))\int_0^1 v_1(\theta\bar h_2(t)t)\,d\theta}{(1-v_0(t))^2}\biggr]\bar h_2(t),
\end{aligned}$$

and

$$\bar\delta=\min\bar\delta_i, \tag{2.24}$$

where $\bar\delta_i$ is the least solution (if any exists) in D_0 of the equation

$$\bar h_i(t)-1=0. \tag{2.25}$$

The functions $\bar h_i$ are motivated by method (1.3), since

$$y_0-\alpha=x_0-\alpha-H'(x_0)^{-1}H(x_0)+\tfrac13 H'(x_0)^{-1}H(x_0)$$

implies

$$\begin{aligned}
\|y_0-\alpha\|&\le\frac{\int_0^1 v((1-\theta)d_0)\,d\theta+\frac13\int_0^1 v_1(\theta d_0)\,d\theta}{1-v_0(d_0)}\\
&\le\bar h_1(d_0)d_0\le d_0<\bar\delta,
\end{aligned}$$

and

$$\begin{aligned}
z_0-\alpha&=x_0-\alpha-H'(x_0)^{-1}H(x_0)-\tfrac38\bigl(5I-8H'(x_0)^{-1}H'(y_0)\\
&\quad+3(H'(x_0)^{-1}H'(y_0))^2\bigr)H'(x_0)^{-1}H(x_0)\\
&=x_0-\alpha-H'(x_0)^{-1}H(x_0)-\tfrac38\bigl(3(H'(x_0)^{-1}H'(y_0)-I)^2\\
&\quad-2(H'(x_0)^{-1}H'(y_0)-I)\bigr)H'(x_0)^{-1}H(x_0)
\end{aligned}$$

yields

$$
\begin{aligned}
\|z_0-\alpha\| \le {} & \bigg[\frac{\int_0^1 v((1-\theta)d_0)\,d\theta}{1-v_0(d_0)} + \frac{3}{8}\bigg(3\bigg(\frac{v_0(d_0)+v_0(\|y_0-\alpha\|)}{1-v_0(d_0)}\bigg)^2 \\
& + 2\frac{v_0(d_0)+v_0(\|y_0-\alpha\|)}{1-v_0(d_0)}\bigg)\frac{\int_0^1 v_1(\theta d_0)\,d\theta}{1-v_0(d_0)}\bigg]d_0 \\
\le {} & \bar{h}_2(d_0)d_0 \le d_0,
\end{aligned}
$$

and further

$$
\begin{aligned}
x_1-\alpha = {} & z_0-\alpha - H'(z_0)^{-1}H(z_0) \\
& + H'(z_0)^{-1}(H'(x_0)-H'(z_0))H'(x_0)^{-1}H(z_0) \\
& - 3H'(x_0)(H'(x_0)-H'(y_0))H'(x_0)^{-1}H(z_0)
\end{aligned}
$$

leads to

$$
\begin{aligned}
\|x_1-\alpha\| \le {} & \bigg[\frac{\int_0^1 v((1-\theta)\|z_0-\alpha\|)\,d\theta}{1-v_0(\|z_0-\alpha\|)} \\
& + \frac{(v_0(d_0)+v_0(\|z_0-\alpha\|))\int_0^1 v_1(\theta\|z_0-\alpha\|)\,d\theta}{(1-v_0(\|z_0-\alpha\|))(1-v_0(d_0))} \\
& + 3\frac{(v_0(d_0)+v_0(\|y_0-\alpha\|))\int_0^1 v_1(\theta\|z_0-\alpha\|)\,d\theta}{(1-v_0(d_0))^2}\bigg]\|z_0-\alpha\| \\
\le {} & \bar{h}_3(d_0)d_0 \le d_0.
\end{aligned}
$$

Hence, we arrive at the corresponding result for method (1.3):

THEOREM 2.2. *Suppose conditions* (A) *hold for* $R=\bar{\delta}$. *Then the conclusions of Theorem* 2.1 *hold for method* (1.3) *with the* h_i *functions replaced by* $\bar{h}_i$.

REMARK 2.1. The *computational order of convergence* (COC) [13] is defined as

$$
\text{COC} = \log\left\|\frac{d_{i+2}}{d_{i+1}}\right\| \Big/ \log\left\|\frac{d_{i+1}}{d_i}\right\|, \qquad i=1,2,\ldots, \tag{2.26}
$$

and the *approximate computational order of convergence* (ACOC) [4] is

$$
\text{ACOC} = \log\left\|\frac{\bar{d}_{j+2}}{\bar{d}_{j+1}}\right\| \Big/ \log\left\|\frac{\bar{d}_{j+1}}{\bar{d}_j}\right\|, \qquad j=1,2,\ldots, \tag{2.27}
$$

where $\bar{d}_j = x_j - x_{j-1}$. Hence, we provide a practical convergence order which avoids higher derivatives.

3. Numerical results. Estimates (2.6) and (2.24) are used to find δ and $\bar{\delta}$, respectively.

EXAMPLE 1. We consider the example given in the introduction. Note that $\alpha = 1/\pi$ is a zero of the function H. Then we can choose $v_0(t) = Lt$, $v(t) = Lt$ and $v_1(t) = L/2$, where $L = \frac{2}{2\pi+1}(80 + 16\pi + (11 + 12\log 2)\pi^2)$. So, we obtain the radii

$$\delta_1 = 1.0355\times 10^{-1},\ \delta_2 = 1.5843\times 10^{-5},\ \delta_3 = 1.3069\times 10^{-5},\ \delta = 1.3069\times 10^{-5},$$
$$\bar{\delta}_1 = 1.0355\times 10^{-1},\ \bar{\delta}_2 = 1.5850\times 10^{-3},\ \bar{\delta}_3 = 2.5376\times 10^{-4},\ \bar{\delta} = 2.5376\times 10^{-4}.$$

EXAMPLE 2. Consider the function $H : D \to \mathbb{R}^3$ defined by

$$H(x) = (10x_1 + \sin(x_1 + x_2) - 1, 8x_2 - \cos^2(x_3 - x_2) - 1, 12x_3 + \sin(x_3) - 1)^T,$$

where $x = (x_1, x_2, x_3)^T$.

The Fréchet derivative of $H(x)$ is given by

$$H'(x) = \begin{bmatrix} 10 + \cos(x_1 + x_2) & \cos(x_1 + x_2) & 0 \\ 0 & 8 + \sin 2(x_2 - x_3) & -\sin 2(x_2 - x_3) \\ 0 & 0 & 12 + \cos(x_3) \end{bmatrix}.$$

We can choose $v_0(t) = v(t) = 0.269812t$ and $v_1(t) = 1.08139$. So, we obtain

$$\delta_1 = 1.5802,\ \delta_2 = 8.8440 \times 10^{-1},\ \delta_3 = 8.3063 \times 10^{-1},\ \delta = 8.3063 \times 10^{-1},$$
$$\bar{\delta}_1 = 1.5802,\ \bar{\delta}_2 = 7.7517 \times 10^{-1},\ \bar{\delta}_3 = 5.7843 \times 10^{-1},\ \bar{\delta} = 5.7843 \times 10^{-1}.$$

EXAMPLE 3. Next, we consider an equation due to Kepler:

$$H(x) = x - \beta\sin(x) - K = 0,$$

where $0 \le \beta < 1$, $0 \le K \le \pi$. Different values of β and K are given in [5]. Set $K = 0.1$ and $\beta = 0.27$. Then we have $\alpha \approx 0.13682853547099\ldots$. Notice that

$$H'(x) = 1 - \beta\cos(x),$$

so

$$\begin{aligned} |H'(\alpha)^{-1}(H'(x) - H'(y))| &= \frac{|\beta(\cos(x) - \cos(y))|}{|1 - \beta\cos(\alpha)|} \\ &= \frac{2\beta\big|\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})\big|}{|1 - \beta\cos(\alpha)|} \\ &\le \frac{\beta}{|1 - \beta\cos(\alpha)|}|x - y| \end{aligned}$$

and

$$|H'(\alpha)^{-1}H'(x)| = \frac{|1 - \beta\cos(x)|}{|1 - \beta\cos(\alpha)|} \le \frac{1 + \beta}{|1 - \beta\cos(\alpha)|}.$$

Then we can choose $v_0(t) = v(t) = 0.3685888t$ and $v_1(t) = 1.7337327$. The calculated values of parameters are given by

$\delta_1 = 7.6343\times10^{-1}$, $\delta_2 = 4.6114\times10^{-1}$, $\delta_3 = 4.2457\times10^{-1}$, $\delta = 4.2457\times10^{-1}$,
$\bar{\delta}_1 = 7.6343\times10^{-1}$, $\bar{\delta}_2 = 5.1284\times10^{-1}$, $\bar{\delta}_3 = 3.2891\times10^{-1}$, $\bar{\delta} = 3.2891\times10^{-1}$.

Example 4. Consider $C[0,1] = Y = X$ and $D = \bar{B}(0,1)$. Define a function H on D by

$$H(\varphi)(x) = \varphi(x) - 10\int_0^1 x\theta\varphi(\theta)^3\,d\theta.$$

Then

$$H'(\varphi(\xi))(x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)\,d\theta \qquad \text{for each } \xi \in D.$$

Since $\alpha = 0$, we can choose $v_0(t) = 15t, v(t) = 30t, v_1(t) = 30$. Then we obtain

$\delta_1 = 3.0000\times10^{-1}$, $\delta_2 = 1.9135\times10^{-4}$, $\delta_3 = 1.5863\times10^{-4}$, $\delta = 1.5863\times10^{-4}$,
$\bar{\delta}_1 = 3.0000\times10^{-1}$, $\bar{\delta}_2 = 7.2968\times10^{-3}$, $\bar{\delta}_3 = 1.4109\times10^{-3}$, $\bar{\delta} = 1.4109\times10^{-3}$.

Example 5. Consider the Hammerstein equation

$$x(s) = \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt, \tag{3.1}$$

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

Clearly, we have $\alpha(s) = 0$. Define $H : D \subseteq C[0,1] \to C[0,1]$ by

$$H(x)(s) = x(s) - \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt.$$

Observe that

$$\left\|\int_0^1 G(s,t)dt\right\| \le 1/8.$$

Since

$$H'(x)y(s) = y(s) - \int_0^1 G(s,t)\big(\tfrac{3}{2}x(t)^{1/2} + x(t)\big)\,dt,$$

and $H'(\alpha(s)) = 1$, we have

$$\|H'(\alpha)^{-1}(H'(x) - H'(y))\| \le \tfrac{5}{16}\|x - y\|. \tag{3.2}$$

Replacing y by x_0 we have

$$\|H'(\alpha)^{-1}(H'(x) - H'(x_0))\| \le \tfrac{5}{16}\|x - x_0\|.$$

Therefore, we can take

$$v_0(x) = v(x) = L_0 x, \quad v_1(x) = Lx, \quad \text{where} \quad L_0 = L = \tfrac{5}{16}.$$

Hence, we obtain

$$\begin{aligned}
&\delta_1 = 1.7454, \quad \delta_2 = 1.2208, \quad \delta_3 = 1.1917, \quad \delta = 1.1917,\\
&\bar{\delta}_1 = 1.7454, \quad \bar{\delta}_2 = 1.0594, \quad \bar{\delta}_3 = 9.8115 \times 10^{-1}, \quad \bar{\delta} = 9.8115 \times 10^{-1}.
\end{aligned}$$

EXAMPLE 6. Lastly, we intend to show the sixth order convergence of the methods under study by calculating the approximate computational order of convergence (ACOC) using formula (2.27). To do that, we apply the methods (1.2) and (1.3) to solve systems of nonlinear equations in $\mathbb{R}^m$. Computations are performed in *Mathematica* using multiple-precision arithmetic. For every method, we record the number n of iterations needed for the stopping criterion

$$\|H(x_n)\| < 10^{-350}$$

to be satisfied. This precision is required to compute the approximate computational order of convergence (ACOC) of higher order methods. This is because higher order methods gain a large number of significant digits of a solution in just a few iterations. The command used to get such precision is N[expression,n], which gives the numerical value of the expression with n-digit precision.

Numerical results are displayed in Table 1, and include:

- The required number n of iterations.
- The value of $\|H(x_n)\|$ of approximation to the corresponding solution, where $N(-h)$ denotes $N \times 10^{-h}$.
- The approximate computational order of convergence (ACOC).
- The elapsed CPU-time in seconds recorded by taking the mean of 50 performances of the program.

Let us consider the system of nonlinear equations

$$\begin{cases} x_i^2 x_{i+1} - 1 = 0, & 1 \le i \le m-1, \\ x_i^2 x_1 - 1 = 0, & i = m. \end{cases} \tag{3.3}$$

with initial value $x_0 = \{2, \overset{m\,\text{times}}{\cdots}, 2\}^T$. The required solution of the systems for $m = 8, 25, 50, 100$ is $\alpha = \{1, \overset{m\,\text{times}}{\cdots}, 1\}^T$.

From the numerical results shown in Table 1 it is clear that methods (1.2) and (1.3) have stable convergence behavior. Similar numerical tests, carried out for a number of other problems, confirmed the above conclusions to a large extent.

Table 1. Performance of methods

Methods	n	$\|H(x_n)\|$	ACOC	CPU-time
$m = 8$				
(1.2)	4	1.2961(−304)	6	0.0475
(1.3)	4	1.1849(−168)	6	0.0472
$m = 25$				
(1.2)	4	2.2913(−304)	6	0.1420
(1.3)	4	2.0947(−168)	6	0.0939
$m = 50$				
(1.2)	4	3.2404(−304)	6	0.4521
(1.3)	4	2.9624(−168)	6	0.2187
$m = 100$				
(1.2)	4	4.5826(−304)	6	0.9357
(1.3)	4	4.1894(−168)	6	0.4722

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