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EXTENDED EFFICIENT HIGH CONVERGENCE ORDER SCHEMES FOR EQUATIONS

Abstract. We investigate the ball of convergence using only the first derivative for two sixth order algorithms for solving equations that are run under the equal set of circumstances. In addition, we provide a calculable ball comparison between the two schemes under consideration. Our technique is based on the first derivative that only appears in the schemes.

1. Introduction. Let us consider a Fréchet differentiable operator $\Phi : \Omega \subseteq X \to Y$, where X, Y are Banach spaces and Ω ($\neq \emptyset$) is convex and open. In science and other practical fields, equations of the type

$$\Phi(v) = 0 \tag{1.1}$$

are regularly used to address a wide range of problems. Obtaining solutions to such equations is a complicated process. The solutions can only be found analytically in a small number of cases. As a result, iterative procedures are often employed to solve these equations. It is, however, a challenging task to create an effective iterative strategy for (1.1). The traditional Newton's iterative approach is most often considered to solve such problems. In addition, a large amount of research on higher order modifications of conventional processes such as Newton's, Chebyshev's, Jarratt's, etc. has been conducted [1–4, 6–9, 11, 19, 21, 22, 26, 28, 33].

A number of writers, for example, Homeier [20], Frontini and Sormani [17], Cordero and Torregrosa [12], Noor and Waseem [25] and Grau et al. [18],

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created third convergence order techniques, each of which requires one Φ and two Φ' evaluations. In [13], two cubically convergent iterative procedures were designed by Cordero and Torregrosa. Another third order convergent scheme based on two evaluations of Φ, one of Φ' and one matrix inversion was presented by Darvishi and Barati [15]. In addition, Cordero et al. [14] extended Jarratt's scheme [22] for addressing nonlinear systems. Grau et al. [18] and Darvishi and Barati [16] also suggested schemes with convergence order 4. Sharma et al. [31] composed two weighted-Newton steps to generate an efficient fourth order weighted-Newton scheme for nonlinear systems. Also, fourth and sixth order convergent iterative algorithms were developed by Sharma and Arora [30] to solve nonlinear systems. Other results on different iterative processes together with their convergence ball and dynamical behaviors are discussed in [5, 10, 23, 24, 27, 29, 32].

The main purpose of this article is to increase the usefulness of the sixth convergence order schemes from [30] and [34], respectively. In addition, we compare their convergence balls. The schemes we are going to study are

$$\begin{aligned} y_n &= v_n - \gamma\Phi'(v_n)^{-1}\Phi(v_n),\\ z_n &= v_n - \big(\tfrac{23}{8}I - 3\Phi'(v_n)^{-1}\Phi'(y_n)\\ &\qquad + \tfrac{9}{8}(\Phi'(v_n)^{-1}\Phi'(y_n))\big)\Phi'(v_n)^{-1}\Phi(v_n),\\ v_{n+1} &= z_n - \big(\tfrac{5}{2}I - \tfrac{3}{2}\Phi'(v_n)^{-1}\Phi'(y_n)\big)\Phi'(v_n)^{-1}\Phi(z_n), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= v_n - \gamma\Phi'(v_n)^{-1}\Phi(v_n),\\ z_n &= v_n - \big(I + \tfrac{21}{8}\Phi'(v_n)^{-1}\Phi'(y_n) - \tfrac{9}{2}(\Phi'(v_n)^{-1}\Phi'(y_n))^2\\ &\qquad + \tfrac{15}{8}(\Phi'(v_n)^{-1}\Phi'(y_n))^3\big)\Phi'(v_n)^{-1}\Phi(v_n),\\ v_{n+1} &= z_n - \Big(3I - \tfrac{5}{2}\Phi'(v_n)^{-1}\Phi'(y_n)\\ &\qquad + \big(\tfrac{1}{2}\Phi'(v_n)^{-1}\Phi'(y_n)\big)^2\Big)\Phi'(v_n)^{-1}\Phi(z_n). \end{aligned} \tag{1.3}$$

If $\gamma = \frac{2}{3}$, schemes (1.2) and (1.3) are reduced to the schemes designed in [30] and [34], respectively. The convergence of these schemes has been shown by applying the expensive Taylor formula, which reduces their scope of utility. We consider the following function to explain our viewpoint:

$$\Phi(v) = \begin{cases} v^3\ln(v^2) + v^5 - v^4 & \text{if } v \neq 0,\\ 0 & \text{if } v = 0, \end{cases} \tag{1.4}$$

where $X = Y = \mathbb{R}$ and F is defined on $\Omega = \big[-\frac{1}{2}, \frac{3}{2}\big]$. Then the unboundedness of Φ''' makes the earlier convergence theorems ineffective for schemes (1.2) and (1.3). Also, existing results provide little information regarding the bounds on error, domain of convergence, or the location of the solution. It is critical to conduct the ball analysis of an iterative scheme in detail in order to determine convergence radii, approximate error bounds, and the region where x_* is the only solution. Another benefit of this analysis is that it simplifies the difficult task of selecting v_0. This motivates us to investigate and compare the convergence balls of (1.2) and (1.3) when subjected to the identical set of constraints. In addition to providing an error estimate $\|v_n - x_*\|$ and the convergence radii, the convergence theorems that we present also offer a precise location for the solution.

The following is a summary of the contents of this article: Section 2 contains the ball of convergence of schemes (1.2) and (1.3). Section 3 includes the numerical experiments.

2. Ball of convergence. Some scalar parameters and functions will be introduced for the ball of convergence analysis, first for scheme (1.2). Set $T = [0, \infty)$.

Suppose the following:

(1) $\omega_0 : T \to T$ is a continuous non-decreasing function such that $\omega_0(t) - 1$ has a smallest root $R_0 \in T_0 \setminus \{0\}$. Set $T_0 = [0, R_0)$.

(2) $\omega : [0, 2R_0) \to T$, $\omega_1 : T_0 \to T$ are continuous non-decreasing functions such that $g_1(t) - 1$ has a smallest root $r_1 \in T_0 \setminus \{0\}$, where $g_1 : T_0 \to T$ is given by

$$g_1(t) = \frac{\int_0^1 \omega((1-\theta)t)\ d\theta + |1-\gamma| \int_0^1 \omega_1(\theta t)\ d\theta}{1 - \omega_0(t)}.$$

(3) $g_2 : T_0 \to T$ is such that $g_2(t) - 1$ has a smallest root $r_2 \in T_0 \setminus \{0\}$, where

$$\begin{aligned} g_2(t) = {} & \frac{\int_0^1 \omega((1-\theta)t)\, d\theta}{1 - \omega_0(t)} \\ & + \frac{3}{8}\left(3\left(\frac{\omega_0(t) + \omega_0(g_1(t)t)}{1 - \omega_0(t)}\right)^2 \right. \\ & \left. + 2\frac{\omega_0(t) + \omega_0(g_1(t)t)}{1 - \omega_0(t)}\right) \frac{\int_0^1 \omega_1(\theta t)\, d\theta}{1 - \omega_0(t)}. \end{aligned}$$

(4) $\omega_0(g_2(t)t) - 1$ has a smallest root $R_1 \in T_0 \setminus \{0\}$. Set $R = \min\{R_0, R_1\}$ and $T_1 = [0, R)$.

(5) $g_3 : T_1 \to T$ is such that $g_3(t) - 1$ has a smallest root $r_3 \in T_1 \setminus \{0\}$, where

$$\begin{aligned} g_3(t) = {} & \left[\frac{\int_0^1 \omega((1-\theta)g_2(t)t)\, d\theta}{1 - \omega_0(g_2(t)t)} + \frac{(\omega_0(t) + \omega_0(g_2(t)t)) \int_0^1 \omega_1(\theta g_2(t)t)\, d\theta}{(1 - \omega_0(t))(1 - \omega_0(g_2(t)))} \right. \\ & \left. + \frac{3(\omega_0(t) + \omega_0(g_2(t)t)) \int_0^1 \omega_1(\theta g_2(t)t)\, d\theta}{(1 - \omega_0(t))^2}\right] g_2(t). \end{aligned}$$

The parameter r defined by

$$r = \min\{r_i : i = 1, 2, 3\}, \tag{2.1}$$

will be shown to be a convergence radius for scheme (1.2). Set $M = [0, r)$. By the definition of r it follows that for all $t \in M$,

$$0 \le \omega_0(t) < 1, \tag{2.2}$$

$$0 \le \omega_0(g_2(t)t) < 1, \tag{2.3}$$

$$0 \le g_i(t) < 1. \tag{2.4}$$

The notation $\overline{B}(x_*, \lambda)$ is used for the closure of the ball $B(x_*, \lambda)$ with radius $\lambda > 0$ and center $x_* \in \Omega$.

We suppose from now on that x_* is a simple root of Φ, the "ω" functions are as defined previously, and the following *conditions* (A) hold:

(A_1) For all $x \in \Omega$,

$$\|\Phi'(x_*)^{-1}(\Phi'(x) - \Phi'(x_*))\| \le \omega_0(\|x - x_*\|).$$

(A_2) Set $\Omega_0 = \Omega \cap B(x_*, R_0)$. For all $x, y \in \Omega_0$,

$$\|\Phi'(x_*)^{-1}(\Phi'(x) - \Phi'(y))\| \le \omega(\|x - y\|),$$
$$\|\Phi'(x_*)^{-1}\Phi'(x)\| \le \omega_1(\|x - x_*\|).$$

(A_3) $\overline{B}(x_*, \tilde{r}) \subset \Omega$ for some $\tilde{r}$ to be defined later.

(A_4) There exists $r_* \ge \tilde{r}$ satisfying

$$\int_0^1 \omega_0(\theta r_*)\, d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \overline{B}(x_*, r_*)$. Next, we state a result on the ball of convergence for scheme (1.2) utilizing conditions (A).

THEOREM 2.1. *Suppose conditions* (A_1)–(A_4) *hold for* $\tilde{r} = r$. *Then sequence* $\{v_n\}$ *given by scheme* (1.2) *is well defined in* $B(x_*, r)$, *stays in* $B(x_*, r)$ *and converges to* x_* *provided that* $v_0 \in B(x_*, r) \setminus \{x_*\}$. *Moreover,*

$$\|y_n - x_*\| \le g_1(\|v_n - x_*\|)\|v_n - x_*\| \le \|v_n - x_*\| < r, \tag{2.5}$$

$$\|z_n - x_*\| \le g_2(\|v_n - x_*\|)\|v_n - x_*\| \le \|v_n - x_*\|, \tag{2.6}$$

$$\|v_{n+1} - x_*\| \le g_3(\|v_n - x_*\|)\|v_n - x_*\| \le \|v_n - x_*\|, \tag{2.7}$$

where the functions g_i *and the radius* r *are as defined previously. Furthermore, the only root of* $\Phi(v) = 0$ *in the set* Ω_1 *defined in* (A_4) *is* x_*.

Proof. Let $z \in B(x_*, r) \setminus \{x_*\}$. Using ($A_1$), (2.1) and (2.2), we get

$$\|\Phi'(x_*)^{-1}(\Phi'(z) - \Phi'(x_*))\| \le \omega_0(\|z - x_*\|) \le \omega_0(r) < 1,$$

which together with a lemma due to Banach on invertible operators [28] implies $\Phi'(z)^{-1} \in L(Y, X)$ with

$$\|\Phi'(z)^{-1}\Phi'(x_*)\| \leq \frac{1}{1-\omega_0(\|z-x_*\|)}. \tag{2.8}$$

Notice that y_0, z_0, v_1 are well defined by scheme (1.2). We can write in turn by the first substep of scheme (1.2), (A_2) and (2.8) (for $z = v_0$) that

$$\begin{aligned}
\|y_0 - x_*\| &= \|v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) + (1-\gamma)\Phi'(v_0)^{-1}\Phi(v_0)\| \\
&\leq \|\Phi'(v_0)^{-1}\Phi'(x_*)\| \\
&\quad \times \left\| \int_0^1 \Phi'(x_*)^{-1}(\Phi'(x_* + \theta(v_0 - x_*)) - \Phi'(v_0))\, d\theta \right\| \|v_0 - x_*\| \\
&\quad + |1-\gamma|\, \|\Phi'(v_0)^{-1}\Phi'(x_*)\|\, \|\Phi'(x_*)^{-1}\Phi(v_0)\| \\
&\leq \frac{(\int_0^1 \omega((1-\theta)\|v_0 - x_*\|)\, d\theta + |1-\gamma| \int_0^1 \omega_1(\theta\|v_0 - x_*\|)\, d\theta)\|v_0 - x_*\|)}{1-\omega_0(\|v_0 - x_*\|)} \\
&\leq g_1(\|v_0 - x_*\|)\|v_0 - x_*\| \leq \|v_0 - x_*\| < r.
\end{aligned} \tag{2.9}$$

Similarly, by the second substep of scheme (1.2) and (2.9), we have

$$\begin{aligned}
\|z_0 - x_*\| &= \Big\|v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) \\
&\qquad - \tfrac{3}{8}\big(5I - 8\Phi'(v_0)^{-1}\Phi'(y_0) + 3(\Phi'(v_0)^{-1}\Phi'(y_0))^2\big)\Phi'(v_0)^{-1}\Phi(v_0)\Big\| \\
&= \Big\|v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) \\
&\quad - \tfrac{3}{8}\Big(3(\Phi'(v_0)^{-1}(\Phi'(y_0) - \Phi'(v_0)))^2 \\
&\quad - 2\Phi'(v_0)^{-1}(\Phi'(y_0) - \Phi'(v_0))\Big)\Phi'(v_0)^{-1}\Phi(v_0)\Big\| \\
&\leq \Bigg[\frac{\int_0^1 \omega((1-\theta)\|v_0 - x_*\|)\, d\theta}{1-\omega_0(\|v_0 - x_*\|)} \\
&\quad + \frac{3}{8}\Bigg(3\bigg(\frac{\omega_0(\|v_0 - x_*\|) + \omega_0(\|y_0 - x_*\|)}{1-\omega_0(\|v_0 - x_*\|)}\bigg)^2 \\
&\quad + 2\frac{\omega_0(\|v_0 - x_*\|) + \omega_0(\|y_0 - x_*\|)}{1-\omega_0(\|v_0 - x_*\|)}\Bigg)\frac{\int_0^1 \omega_1(\theta\|v_0 - x_*\|)\, d\theta}{1-\omega_0(\|v_0 - x_*\|)}\Bigg]\|v_0 - x_*\| \\
&\leq g_2(\|v_0 - x_*\|)\|v_0 - x_*\| \leq \|v_0 - x_*\|.
\end{aligned} \tag{2.10}$$

Moreover, by the third substep of scheme (1.2), (2.9) and (2.10), we get

$$\begin{aligned}
\|v_1 - x_*\| &= \|z_0 - x_* - \Phi'(z_0)^{-1}\Phi(z_0) \\
&\quad + (\Phi'(z_0)^{-1} - \Phi'(v_0)^{-1})\Phi(z_0) - \tfrac{3}{2}\Phi'(v_0)^{-1}(\Phi'(v_0) - \Phi'(y_0))\Phi'(v_0)^{-1}\Phi(z_0)\|
\end{aligned} \tag{2.11}$$

$$\leq \bigg[\frac{\int_0^1 \omega((1-\theta)\|z_0-x_*\|)\,d\theta}{1-\omega_0(\|z_0-x_*\|)}$$
$$+\frac{(\omega_0(\|v_0-x_*\|)+\omega_0(\|z_0-x_*\|))\int_0^1 \omega_1(\theta\|z_0-x_*\|)\,d\theta}{(1-\omega_0(\|v_0-x_*\|))(1-\omega_0(\|z_0-x_*\|))}$$
$$+\frac{3}{2}\,\frac{(\omega_0(\|v_0-x_*\|)+\omega_0(\|y_0-x_*\|))\int_0^1 \omega_1(\theta\|z_0-x_*\|)\,d\theta}{(1-\omega_0(\|v_0-x_*\|))^2}\bigg]\|z_0-x_*\|$$
$$\leq g_3(\|v_0-x_*\|)\|v_0-x_*\| \leq \|v_0-x_*\|,$$

where we have also used (2.1), (2.4) (for $i=1,2,3$), (2.8) (for $z=v_0,z_0$), (A_2), and (2.9)–(2.11). Hence, items (2.5)–(2.7) hold if $n=0$.

Now simply replace v_0, y_0, z_0, v_1 by v_j, y_j, z_j, v_{j+1} in the previous calculations to complete the induction step for items (2.5)–(2.7). Then, from the estimation

$$\|v_{j+1}-x_*\| \leq b\|v_j-x_*\| < r, \tag{2.12}$$

where $b=g_3(\|v_0-x_*\|)\in[0,1)$, we deduce $v_{j+1}\in B(x_*,r)$ and $\lim_{j\to\infty} v_j = x_*$.

Next, we show the uniqueness of x_*. Set $Q=\int_0^1 \Phi'(x_*+\theta(q-x_*))\,d\theta$ for some $q\in\Omega_1$ with $\Phi(q)=0$. Then, using (A_1) and (A_4), we obtain

$$\|\Phi'(x_*)^{-1}(Q-\Phi'(x_*))\| \leq \int_0^1 \omega_0(\theta\|q-x_*\|)\,d\theta \leq \int_0^1 \omega_0(\theta r_*)\,d\theta < 1. \tag{2.13}$$

So, we deduce $x_*=q$ from $0=\Phi(q)-\Phi(x_*)=Q(q-x_*)$ and the invertibility of Q. ∎

Next, we analyse the ball of convergence of scheme (1.3) in an analogous way. Define

$$\overline{g}_1(t)=g_1,$$
$$\overline{g}_2(t)=\frac{\int_0^1 \omega((1-\theta)t)\,d\theta}{1-\omega_0(t)}$$
$$+\frac{3}{8}\bigg(5\bigg(\frac{\omega_0(t)+\omega_0(\overline{g}_1(t)t)}{1-\omega_0(t)}\bigg)^2+2\frac{\omega_0(t)+\omega_0(\overline{g}_1(t)t)}{1-\omega_0(t)}\bigg)\frac{\int_0^1 \omega_1(\theta t)\,d\theta}{1-\omega_0(t)},$$
$$\overline{g}_3(t)=\bigg[\frac{\int_0^1 \omega((1-\theta)\overline{g}_2(t)t)\,d\theta}{1-\omega_0(g_2(t)t)}+\frac{(\omega_0(t)+\omega_0(\overline{g}_2(t)t))\int_0^1 \omega_1(\theta\overline{g}_2(t)t)\,d\theta}{(1-\omega_0(t))(1-\omega_0(\overline{g}_2(t)t))}$$
$$+\frac{1}{2}\bigg(\bigg(\frac{\omega_0(t)+\omega_0(\overline{g}_2(t)t)}{1-\omega_0(t)}\bigg)^2$$
$$+3\frac{\omega_0(t)+\omega_0(\overline{g}_2(t)t)}{1-\omega_0(t)}\bigg)\frac{\int_0^1 \omega_1(\theta\overline{g}_2(t)t)\,d\theta}{1-\omega_0(t)}\bigg]g_2(t).$$

Suppose the function $\overline{g}_i(t)-1$ has a smallest root in $T_0\setminus\{0\}$ denoted by $\overline{r}_i$.

Set

$$\overline{r} = \min \overline{r}_i. \tag{2.14}$$

Moreover, suppose conditions (A_1)–(A_4) hold with $\tilde{r} = \overline{r}$.

Using estimates (2.9)–(2.11) we show that the functions $\overline{g}_i$ are motivated by the following calculations:

$$\begin{aligned}\|y_n - x_*\| &\leq g_1(\|v_n - x_*\|)\|v_n - x_*\| = \overline{g}_1(\|v_n - x_*\|)\|v_n - x_*\| \\ &\leq \|v_n - x_*\| < \overline{r}.\end{aligned}$$

Moreover, the second substep gives

$$\begin{aligned}\|z_n - x_*\| = \Big\| & v_n - x_* - \Phi'(v_n)^{-1}\Phi(v_n) \\ & - \tfrac{3}{8}\big(7\Phi'(v_n)^{-1}\Phi'(y_n) - 12(\Phi'(v_n)^{-1}\Phi'(y_n))^2 \\ & + 5(\Phi'(v_n)^{-1}\Phi'(y_n))^3\big)\Phi'(v_n)^{-1}\Phi(v_n)\Big\| \\ = \Big\| & v_n - x_* - \Phi'(v_n)^{-1}\Phi(v_n) - \tfrac{3}{8}\big(5(\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n)))^2 \\ & - 2\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n))\big)\Phi'(v_n)^{-1}\Phi'(y_n)\Phi'(v_n)^{-1}\Phi(v_n)\Big\| \\ \leq \Bigg[& \frac{\int_0^1 \omega((1-\theta)\|v_n - x_*\|\,d\theta}{1 - \omega_0(\|v_n - x_*\|)} \\ & + \frac{3}{8}\Bigg(5\Bigg(\frac{\omega_0(\|v_n - x_*\|) + \omega_0(\|y_n - x_*\|)}{1 - \omega_0(\|v_n - x_*\|)}\Bigg)^2 \\ & + 2\frac{\omega_0(\|v_n - x_*\|) + \omega_0(\|y_n - x_*\|)}{1 - \omega_0(\|v_n - x_*\|)}\,\frac{\int_0^1 \omega_1(\theta\|v_n - x_*\|)\,d\theta}{1 - \omega_0(\|v_n - x_*\|)}\Bigg]\|v_n - x_*\| \\ \leq \; & \overline{g}_2(\|v_n - x_*\|)\|v_n - x_*\| \leq \|v_n - x_*\|.\end{aligned}$$

Furthermore, the third substep leads to

$$\begin{aligned}\|v_{n+1} - x_*\| = \Big\| & z_n - x_* - \Phi'(z_n)^{-1}\Phi(z_n) \\ & + \Phi'(z_n)^{-1}(\Phi'(v_n) - \Phi'(z_n))\Phi'(v_n)^{-1}\Phi(z_n) \\ & - \tfrac{1}{2}\big(4I - 5\Phi'(v_n)^{-1}\Phi'(y_n) + (\Phi'(v_n)^{-1}\Phi'(y_n))^2\big)\Phi'(v_n)^{-1}\Phi(z_n)\Big\| \\ = \Bigg\| & z_n - x_* - \Phi'(z_n)^{-1}\Phi(z_n) \\ & + \Phi'(z_n)^{-1}(\Phi'(v_n) - \Phi'(z_n))\Phi'(v_n)^{-1}\Phi(z_n) \\ & - \frac{1}{2}\Big(\big(\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n))\big)^2 \\ & - 3\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n))\Big)\Phi'(v_n)^{-1}\Phi(z_n)\Bigg\|\end{aligned}$$

$$
\begin{aligned}
&\le \Bigg[\frac{\int_0^1 \omega((1-\theta)\|z_n - x_*\|\,d\theta}{1-\omega_0(\|z_n - x_*\|)} \\
&\quad + \frac{(\omega_0(\|v_n - x_*\|) + \omega_0(\|z_n - x_*\|))\int_0^1 \omega_1(\theta\|z_n - x_*\|)\,d\theta}{(1-\omega_0(\|v_n - x_*\|))(1-\omega_0(\|z_n - x_*\|))} \\
&\quad + \frac{1}{2}\Bigg(\Bigg(\frac{\omega_0(\|v_n - x_*\|) + \omega_0(\|y_n - x_*\|)}{1-\omega_0(\|v_n - x_*\|)}\Bigg)^2 \\
&\quad + 3\frac{\omega_0(\|v_n - x_*\|) + \omega_0(\|y_n - x_*\|)}{1-\omega_0(\|v_n - x_*\|)}\Bigg)\frac{\int_0^1 \omega_1(\theta\|z_n - x_*\|)\,d\theta}{1-\omega_0(\|v_n - x_*\|)}\Bigg]\|z_n - x_*\| \\
&\le \overline{g}_3(\|v_n - x_*\|)\|v_n - x_*\| \le \|v_n - x_*\|.
\end{aligned}
$$

Hence, we obtain following result on the convergence ball for scheme (1.3).

THEOREM 2.2. *Suppose that conditions* (A_1)–(A_4) *hold with* $\tilde{r} = \overline{r}$. *Then the conclusions of Theorem* 2.1 *hold for scheme* (1.3) *with* r, g_i *replaced by* $\overline{r}$, $\overline{g}_i$, *respectively.*

3. Numerical examples. We use the suggested approach to estimate the convergence radii for schemes (1.2) and (1.3) provided that $\gamma = \frac{2}{3}$.

EXAMPLE 1. Let $X = Y = C[0,1]$ and $\Omega = \overline{B}(0,1)$. Define Φ on Ω by

$$\Phi(v)(a) = v(a) - 5\int_0^1 au\,v(u)^3\,du,$$

where $v(\cdot) \in C[0,1]$. We have $x_* = 0$. Conditions (A_1)–(A_4) are satisfied for $\omega_0(t) = 7.5t$, $\omega(t) = 15t$ and $\omega_1(t) = 2$. Then the values of r and $\overline{r}$ are produced using formulas (2.1) and (2.4), respectively. These results are summarized in Table 1.

Table 1. Comparison of convergence radii for Example 1

Scheme (1.2)	Scheme (1.3)
$r_1 = 0.022222$	$\overline{r}_1 = 0.022222$
$r_2 = 0.018948$	$\overline{r}_2 = 0.017369$
$r_3 = 0.013823$	$\overline{r}_3 = 0.014756$
$r = 0.013823$	$\overline{r} = 0.014756$

EXAMPLE 2. Let $X = Y = \mathbb{R}^3$ and $\Omega = \overline{B}(0,1)$. Consider Φ on Ω defined for $v = (v_1, v_2, v_3)^t$ as

$$\Phi(v) = \big(e^{v_1} - 1, \tfrac{e-1}{2}v_2^2 + v_2, v_3\big)^t.$$

We have $x_* = (0,0,0)^t$. Conditions (A_1)–(A_4) are satisfied for $\omega_0(t) = (e-1)t$, $\omega(t) = e^{\frac{1}{e-1}}t$ and $\omega_1(t) = 2$. The results are displayed in Table 2.

Table 2. Comparison of convergence radii for Example 2

Scheme (1.2)	Scheme (1.3)
$r_1 = 0.127564$	$\bar{r}_1 = 0.127564$
$r_2 = 0.088919$	$\bar{r}_2 = 0.080947$
$r_3 = 0.064030$	$\bar{r}_3 = 0.068190$
$r = 0.064030$	$\bar{r} = 0.068190$

Example 3. Finally, the motivating issue stated in Section 1 is solved for $x_* = 1$. We choose $\omega_0(t) = \omega(t) = 96.662907t$ and $\omega_1(t) = 2$. The radius can be found in Table 3. It is found out that scheme (1.3) has a larger radius of convergence in all three examples. But we cannot conclude that the scheme (1.3) is always better to use than scheme (1.2).

Table 3. Comparison of convergence radii for Example 3

Scheme (1.2)	Scheme (1.3)
$r_1 = 0.002299$	$\bar{r}_1 = 0.002299$
$r_2 = 0.001586$	$\bar{r}_2 = 0.001443$
$r_3 = 0.001141$	$\bar{r}_3 = 0.001215$
$r = 0.001141$	$\bar{r} = 0.001215$

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