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EXTENDING THE APPLICABILITY OF A SEVENTH-ORDER METHOD FOR EQUATIONS UNDER GENERALIZED CONDITIONS

Abstract. We extend the applicability of a seventh-order method for solving Banach space-valued equations. This is achieved by using generalized conditions on the first derivative which only appears in the method. Earlier works use conditions up to the eighth derivative to establish convergence. Our technique is very general and can be used to extend the applicability of other methods along the same lines.

1. Introduction. Let E_1 and E_2 be Banach spaces and $D \subset E_1$ be an open and convex set. We are concerned with the problem of approximating a solution x_* of the equation

$$\mathcal{G}(x) = 0, \tag{1.1}$$

where $\mathcal{G} : D \subset E_1 \to E_2$ is a nonlinear Fréchet differentiable operator. Interested readers can find related results in [1–19], and the references therein.

We study the local convergence of the seventh-order method defined by

$$\begin{aligned} y_k &= x_k - \tfrac{2}{3}\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k), \\ z_k &= x_k - \big[\tfrac{23}{8}I - \mathcal{G}'(x_k)^{-1}\mathcal{G}'(y_k)\big(3I - \tfrac{9}{8}\mathcal{G}'(x_k)^{-1}\mathcal{G}'(y_k)\big)\big]\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k) \\ x_{k+1} &= z_k - (5I - 3\mathcal{G}'(x_k)^{-1}\mathcal{G}'(y_k))\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k) \end{aligned} \tag{1.2}$$

for each $k = 0, 1, 2, \ldots$.

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The advantages of using this method are given in [17]. The convergence order was established in that work by using Taylor expansions and hypotheses up to the eighth derivative and in the setting of a multidimensional Euclidean space. Notice, however, that only the first derivative appears in the method. Hence, method (1.2) can be used under these restrictions only when these high-order derivatives exist. But method (1.2) may converge. That is why it is important to drop these conditions.

The assumptions on derivatives of order up to 8 reduce the applicability of the method. For example: Let $X = Y = \mathbb{R}$, $D = [-1/2, 3/2]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then $f(1) = 0$, and

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

But the function $f'''(t)$ is not bounded on D. Thus, the convergence of method (1.2) is not guaranteed by the analysis in [17]. Moreover, computable error bounds on $\|x_n - x_*\|$ or uniqueness results were not given there either. We address all these problems in this paper. In particular, we only use hypotheses on the first derivative. Moreover, the convergence is given in a Banach space setting. Furthermore, computable error bounds as well as the uniqueness of the solution results are provided. Hence, we extend the applicability of the method.

Finally, our approach is so general that it can be used to extend the applicability of other methods [1–16, 18, 19] in a similar way.

The local convergence analysis is developed in Section 2. Moreover, numerical examples can be found in Section 3.

2. Convergence. We denote by $S[z, \gamma]$ the closure of the ball $S(z, \gamma)$ centered at $z \in E_1$ and of radius $\gamma > 0$.

We first introduce some real parameters and functions to be used in the local convergenceanalysis of method (1.2). Let $B = [0, \infty)$.

Suppose the following:

(1) $\xi_0 : B \to B$ is a continuous and nondecreasing (CN) function such that the function $\xi_0(t) - 1$ has a minimal zero $\rho_0 \in B - \{0\}$. Set $B_0 = [0, \rho_0)$.
(2) $\xi : B_0 \to B$ and $\xi_1 : B_0 \to B$ are CN functions such that the function $\eta_1(t) - 1$ has a minimal zero $\rho_1 \in B_0 - \{0\}$, where $\eta_1 : B_0 \to B$ is defined by

$$\eta_1(t) = \frac{\int_0^1 \xi((1-\tau)t)\, d\tau + \frac{1}{3}\int_0^1 \xi_1(\tau t)\, d\tau}{1 - \xi_0(t)}.$$

(3) The function $\eta_2(t)-1$ has a minimal zero $\rho_2 \in B_0 - \{0\}$, where $\eta_2 : B_0 \to B$ is defined by

$$\eta_2(t) = \frac{\int_0^1 \xi((1-\tau)t)\,d\tau}{1-\xi_0(t)} + \frac{3}{8}\left(3\left(\frac{\xi_0(t)+\xi_0(\eta_1(t)t)}{1-\xi_0(t)}\right)^2 + 2\frac{\xi_0(t)+\xi_0(\eta_1(t)t)}{1-\xi_0(t)}\right)\frac{\int_0^1 \xi_1(\tau t)\,d\tau}{1-\xi_0(t)}.$$

(4) The function $\xi_0(\eta_2(t)t)-1$ has a minimal zero $r_1 \in B_0 - \{0\}$. Let $r = \min\{\rho_0, r_1\}$ and $B_1 = [0, r)$.

(5) The function $\eta_3(t)-1$ has a minimal zero $\rho_3 \in B_1 - \{0\}$, where $\eta_3 : B_1 \to B$ is defined by

$$\eta_3(t) = \Bigg[\frac{\int_0^1 \xi((1-\tau)\eta_2(t)t)\,d\tau}{1-\xi_0(\eta_2(t)t)} + \frac{(\xi_0(t)+\xi_0(\eta_2(t)t))\int_0^1 \xi_1(\tau\eta_2(t)t)\,d\tau}{(1-\xi_0(t))(1-\xi_0(\eta_2(t)t))} + \frac{3(\xi_0(t)+\xi_0(\eta_1(t)t))\int_0^1 \xi_1(\tau\eta_2(t)t)\,d\tau}{2(1-\xi_0(t))^2}\Bigg]\eta_2(t).$$

The parameter ρ defined by

$$\rho = \min\{\rho_m : m = 1,2,3\} \tag{2.1}$$

will be shown in Theorem 2.1 to be a radius of convergence for method (1.2).

Let $B_2 = [0, \rho)$. Then

$$0 \le \xi_0(t) < 1, \tag{2.2}$$

$$0 \le \xi_0(\eta_2(t)t) < 1, \tag{2.3}$$

$$0 \le \eta_m(t) < 1 \tag{2.4}$$

for $m = 1,2,3$ and each $t \in B_2$.

The following conditions are needed for functions η_m as previously defined, and x_* a simple solution of the equation $\mathcal{G}(x) = 0$.

(h1) For each $v \in D$,

$$\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(v)-\mathcal{G}'(x_*))\| \le \xi_0(\|v-x_*\|).$$

Set $D_0 = S[x_*, \rho_0] \cap D$.

(h2) For each $v, u \in D_0$,

$$\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(v)-\mathcal{G}'(u))\| \le \xi(\|v-u\|),$$

$$\|\mathcal{G}'(x_*)^{-1}\mathcal{G}'(v)\| \le \xi_1(\|v-x_*\|).$$

(h3) $S[x_*, \rho] \subset D$.

Next, we prove a local convergence result utilizing conditions (h1)–(h3) as well as the preceding notation.

Theorem 2.1. *Suppose that conditions* (h1)–(h3) *hold. Then*

$$\lim_{n\to\infty} x_n = x_* \quad \textit{provided} \quad x_0 \in S(x_*,\rho).$$

Proof. The following items are proven by induction:

$$\{x_n\} \subset S(x_*,\rho), \tag{2.5}$$

$$\|y_n - x_*\| \le \eta_1(q_n)q_n \le q_n < r, \tag{2.6}$$

$$\|z_n - x_*\| \le \eta_2(q_n)q_n \le q_n, \tag{2.7}$$

$$q_{n+1} \le \eta_3(q_n)q_n \le q_n, \tag{2.8}$$

where $q_n = \|x_n - x_*\|$ and ρ is given in (2.1). Let $y \in S(x_*,\rho) - \{x_*\}$. By condition (h1) and (2.1), we get

$$\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(y) - \mathcal{G}'(x_*))\| \le \xi_0(\|y - x_*\|) \le \xi_0(\rho) < 1. \tag{2.9}$$

Thus, $\mathcal{G}'(y)^{-1} \in L(E_1, E)$ by a perturbation lemma for invertible linear operators [10] attributed to Banach, and

$$\|\mathcal{G}'(y)^{-1}\mathcal{G}'(x_*)\| \le \frac{1}{1 - \xi_0(\|y - x_*\|)}. \tag{2.10}$$

The values y_0 and z_0 are also well defined by the first two substeps of the scheme (1.2) for $n = 0$. Then, we can write

$$\begin{aligned} y_0 - x_* &= x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + \tfrac{1}{3}\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\ &= \mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_*)\int_0^1 \mathcal{G}'(x_*)^{-1}\big(\mathcal{G}'(x_* + \tau(x_0 - x_*)) - \mathcal{G}'(x_0)\big)\, d\tau\, (x_0 - x_*) \\ &\quad + \tfrac{1}{3}\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} z_0 - x_* &= x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\ &\quad - \big(\tfrac{15}{8}I - 3\mathcal{G}'(x_0)^{-1}\mathcal{G}'(y_0) + \tfrac{9}{8}(\mathcal{G}'(x_0)^{-1}\mathcal{G}'(y_0))^2\big)\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\ &= x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\ &\quad - \tfrac{3}{8}[3\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(y_0) - \mathcal{G}'(x_0))^2 \\ &\quad - 2\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(y_0) - \mathcal{G}'(x_0))]\mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_0). \end{aligned} \tag{2.12}$$

Using (2.1), (2.4) (for $m = 1, 2$), conditions (h2), (2.10) (for $y = x_0, y_0$), (2.11), (2.12), and the triangle inequality we obtain in turn

$$\begin{aligned} \|y_0 - x_*\| &\le \frac{(\int_0^1 \xi((1-\tau)q_0)\, d\tau + \frac{1}{3}\int_0^1 \xi_1(\tau q_0)\, d\tau)q_0}{1 - \xi_0(q_0)} \\ &\le \eta_1(q_0)q_0 \le q_0 < \rho, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|z_0 - x_*\| \le \Bigg[&\frac{\int_0^1 \xi((1-\tau)q_0)\,d\tau}{1-\xi_0(q_0)} \\ &+ \frac{3}{8}\Bigg(3\Bigg(\frac{(\xi_0(q_0)+\xi_0(\|y_0-x_*\|))}{1-\xi_0(q_0)}\Bigg)^2 \\ &+ 2\frac{(\xi_0(q_0)+\xi_0(\|y_0-x_*\|))}{1-\xi_0(q_0)}\Bigg)\frac{\int_0^1 \xi_1(\tau q_0)\,d\tau}{1-\xi_0(q_0)}\Bigg] q_0 \\ \le{}& \eta_2(q_0)q_0 \le q_0,\end{aligned} \tag{2.14}$$

proving $y_0, z_0 \in S(x_*, \rho) \subset D$ (by condition (h3)) and the estimates (2.6) and (2.7), respectively. Moreover, by condition (h2), estimates (2.13), (2.14), the triangle inequality, and the third substep of method (1.2) for $n = 0$ defining x_1 we have

$$\begin{aligned}x_1 - x_* = {}& z_0 - x_* - \mathcal{G}'(z_0)^{-1}\mathcal{G}(z_0) \\ &+ \mathcal{G}'(z_0)^{-1}(\mathcal{G}'(x_0) - \mathcal{G}'(z_0))\mathcal{G}'(x_0)^{-1}\mathcal{G}(z_0) \\ &- \tfrac{3}{2}\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_0) - \mathcal{G}'(y_0))\mathcal{G}'(x_0)^{-1}\mathcal{G}(z_0).\end{aligned} \tag{2.15}$$

Hence, it follows that

$$\begin{aligned}q_1 \le \Bigg[&\frac{\int_0^1 \xi((1-\tau)\|z_0-x_*\|)\,d\tau}{1-\xi_0(\|z_0-x_*\|)} \\ &+ \frac{(\xi_0(q_0)+\xi_0(\|z_0-x_*\|))}{(1-\xi_0(q_0))(1-\xi_0(\|z_0-x_*\|))}\int_0^1 \xi_1(\tau\|z_0-x_*\|)\,d\tau \\ &+ \frac{3}{2}\frac{(\xi_0(q_0)+\xi_0(\|y_0-x_*\|))}{1-\xi_0(q_0)}\int_0^1 \xi_1(\tau\|z_0-x_*\|)\,d\tau\Bigg]\|z_0-x_*\| \\ \le{}& \eta_3(q_0)q_0 \le q_0,\end{aligned} \tag{2.16}$$

proving that $x_1 \in S(x_*, \rho)$ and estimates (2.5) and (2.8) hold for $n = 1$ and $n = 0$, respectively. Hence, items (2.5)–(2.8) hold for $n = 0$. Then, replace x_0, y_0, z_0, x_1 by x_j, y_j, z_j, x_{j+1}, respectively in the preceding calculations to complete the induction for items (2.5)–(2.8). It follows from the estimation

$$q_{j+1} \le \lambda q_j < \rho, \tag{2.17}$$

where $\lambda = \eta_3(q_0) \in [0, 1)$, that $x_{j+1} \in S(x_*, \rho)$ and $\lim_{j\to\infty} x_j = x_*$. ∎

Next, a result is given concerning the uniqueness of the solution x_*.

Proposition 2.2. *Suppose:*

(1) *$x_* \in S(x_*, r) \subset D$ is a simple solution of the equation $\mathcal{G}(x) = 0$ for some $r > 0$, and condition* (h1) *holds.*

(2) *There exists $\rho_* \geq r$ such that*

$$\int_0^1 \xi_0(\tau\rho_*)\,d\tau < 1. \tag{2.18}$$

Set $D_1 = S[x_*, \rho_*] \cap D$. Then x_* is the only solution of the equation $\mathcal{G}(x) = 0$ in D_1.

Proof. Let $T = \int_0^1 \mathcal{G}'(v + \tau(x_* - v))\,d\tau$ for some $v \in D_1$ with $\mathcal{G}(v) = 0$. Then it follows from (h1) and (2.18) that

$$\begin{aligned} \|\mathcal{G}'(x_*)^{-1}(T - \mathcal{G}'(x_*))\| &\leq \int_0^1 \xi_0(\tau\|v - x_*\|)\,d\tau \\ &\leq \int_0^1 \xi_0(\tau\rho_*)\,d\tau. \end{aligned} \tag{2.19}$$

Hence, $T^{-1} \in L(E_1, E)$ and from the identity $0 = \mathcal{G}(v) - \mathcal{G}(x_*) = T(v - x_*)$, we conclude $v = x_*$. ∎

Remark 2.3. (a) Notice that only condition (h1) is used in Proposition 2.2. But if we suppose that all the conditions (h1)–(h3) hold, then we can set $r = \rho$.

(b) Given (h1) and the estimate

$$\begin{aligned} \|\mathcal{G}'(x_*)^{-1}\mathcal{G}'(x)\| &= \|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(x_*)) + I\| \\ &\leq 1 + \|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(x_*))\| \leq 1 + \xi_0(\|x - x_*\|) \end{aligned}$$

the second condition in (h2) can be dropped and ξ_1 can be replaced by

$$\xi_1(t) = 1 + \xi_0(t)$$

or

$$\xi_1(t) = 2,$$

since $t \in [0, \rho_0)$.

(c) The results obtained here can be used for operators $\mathcal{G}$ satisfying autonomous differential equations [1–4] of the form

$$\mathcal{G}'(x) = P(\mathcal{G}(x))$$

where P is a continuous operator. Then, since $\mathcal{G}'(x_*) = P(\mathcal{G}(x_*)) = P(0)$, we can apply the results without actually knowing x_*. For example, let $\mathcal{G}(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

(d) Let $\xi_0(t) = K_0 t$ and $\xi(t) = Kt$. In [2,3] we showed that $\rho_A = \frac{2}{2K_0 + K}$ is a convergence radius of Newton's method

$$x_{n+1} = x_n - \mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n) \quad \text{for each } n = 0, 1, 2, \ldots \tag{2.20}$$

under conditions (h1) and (h2). It follows from the definition of ρ in (2.1) that it cannot be larger than the convergence radius ρ_A of the second order

Newton's method (2.20). As already noted in [2,3], ρ_A is at least as large as the convergence radius given by Rheinboldt [15],

$$\rho_R = \frac{2}{3K_1}, \tag{2.21}$$

where K_1 is the Lipschitz constant on D. The same value for ρ_R was given by Traub [18]. In particular, for $K_0 < K_1$ we have

$$\rho_R < \rho_A$$

and

$$\rho_R/\rho_A \to 1/3 \quad \text{as } K_0/K_1 \to 0.$$

That is, the radius of convergence ρ_A is at most three times larger than Rheinboldt's.

3. Numerical examples. We compute the radius of convergence for three examples.

Example 3.1. Consider the nonlinear system

$$\begin{aligned} x^2 + y - 2 &= 0, \\ x + y^2 - 2 &= 0. \end{aligned}$$

The associated nonlinear mapping $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathcal{G}(x, y) = \begin{bmatrix} \mathcal{G}_1(x, y) \\ \mathcal{G}_2(x, y) \end{bmatrix}$$

where $\mathcal{G}_1(x, y) = x^2 + y - 2$ and $\mathcal{G}_2(x, y) = x + y^2 - 2$. Notice that $x_* = (1, 1)^T$. Set $D = S(x_*, 1/2)$. We use the max norm for matrices. By the definition of the mapping $\mathcal{G}$, we have

$$\mathcal{G}'(x, y) = \begin{bmatrix} \partial\mathcal{G}_1/\partial x & \partial\mathcal{G}_1/\partial y \\ \partial\mathcal{G}_2/\partial x & \partial\mathcal{G}_2/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 1 \\ 1 & 2y \end{bmatrix}$$

so $\mathcal{G}'(x_*) = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and $\mathcal{G}'^{-1}(x_*) = \frac{1}{3}\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$. In view of these calculations, conditions (h1)–(h3) are satisfied if we set $\xi_0(t) = \xi(t) = 2t$, and $\xi_1(t) = 2$. Then, by the definition of η_1, η_2, η_3 and the radius ρ given by (2.1), we obtain

$$\rho_1 = 0.2222, \quad \rho_2 = 0.6494, \quad \rho = \rho_3 = 0.2141.$$

Example 3.2. Let $E = E_1 = C[0, 1]$, $D = S[0, 1]$ and define the operator $\mathcal{G} : D \to E_2$ by

$$\mathcal{G}(\psi)(x) = \psi(x) - 5\int_0^1 x\theta\psi(\theta)^3\, d\theta. \tag{3.1}$$

Then we obtain

$$\mathcal{G}'(\psi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\psi(\theta)^2\xi(\theta)\,d\theta \quad \text{for each } \xi \in D.$$

Thus, we get $x_* = 0$, so that we can take $\xi_0(t) = 7.5t$, $\xi(t) = 15t$, and $\xi_1(t) = 2$. Then the radii are

$$\rho_1 = 0.022222, \quad \rho_2 = 0.01894838, \quad \rho = \rho_3 = 0.01546.$$

EXAMPLE 3.3. In the academic example of the introduction, we take $\xi_0(t) = \xi(t) = 96.6629073t$ and $\xi_1(t) = 2$. Then the radii are

$$\rho_1 = 0.00229894, \quad \rho_2 = 0.00158586, \quad \rho = \rho_3 = 0.00128222.$$

References

[1] I. K. Argyros, *Computational Theory of Iterative Methods*, Stud. Comput. Math. 15, Elsevier, New York, 2007.

[2] I. K. Argyros, *Unified convergence criteria for iterative Banach space valued methods with applications*, Mathematics 9 (2021), art. 1942, 15 pp.

[3] I. K. Argyros and Á. A. Magreñán, *A Contemporary Study of Iterative Methods*, Elsevier/Academic Press, New York, 2018.

[4] I. K. Argyros and Á. A. Magreñán, *Iterative Methods and their Dynamics with Applications*, CRC Press, New York, 2017.

[5] R. Behl, S. Bhalla, E. Martínez and M. A. Alsulami, *Derivative-free King's scheme for multiple zeros of nonlinear functions*, Mathematics 9 (2021), art. 1242, 14 pp.

[6] R. Behl, S. Bhalla, Á. A. Magreñán and A. Moysi, *An optimal derivative free family of Chebyshev–Halley's method for multiple zeros*, Mathematics 9 (2021), art. 546, 19 pp.

[7] A. Cordero, C. Jordán, E. Sanabria-Codesal and J. R. Torregrosa, *Highly efficient algorithms for solving nonlinear systems with arbitrary order of convergence $p + 3$, $p \geq 5$*, J. Comput. Appl. Math. 330 (2018), 748–758.

[8] J. A. Ezquerro, M. A. Hernández, N. Romero and A. I. Velasco, *On Steffensen's method on Banach spaces*, J. Comput. Appl. Math. 249 (2013), 9–23.

[9] M. A. Hernández and M. J. Rubio, *A uniparametric family of iterative processes for solving nondifferentiable equations*, J. Math. Anal. Appl. 275 (2002), 821–834.

[10] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, 2nd ed., Pergamon Press, Oxford, 1982.

[11] H. T. King and J. F. Traub, *Optimal order of one-point and multipoint iteration*, J. ACM 21 (1974), 643–651.

[12] Á. A. Magreñán, I. K. Argyros, J. J. Rainer and J. A. Sicilia, *Ball convergence of a sixth-order Newton-like method based on means under weak conditions*, J. Math. Chem. 56 (2018), 2117–2131.

[13] Á. A. Magreñán and J. M. Gutiérrez, *Real dynamics for damped Newton's method applied to cubic polynomials*, J. Comput. Appl. Math. 275 (2015), 527–538.

[14] H. Ren, Q. Wu and W. Bi, *A class of two-step Steffensen type methods with fourth-order convergence*, Appl. Math. Comput. 209 (2009), 206–210.

[15] W. C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, in: Mathematical Models and Numerical Methods, A. N. Tikhonov et al. (eds.), Banach Center Publ. 3, Polish Sci. Publ., Warszawa, 1977, 129–142.

[16] J. R. Sharma and H. Arora, *A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations*, Numer. Algorithms 4 (2014), 917–933.

[17] J. R. Sharma and H. Arora, *Efficient Jarratt-like methods for solving systems of nonlinear equations*, Calcolo 51 (2014), 193–210.

[18] J. F. Traub, *Iterative Methods for the Solution of Equations*. Prentice-Hall, Englewood Cliffs, NJ, 1964.

[19] X. Y. Xiao and H. W. Yin, *Increasing the order of convergence for iterative methods to solve nonlinear systems*, Calcolo 53 (2016), 285–300.

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