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MATHEMATICAL LOGIC AND FOUNDATIONS

On the Erdős–Dushnik–Miller theorem without AC

by

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Summary. In ZFA (Zermelo–Fraenkel set theory with the Axiom of Extensionality weakened to allow the existence of atoms), we prove that the strength of the proposition EDM ("If $G = (V_G, E_G)$ is a graph such that V_G is uncountable, then for every coloring $f : [V_G]^2 \to \{0, 1\}$ either there is an uncountable set monochromatic in color 0, or there is a countably infinite set monochromatic in color 1") is *strictly between* $\mathsf{DC}_{\aleph_1}$ (where $\mathsf{DC}_{\aleph_1}$ is Dependent Choices for $\aleph_1$, a weak choice form stronger than Dependent Choices (DC)) and Kurepa's principle ("Any partially ordered set such that all of its antichains are finite and all of its chains are countable is countable"). Among other new results, we study the relations of EDM to BPI (Boolean Prime Ideal Theorem), RT (Ramsey's theorem), De Bruijn–Erdős' theorem for n-colorings, König's lemma and several other weak choice forms. Moreover, we answer a part of a question raised by Lajos Soukup.

1. Introduction. In 1941, Dushnik and Miller established the proposition "If κ is an uncountable cardinal, then for every coloring $f : [\kappa]^2 \to \{0, 1\}$, either there is a set of cardinality κ monochromatic in color 0, or there is a countably infinite set monochromatic in color 1" in ZFC, and gave credit to Erdős for the proof of the result for the case in which κ is a singular cardinal—see [7, Theorem 5.22 and footnote 6 on page 606]. The above result is generally known as the Erdős–Dushnik–Miller theorem. In ZFC, the theorem applies to prove that "Every partially ordered set such that all of its chains are finite and all of its antichains are countable is countable" (abbreviated here as "$\mathsf{CAC}^{\aleph_0}$") and Kurepa's result on partially ordered sets stated in the abstract (abbreviated here as "$\mathsf{CAC}_1^{\aleph_0}$"). Kurepa [16] explicitly proved $\mathsf{CAC}_1^{\aleph_0}$

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in ZFC in response to Sierpiński's question [18, pp. 190–191]. Banerjee [1] and Banerjee and Gyenis [3] studied some relations of $\mathsf{CAC}^{\aleph_0}$ and $\mathsf{CAC}_1^{\aleph_0}$ with weak forms of the Axiom of Choice (AC). Recently, Tachtsis [19] investigated the deductive strength of $\mathsf{CAC}_1^{\aleph_0}$ without AC in more detail. Among various results, Tachtsis [19] proved that $\mathsf{CAC}_1^{\aleph_0}$ holds in $\mathsf{ZF}+\mathsf{DC}_{\aleph_1}$ ([1]) and $\mathsf{CAC}_1^{\aleph_0}$ holds in the Mostowski linearly ordered model (labeled as Model $\mathcal{N}_3$ in [9]) as well as the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [9]). Inspired by the research work of [19], we study the deductive strength of EDM without AC. Let X be a set. We note that without AC, there are two definitions of *uncountable* sets:

(1) *X is uncountable* if $|X| \not\leq \aleph_0$ (i.e., there is no injection from X into $\aleph_0$).
(2) *X is uncountable* if $\aleph_0 < |X|$ (i.e., there is an injection from $\aleph_0$ into X and there is no injection from X into $\aleph_0$).

We note that all the results in this paper are obtained with the first definition of uncountable sets. Lajos Soukup asked the following question.

Question 1.1. What is the relationship between $\mathsf{CAC}_1^{\aleph_0}$ and $\mathsf{CAC}^{\aleph_0}$ in ZF and ZFA?

Main result. Fix $X \in \{\mathsf{CAC}_1^{\aleph_0}, \mathsf{CAC}^{\aleph_0}\}$. The first author proves that the strength of EDM is strictly between $\mathsf{DC}_{\aleph_1}$ and X, and $\mathsf{CAC}_1^{\aleph_0}$ does not imply $\mathsf{CAC}^{\aleph_0}$ in ZFA (cf. Theorems 4.1, 4.2, 4.4, Corollary 3.6).

Other results. The first author observes the following in ZF:

(1) $\mathsf{CAC}^{\aleph_0}$ implies $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ ("Every countably infinite family of countably infinite sets has a choice function"). Thus $\mathsf{CAC}^{\aleph_0}$ is not provable in ZF (Proposition 3.4).
(2) WOAM ("Every set is either well-orderable or has an amorphous subset") implies RT for any locally countable connected graph (Proposition 3.7).
(3) EDM is strictly stronger than RT (Theorem 4.1(3, 4)).
(4) WOAM + RT implies EDM (Theorem 4.2(1)).
(5) $\mathsf{CAC}^{\aleph_0} + \mathsf{CAC}_1^{\aleph_0}$ + A (Antichain Principle) does not imply EDM in ZFA (Theorem 4.4).

1.1. The uniqueness of algebraic closures, and Łoś's theorem. Pincus [9, Note 41] proved that the statement "If a field has an algebraic closure, then it is unique up to isomorphism" [9, Form 233] does not imply "There are no amorphous sets" in ZFA. Recently, Tachtsis [21] constructed a model of $\mathsf{ZFA}+\neg\mathsf{AC}$ to prove that $\mathsf{AC}^{\mathsf{LO}}$ ("Every linearly ordered family of non-empty sets has a choice function") does not imply $\mathsf{ŁT}$ ("If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}}\rangle$

([1]) ZF denotes Zermelo–Fraenkel set theory without AC. Complete definitions of the choice forms will be given in Section 2.

is a non-trivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ is an ultrafilter on a non-empty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent").

We observe the following:

$\mathsf{AC}^{\mathsf{LO}} + \mathsf{EDM} +$ Form 233 does not imply $\mathsf{ŁT}$ in ZFA (Theorem 5.2).

1.2. Remarks. Blass [4] investigated the strength of RT in the hierarchy of choice forms. In Section 6, applying the above-mentioned results and mainly inspired by the results of [4], we remark that EDM is independent of each of BPI, KW (Kinna–Wagner Selection Principle), $\mathsf{AC}_{\mathsf{WO}}$ (Axiom of Choice for non-empty, well-orderable sets), "There are no amorphous sets", the n-coloring theorem (De Bruijn–Erdős' theorem for n-colorings), and A (Antichain Principle) in ZFA. Moreover, $\mathsf{ŁT}$ and EDM are mutually independent in ZF. In [19], Tachtsis proved that $\mathsf{CAC}_1^{\aleph_0}$ and DT (Dilworth's theorem) are mutually independent in ZFA. A natural question which arises is about the relation of $\mathsf{CAC}^{\aleph_0}$ and EDM to DT. We also remark that DT is independent of EDM and $\mathsf{CAC}^{\aleph_0}$ in ZFA ([2]).

2. Basics and diagram of results

Definition 2.1. Suppose X and Y are two sets.

- We write $|X| \leq |Y|$ or $|Y| \geq |X|$ if there is an injection $f : X \to Y$.
- We write $|X| = |Y|$ if there is a bijection $f : X \to Y$.
- We write $|X| < |Y|$ or $|Y| > |X|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$.
- If $f : X \to Y$ is a function, then we denote the range of f by $\operatorname{ran}(f)$ and the domain of f by $\operatorname{dom}(f)$.

Definition 2.2. Let $(P, \leq)$ be a partially ordered set, or poset for short. A subset $D \subseteq P$ is a *chain* if $(D, \leq\restriction D)$ is linearly ordered. A subset $A \subseteq P$ is an *antichain* if no two elements of A are comparable under $\leq$. The size of the largest antichain of $(P, \leq)$ is known as its *width*. A subset $C \subseteq P$ is *cofinal* in P if for every $x \in P$ there is an element $c \in C$ such that $x \leq c$. A *tree* is a connected undirected graph without circuits one of whose vertices is designated as the origin. We note that a tree may also be defined as a poset $(P, <)$ with a least element and with the property that for any element $x \in P$ the set of predecessors of x is a finite set that is linearly ordered by $<$. The number of vertices on the unique path connecting a vertex v to the origin is the *level* of v, denoted by $l(v)$. A vertex v' is a *successor* of a vertex v if v and v' are connected by an edge and $l(v') = l(v) + 1$. A tree is *locally finite*

([2]) We note that Theorem 5.2 is a combined effort of both authors under the proper guidance of the referee (see Acknowledgements). Moreover, all remarks in Section 6 (except Remark 6.1(7)) are due to both authors.

if each vertex has only finitely many successors. An *ω-tree* is a locally finite tree with at least one vertex in level n for each $n \in \omega$ (cf. [9, Note 21]). An infinite set X is *amorphous* if X cannot be written as a disjoint union of two infinite subsets. A set X is *Dedekind-finite* if $\aleph_0 \not\leq |X|$. Otherwise, X is *Dedekind-infinite*. We say that a graph $G = (V_G, E_G)$ is *locally countable* if for every $v \in V_G$, the set of neighbors of v is countable. The graph G is *connected* if any two vertices are joined by a path of finite length.

Definition 2.3 (A list of choice forms).

(1) The *Axiom of Choice*, AC [9, Form 1]: Every family of non-empty sets has a choice function.
(2) The *Boolean Prime Ideal Theorem*, BPI [9, Form 14]: Every Boolean algebra has a prime ideal.
(3) The *Kinna–Wagner Selection Principle*, KW [9, Form 15]: For every set M there is a function f such that for all $A \in M$, if $|A| > 1$ then $\emptyset \neq f(A) \subsetneq A$.
(4) The *Axiom of Multiple Choice*, MC [9, Form 67]: Every family $\mathcal{A}$ of non-empty sets has a *multiple choice function*, i.e., there is a function f with domain $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $\emptyset \neq f(A) \in [A]^{<\omega}$.
(5) $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ [9, Form 350]: Every denumerable, i.e. countably infinite, family of denumerable sets has a multiple choice function.
(6) $\mathsf{AC}_{\mathsf{WO}}$ [9, Form 60]: Every set of non-empty, well-orderable sets has a choice function.
(7) $\mathsf{AC}^{\mathsf{LO}}$ [9, Form 202]: Every linearly ordered set of non-empty sets has a choice function.
(8) $\mathsf{PAC}_{\mathsf{fin}}^{\aleph_1}$ (cf. [1]): Every $\aleph_1$-sized family $\mathcal{A}$ of non-empty finite sets has an $\aleph_1$-sized subfamily $\mathcal{B}$ with a choice function.
(9) $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ [9, Form 32A]: Every denumerable family of denumerable sets has a choice function. We recall that $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ is equivalent to $\mathsf{PAC}_{\aleph_0}^{\aleph_0}$ [9, Form 32B] (Every denumerable family $\mathcal{A}$ of denumerable sets has an infinite subfamily $\mathcal{B}$ with a choice function).
(10) $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ [9, Form 10]: Every denumerable family of non-empty finite sets has a choice function.
(11) AC_n^- for each $n \in \omega \setminus \{0, 1\}$ [9, Form 342(n)]: Every infinite family $\mathcal{A}$ of n-element sets has a *partial choice function*, i.e., $\mathcal{A}$ has an infinite subfamily $\mathcal{B}$ with a choice function.
(12) WOAM [9, Form 133]: Every set is either well-orderable or has an amorphous subset.
(13) The *Principle of Dependent Choice*, DC [9, Form 43]: If S is a relation on a non-empty set A and $(\forall x \in A)(\exists y \in A)(xSy)$ then there is a sequence $(a_n)_{n\in\omega}$ of elements of A such that $(\forall n \in \omega)(a_n S a_{n+1})$.

(14) DC_κ where $\kappa = \aleph_\alpha$ for some ordinal α [9, Form 87(α)]: Let S be a non-empty set and let R be a binary relation such that for every $\beta < \kappa$ and every β-sequence $s = (s_\epsilon)_{\epsilon<\beta}$ of elements of S there exists $y \in S$ such that sRy. Then there is a function $f : \kappa \to S$ such that for every $\beta < \kappa$, $(f{\restriction}\beta)Rf(\beta)$. We note that $\mathsf{DC}_{\aleph_0}$ is a reformulation of DC.
(15) $\mathsf{DF} = \mathsf{F}$ [9, Form 9]: Every Dedekind-finite set is finite.
(16) $\mathsf{W}_{\aleph_\alpha}$ (cf. [13, Chapter 8]): For every X, either $|X| \le \aleph_\alpha$ or $|X| \ge \aleph_\alpha$.
(17) [9, Form 233]: If a field has an algebraic closure, then the closure is unique up to isomorphism.
(18) WUT [9, Form 231]: The union of a well-orderable collection of well-orderable sets is well-orderable.
(19) $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ [9, Form 122]: Every well-ordered set of non-empty finite sets has a choice function.
(20) The *Countable Union Theorem*, CUT [9, Form 31]: The union of a countable family of countable sets is countable.
(21) CS: Every poset without a maximal element has two disjoint cofinal subsets.
(22) CWF: Every poset has a cofinal well-founded subset.
(23) The *Antichain Principle*, A: Every poset has a maximal antichain.
(24) The *n-coloring theorem*, $\mathcal{P}_n$: If all finite subgraphs of a graph G are n-colorable then G is n-colorable.
(25) *Dilworth's theorem*, DT: If $(P, \le)$ is a poset of width k for some $k \in \omega$, then P can be partitioned into k chains.
(26) *Ramsey's theorem*, RT [9, Form 17]: For every infinite set A and for every partition of the set $[A]^2$ into two sets X and Y, there is an infinite subset $B \subseteq A$ such that either $[B]^2 \subseteq X$ or $[B]^2 \subseteq Y$ (3).
(27) The *Chain/Antichain Principle*, CAC [9, Form 217]: Every infinite poset has an infinite chain or an infinite antichain.
(28) *Łoś's theorem*, $\mathsf{ŁT}$ [9, Form 253]: If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}} \rangle$ is a non-trivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ is an ultrafilter on a non-empty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.

Definition 2.4 (A list of combinatorial statements).

(1) EDM: If $G = (V_G, E_G)$ is a graph such that V_G is uncountable, then for every coloring $f : [V_G]^2 \to \{0, 1\}$ either there is an uncountable set monochromatic in color 0, or there is a countably infinite set monochromatic in color 1.
(2) EDM': EDM restricted to graphs based on a well-ordered set of vertices.

(3) Equivalently, for every infinite graph $G = (V, E)$ and for all $c : [V]^2 \to 2$, there exists an infinite set $Y \subseteq V$ such that $[Y]^2$ is c-monochromatic.

(3) $\mathsf{CAC}^{\aleph_0}$: Every poset such that all of its chains are finite and all of its antichains are countable is countable.
(4) $(\mathsf{CAC}^{\aleph_0})'$: $\mathsf{CAC}^{\aleph_0}$ restricted to posets based on a well-ordered set of elements.
(5) $\mathsf{CAC}_1^{\aleph_0}$: Every poset such that all of its antichains are finite and all of its chains are countable is countable.
(6) $(\mathsf{CAC}_1^{\aleph_0})'$: $\mathsf{CAC}_1^{\aleph_0}$ restricted to posets based on a well-ordered set of elements.
(7) $\mathsf{CACT}^{\aleph_0}$: $\mathsf{CAC}^{\aleph_0}$ restricted to ω-trees.
(8) $(\mathsf{CACT}^{\aleph_0})'$: $\mathsf{CAC}^{\aleph_0}$ restricted to ω-trees based on a well-ordered set of elements.
(9) $\mathsf{CACT}_1^{\aleph_0}$: $\mathsf{CAC}_1^{\aleph_0}$ restricted to ω-trees.
(10) $(\mathsf{CACT}_1^{\aleph_0})'$: $\mathsf{CAC}_1^{\aleph_0}$ restricted to ω-trees based on a well-ordered set of elements.
(11) For a set A, $\mathrm{Sym}(A)$ and $\mathrm{FSym}(A)$ denote the set of all permutations of A and the set of all $\phi \in \mathrm{Sym}(A)$ such that $\{x \in A : \phi(x) \neq x\}$ is finite. For a set A of size at least $\aleph_\alpha$, $\aleph_\alpha\mathrm{Sym}(A)$ denotes the set of all $\phi \in \mathrm{Sym}(A)$ such that $\{x \in A : \phi(x) \neq x\}$ has cardinality at most $\aleph_\alpha$ (cf. [20, Section 2]).

2.1. Permutation models and Mostowski's intersection lemma. We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ where A is a set of atoms, $\mathcal{G}$ is a group of permutations of A, and $\mathcal{F}$ is a normal filter of subgroups of $\mathcal{G}$. The Fraenkel–Mostowski model, or the permutation model $\mathcal{N}$ with respect to M, $\mathcal{G}$ and $\mathcal{F}$ is defined by

$$\mathcal{N} = \{x \in M : (\forall t \in \mathsf{TC}(\{x\}))(\mathrm{sym}_{\mathcal{G}}(t) \in \mathcal{F})\}$$

where for a set $x \in M$, $\mathrm{sym}_{\mathcal{G}}(x) = \{g \in \mathcal{G} : g(x) = x\}$ and $\mathsf{TC}(x)$ is the transitive closure of x in M. If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in \mathcal{I}\}$ generates a normal filter (say $\mathcal{F}_{\mathcal{I}}$) over $\mathcal{G}$, where $\mathrm{fix}_{\mathcal{G}}(E) = \{\phi \in \mathcal{G} : (\forall y \in E)(\phi(y) = y)\}$. Let $\mathcal{N}$ be the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We recall that $\mathcal{N}$ is a model of ZFA (cf. [13, Theorem 4.1, p. 46]). We say $E \in \mathcal{I}$ is a *support* of a set $\sigma \in \mathcal{N}$ if $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{sym}_{\mathcal{G}}(\sigma)$. We recall some terminology from [5, Sections 1, 2]. Let $\Delta(E) = \{\sigma : \mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{sym}_{\mathcal{G}}(\sigma)\}$ for $E \in \mathcal{I}$. We say that $\mathcal{N}$ satisfies *Mostowski's intersection lemma* if $\Delta(E \cap F) = \Delta(E) \cap \Delta(F)$ for all $E, F \in \mathcal{I}$. In this paper:

- We follow the labeling of the models from [9]. $\mathcal{N}_1$ is the basic Fraenkel model, $\mathcal{N}_2$ is the second Fraenkel model, $\mathcal{N}_3$ is the Mostowski linearly ordered model, and $\mathcal{N}_{41}$ is a variation of $\mathcal{N}_3$ (cf. [9]).
- Fix any $n \in \omega \setminus \{0, 1\}$. We denote by $\mathcal{N}_{HT}^1(n)$ the permutation model constructed in [10, Theorem 8].

LEMMA 2.5. *An element x of $\mathcal{N}$ is well-orderable in $\mathcal{N}$ if and only if* $\mathrm{fix}_{\mathcal{G}}(x) \in \mathcal{F}_{\mathcal{I}}$ (cf. [13, equation (4.2), p. 47]). *Thus, an element x of $\mathcal{N}$ with support E is well-orderable in $\mathcal{N}$ if* $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{fix}_{\mathcal{G}}(x)$.

We refer the reader to [9, Note 103, pp. 283–286] for the definitions of the terms "injective cardinality $|x|_-$ of x", "injectively boundable statement" and "boundable statement".

THEOREM 2.6 (Pincus' Transfer Theorem; cf. [17, Theorem 3A3]). *If Φ is a conjunction of injectively boundable statements which hold in the Fraenkel–Mostowski model V_0, then there is a* ZF *model $V \supset V_0$, with the same ordinals and cofinalities as V_0, where Φ holds.*

LEMMA 2.7 (Brunner; cf. [6, Lemma 4.1]). *Let A be a set of atoms, $\mathcal{G}$ be a group of permutations of A, and $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ which is generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Let $\mathcal{N}$ be the Fraenkel–Mostowski model determined by A, $\mathcal{G}$, and $\mathcal{F}$. If $\mathcal{N}$ satisfies Mostowski's intersection lemma where A is Dedekind-finite, then for every set x in $\mathcal{N}$, either x is well-orderable or there exists an infinite subset of A which embeds into x.*

2.2. Diagram. We summarize the main results using the first definition of uncountable sets. We fix any $2 \le n \in \omega$.

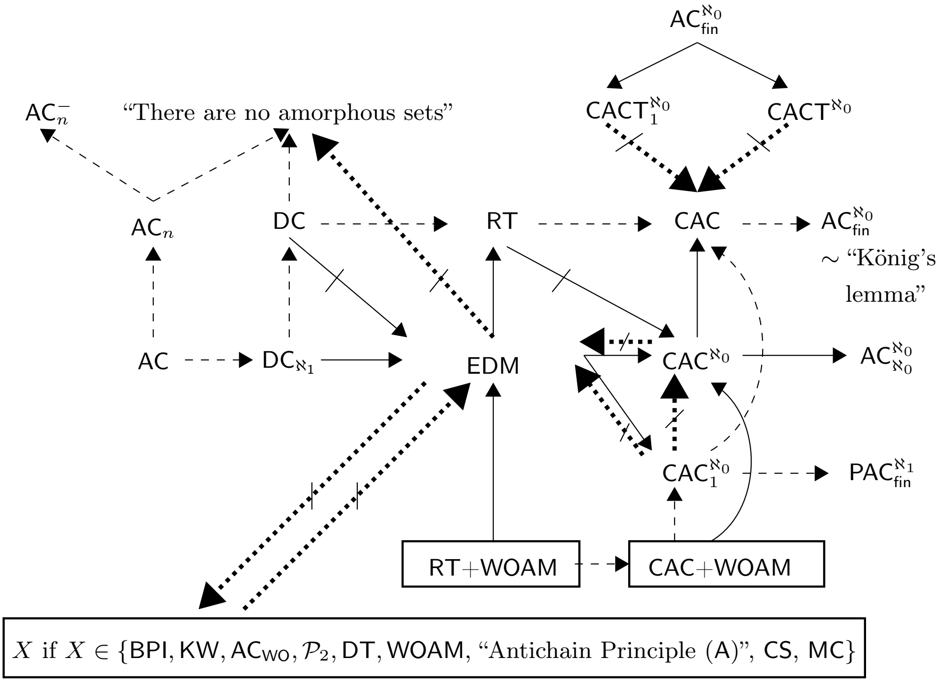

Fig. 1. In the figure, we denote "equivalent to" by $\sim$. We note that $(\forall n \ge 3)(\mathcal{P}_n \sim \mathsf{BPI})$.

In Figure 1, known results are depicted with dashed arrows, new implications or non-implications in ZF are indicated with simple black arrows, and new non-implications in ZFA are marked with thick dotted black arrows.

3. Known and basic results

3.1. Known results

FACT 3.1 (ZF). *The following hold:*

(1) RT *holds for every infinite well-orderable set, and if* RT *holds for an infinite set* Y*, then* RT *holds for any set* $X \supseteq Y$ [25, Theorem 1.7]*; moreover,* $\mathsf{DF} = \mathsf{F}$ *implies* RT [9].
(2) WOAM *implies* CUT [15, Proposition 8(i)]. *So,* WOAM *implies "*$\aleph_1$ *is regular".*
(3) $\mathsf{CAC}_1^{\aleph_0}$ *implies* CAC [19, Theorem 4(11)] *and* CAC *implies* $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ [21, Lemma 4.4].
(4) $\mathsf{WOAM} + \mathsf{CAC}$ *implies* $\mathsf{CAC}_1^{\aleph_0}$ [19, Theorem 8(1)].
(5) $\mathsf{CAC}_1^{\aleph_0}$ *implies* $\mathsf{PAC}_{\mathsf{fin}}^{\aleph_1}$ *and* DC *does not imply* $\mathsf{CAC}_1^{\aleph_0}$ [1, Theorem 4.5, Corollary 4.6].

We recall the following result communicated to us by Tachtsis.

FACT 3.2 (cf. [1, Lemma 4.1, Corollary 4.2]). $(\mathsf{CAC}_1^{\aleph_0})'$ *holds in any permutation model.*

3.2. Basic propositions

PROPOSITION 3.3. *The following hold:*

(1) *"*$\aleph_1$ *is regular" implies* EDM' *in* ZF.
(2) *"*$\aleph_1$ *is regular" implies* $(\mathsf{CAC}^{\aleph_0})'$, $(\mathsf{CACT}_1^{\aleph_0})'$ *as well as* $(\mathsf{CACT}^{\aleph_0})'$ *in* ZF.
(3) *"*$\aleph_1$ *is regular"* $+$ $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ *implies* $\mathsf{CACT}_1^{\aleph_0}$ *and* $\mathsf{CACT}^{\aleph_0}$ *in* ZF.
(4) X *holds in any permutation model if*

$$X \in \{\mathsf{EDM}', (\mathsf{CAC}^{\aleph_0})', (\mathsf{CACT}_1^{\aleph_0})', (\mathsf{CACT}^{\aleph_0})'\}.$$

(5) $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ *implies* $\mathsf{CACT}_1^{\aleph_0}$ *and* $\mathsf{CACT}^{\aleph_0}$ *in* ZF.

Proof. (1) We modify the arguments due to Tachtsis from [1, Lemma 4.1]. Let $G = (V_G, E_G)$ be a graph based on a well-ordered set of vertices. Fix a well-ordering $\preceq$ of V_G. Let $f : [V_G]^2 \to \{0, 1\}$ be a coloring such that all sets monochromatic in color 0 are countable and all sets monochromatic in color 1 are finite. By way of contradiction, assume that V_G is uncountable. We will construct an infinite set monochromatic in color 1 in G, reaching a contradiction. Since V_G is well-ordered by $\preceq$, we can construct (via transfinite induction) a maximal set monochromatic in color 0, C_0 say, without invoking any form of choice. Since C_0 is countable, it follows that $V_G - C_0$ is uncountable and for every vertex $v \in V_G - C_0$, there is $c \in C_0$ such that $f(\{v, c\}) = 1$. We write $V_G - C_0 = \bigcup\{W_p : p \in C_0\}$, where $W_p = \{v \in V_G - C_0 : f(\{v, p\}) = 1\}$. Since $V_G - C_0$ is uncountable and C_0 is countable, it follows by "$\aleph_1$ is regular" that W_p is uncountable for some p in C_0. Let p_0 be the least (with respect to $\preceq$) such vertex of C_0. Next, we construct a maximal set monochromatic in color 0 in (the uncountable set) W_{p_0}, C_1 say, and let (similarly to the above argument) p_1 be the least (with respect to $\preceq$) vertex of C_1 such that the set

$$W_{p_1} = \{v \in W_{p_0} - C_1 : f(\{v, p_1\}) = 1\}$$

is uncountable. Continuing this process step by step and noting that the process cannot stop at a finite stage, we obtain a countably infinite set of vertices $\{p_n : n \in \omega\}$ monochromatic in color 1, contradicting the assumption that all sets monochromatic in color 1 are finite. Therefore, V_G is countable.

(2)–(5) These follow from (1) and the fact that the statement "$\aleph_1$ is a regular cardinal" holds in every permutation model (cf. [8, Corollary 1]) and $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ is equivalent to "Every ω-tree is countable" in ZF. ∎

PROPOSITION 3.4. (ZF) $\mathsf{CAC}^{\aleph_0}$ *implies* $\mathsf{AC}_{\aleph_0}^{\aleph_0}$.

Proof. Since $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ is equivalent to its partial version $\mathsf{PAC}_{\aleph_0}^{\aleph_0}$ (cf. Definition 2.3), it suffices to show $\mathsf{PAC}_{\aleph_0}^{\aleph_0}$. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable family of non-empty, denumerable sets. Without loss of generality, assume that $\mathcal{A}$ is disjoint. For the sake of contradiction, we assume that $\mathcal{A}$ has no partial choice function. Define a binary relation $\leq$ on $A = \bigcup \mathcal{A}$ as follows: for all $a, b \in A$, let $a \leq b$ if and only if $a = b$ or $a \in A_n$, $b \in A_m$ and $n < m$. Clearly, $\leq$ is a partial order on A. Since any two elements of A are $\leq$-comparable if and only if they belong to distinct A_i's, and $\mathcal{A}$ has no partial choice function, all chains in $(A, \leq)$ are finite. Next, if $C \subset A$ is an antichain in $(A, \leq)$, then $C \subseteq A_i$ for some $i \in \omega$. Thus, all antichains in $(A, \leq)$ are countable as A_i is denumerable for all $i \in \omega$. By $\mathsf{CAC}^{\aleph_0}$, A is countable (and hence well-orderable), contradicting $\mathcal{A}$'s having no partial choice function. ∎

PROPOSITION 3.5. *Let A be a set of atoms. Let $\mathcal{G}$ be the group of permutations of A such that either each $\eta \in \mathcal{G}$ moves only finitely many atoms or there exists an $n \in \omega \setminus \{0, 1\}$ such that $\eta^n = 1_A$ for all $\eta \in \mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by A, $\mathcal{G}$, and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. Then the following hold:*

(1) *The Antichain Principle* A *holds in $\mathcal{N}$.*
(2) *If* WUT *holds in $\mathcal{N}$, then both* $\mathsf{CAC}^{\aleph_0}$ *and* $\mathsf{CAC}_1^{\aleph_0}$ *hold in $\mathcal{N}$.*
(3) *If* $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ *holds and* $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ *fails in $\mathcal{N}$, then* $\mathsf{CAC}_1^{\aleph_0}$ *holds and* $\mathsf{CAC}^{\aleph_0}$ *fails in $\mathcal{N}$.*

Proof. Let $(P, \leq)$ be a poset in $\mathcal{N}$. Then the subgroup $H = \mathrm{sym}_{\mathcal{G}}((P, \leq))$ is an element of $\mathcal{F}$. Following the proof of [19, Theorem 3], $\mathrm{Orb}_H(p) = \{\phi(p) : \phi \in H\}$ is an antichain in P for each $p \in P$ and $\mathcal{O} = \{\mathrm{Orb}_H(p) : p \in P\}$ is a well-ordered partition of P.

(1) In $\mathcal{N}$, CS and CWF hold by the methods of [11, Theorem 3.26] and [24, proof of Theorem 10(ii)]. In [12], it has been established that CWF is equivalent to A in ZFA. Thus A holds in $\mathcal{N}$.

Another way to show that A holds in $\mathcal{N}$ is to follow the proof of [13, Theorem 9.2(2)] and use the fact that $\mathrm{Orb}_H(p)$ is an antichain in P for each $p \in P$.

(2) We show $\mathsf{CAC}^{\aleph_0}$ holds in $\mathcal{N}$. Let $(P, \leq)$ be a poset in $\mathcal{N}$ such that all chains in P are finite and all antichains in P are countable (and hence well-orderable). Now, P can be written as a well-orderable disjoint union of antichains. Thus, P is well-orderable in $\mathcal{N}$ since WUT holds in $\mathcal{N}$. So, we are done by Proposition 3.3(4). Similarly, $\mathsf{CAC}_1^{\aleph_0}$ holds in $\mathcal{N}$ by Fact 3.2.

(3) follows from Proposition 3.4 and the arguments of (2). ∎

Corollary 3.6. $\mathsf{CS} + \mathsf{A} + \mathsf{DF} = \mathsf{F} + \mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}} + \mathsf{CAC}_1^{\aleph_0} + \mathsf{DT}$ *does not imply* $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ *in* ZFA. *Consequently,* $\mathsf{CAC}_1^{\aleph_0}$ *does not imply* $\mathsf{CAC}^{\aleph_0}$ *in* ZFA.

Proof. Consider the permutation model (say $\mathcal{M}$) from [19, proof of Theorem 5(4)]. In order to describe $\mathcal{M}$, we start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a countably infinite set A of atoms, which is written as a disjoint union $\bigcup\{B_n : n \in \omega\}$, where $|B_n| = \aleph_0$ for all $n \in \omega$. For each $n \in \omega$, let $\mathcal{G}_n$ be the group of all even permutations of B_n which move only finitely many elements of B_n. Let $\mathcal{G}$ be the weak direct product of the $\mathcal{G}_n$'s for $n \in \omega$. Consequently, *every permutation of A in $\mathcal{G}$ moves only finitely many atoms.* Let $\mathcal{I}$ be the normal ideal of subsets of A generated by all finite unions of B_n. Let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in \mathcal{I}\}$, and $\mathcal{M}$ be the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}$. In $\mathcal{M}$, $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$, $\mathsf{DF} = \mathsf{F}$, and $\mathsf{CAC}_1^{\aleph_0}$ hold whereas $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ fails, and thus $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ fails (cf. [19, proof of Theorem 5(4)]). Since $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ holds in $\mathcal{N}$, and $\{a \in A : g(a) \neq a\}$ is finite for any $g \in \mathcal{G}$, DT holds in $\mathcal{N}$ following the arguments of [23, Theorem 3.4] where Tachtsis proved that DT holds in Lévy's permutation model (labeled as Model $\mathcal{N}_6$ in [9]). The rest follows from Proposition 3.5 and the fact that if $g \in \mathcal{G}$, then $\{a \in A : g(a) \neq a\}$ is finite. ∎

Proposition 3.7. $(\mathsf{ZF} + \mathsf{WOAM})$ RT *holds for any locally countable connected graph* $H = (V_H, E_H)$.

Proof. If V_H is well-orderable, the conclusion follows from Fact 3.1(1). Otherwise, by WOAM, there exists an amorphous subset $V_G \subseteq V_H$. Fix some $r \in V_G$. Let $V_0 = \{r\}$. For each $n \in \omega \setminus \{0\}$, define $V_n = \{v \in V_G : d_G(r, v) = n\}$ where "$d_G(r, v) = n$" means there are n edges in the shortest path joining r and v. By connectedness of G, $V_G = \bigcup_{n\in\omega} V_n$. Since V_G is amorphous, there is at most one $t \in \omega \setminus \{0\}$ such that V_t is infinite. As V_G is amorphous, the power set $\mathcal{P}(V_G)$ of V_G is Dedekind-finite, and thus, for some $n_0 \in \omega \setminus \{0\}$, $V_n = \emptyset$ for all $n \geq n_0$. As V_G is amorphous (and thus also infinite) and $V_G = \bigcup_{n\in\omega} V_n$ is a disjoint union, there exists exactly one $t < n_0$ such that V_t is infinite. Then V_t is countably infinite since $|V_{t-1}| < \omega$, G is locally countable, and the union of a finite family of countable sets is countable in ZF. As V_t is a countably infinite subset of the amorphous set V_G, which is impossible (since V_G is Dedekind-finite, being amorphous), we arrive at a contradiction. ■

4. Erdős–Dushnik–Miller theorem and its variants

THEOREM 4.1. (ZF) *The following hold:*

(1) $\mathsf{DC}_{\aleph_1}$ *implies* EDM. *In particular,* $\mathsf{W}_{\aleph_1}$ + "$\aleph_1$ *is regular" implies* EDM.
(2) *If* $X \in \{\mathsf{CAC}_1^{\aleph_0}, \mathsf{CAC}^{\aleph_0}, \mathsf{PAC}_{\mathsf{fin}}^{\aleph_1}, \mathsf{AC}_{\aleph_0}^{\aleph_0}, \mathsf{CAC}, \mathsf{AC}_{\mathsf{fin}}^{\aleph_0}\}$ *then* EDM *implies* X. *So,* DC *does not imply* EDM.
(3) EDM *implies* RT.
(4) $\mathsf{DF} = \mathsf{F}$ *does not imply* $\mathsf{CAC}^{\aleph_0}$. *Consequently,* RT *does not imply* $\mathsf{CAC}^{\aleph_0}$ *or* EDM.

Proof. (1) Following Proposition 3.3 and the arguments of [19, Theorem 9(1, 2)], we can see that $\mathsf{DC}_{\aleph_1}$ implies EDM in ZF. In particular, let $G = (V_G, E_G)$ be a graph and $f : [V_G]^2 \to \{0, 1\}$ be a coloring such that all sets monochromatic in color 0 are countable and all sets monochromatic in color 1 are finite. By $\mathsf{W}_{\aleph_1}$, $\aleph_1 \leq |V_G|$ or $|V_G| \leq \aleph_1$. For the second case, V_G is well-orderable, and we are done by Proposition 3.3 since $\mathsf{DC}_{\aleph_1}$ implies $\mathsf{W}_{\aleph_1}$ + "$\aleph_1$ is regular". Otherwise, V_G has a subset H with cardinality $\aleph_1$. Since H is well-orderable, it is countable by the arguments of Proposition 3.3; a contradiction.

(2) We prove EDM implies $\mathsf{CAC}^{\aleph_0}$. Let $(P, \leq)$ be a poset satisfying the hypotheses of $\mathsf{CAC}^{\aleph_0}$. Assume that P is uncountable. Let $G = (V_G, E_G)$ be a complete graph such that $V_G = P$ and let $f : [V_G]^2 \to \{0, 1\}$ be a coloring such that $f\{x, y\} = 1$ if $x \leq y$ or $y \leq x$, and $f\{x, y\} = 0$ otherwise. By EDM, either there is an uncountable set monochromatic in color 0 (which is an antichain in $(P, \leq)$) or there is a countably infinite set monochromatic in color 1 (which is a chain in $(P, \leq)$), a contradiction. Similarly, we can prove EDM implies $\mathsf{CAC}_1^{\aleph_0}$. The rest follows from Proposition 3.4 and Fact 3.1.

(3) Let A be an infinite set such that RT fails for A. Let $\{X, Y\}$ be a partition of $[A]^2$ such that there are no infinite subsets B of A with either $[B]^2 \subseteq X$ or $[B]^2 \subseteq Y$. Let $G = (V_G, E_G)$ be a complete graph such that $V_G = A$ and $f : [V_G]^2 \to \{0, 1\}$ be a coloring with $f\{x, y\} = 1$ if $\{x, y\} \in X$ and $f\{x, y\} = 0$ if $\{x, y\} \in Y$. By assumption, all sets monochromatic in

color i are finite for $i \in \{0,1\}$. By EDM, $|V_G| \le \aleph_0$ (since we are using the first definition of uncountable sets), and thus $V_G = A$ is well-orderable. The contradiction follows from the fact that RT holds for A in ZF (cf. Fact 3.1).

(4) Consider the model $\mathcal{N}_{41}$ from [9]. We start with a model M of ZFA + AC where $A = \bigcup\{B_n : n \in \omega\}$ is a disjoint union, each B_n is countably infinite and for each $n \in \omega$, $(B_n, \le_n) \cong (\mathbb{Q}, \le)$ (i.e., ordered like the rationals by $\le_n$). Let $\mathcal{G}$ be the group of all permutations on A such that for all $n \in \omega$ and all $\phi \in \mathcal{G}$, ϕ is an order automorphism of $(B_n, \le_n)$. Let $\mathcal{I}$ be the normal ideal of subsets of A which is generated by finite unions of B_n's, and let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in \mathcal{I}\}$. Let $\mathcal{N}_{41}$ be the Fraenkel–Mostowski model determined by M, $\mathcal{G}$, and $\mathcal{F}$.

In $\mathcal{N}_{41}$, DF = F holds and $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ fails (cf. [22, Theorem 4], [9, Note 112]). Pincus [17] showed that DF = F is equivalent to

$$(\forall x)(|x|_- \le \omega \to (\mathsf{Df}(x) \to \psi(x))),$$

where $\mathsf{Df}(x) \leftrightarrow \neg(\exists y)(y \subseteq x \wedge |y| = \omega)$ and $\psi(x) =$ "x is finite" (we note that $\psi(x)$ is a boundable formula). Thus, DF = F is injectively boundable. Furthermore, $\neg\mathsf{AC}_{\aleph_0}^{\aleph_0}$ is boundable, and hence injectively boundable. Since ϕ = "DF = F $\wedge\ \neg\mathsf{AC}_{\aleph_0}^{\aleph_0}$" is a conjunction of injectively boundable statements, which has a ZFA model, it follows from Theorem 2.6 that ϕ has a ZF model. By Proposition 3.4, we can see that DF = F (and thus RT) does not imply $\mathsf{CAC}^{\aleph_0}$ in ZF. ∎

THEOREM 4.2. *The following hold:*

(1) WOAM + RT *implies* EDM *and* WOAM + CAC *implies* $\mathsf{CAC}^{\aleph_0}$ *in* ZF. *In particular,* EDM *does not imply "There are no amorphous sets" in* ZFA.
(2) *Let A be a set of atoms, $\mathcal{G}$ be any group of permutations of A, and $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ which is generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Let $\mathcal{N}$ be the permutation model determined by A, $\mathcal{G}$, and $\mathcal{F}$. If $\mathcal{N}$ satisfies Mostowski's intersection lemma where A is Dedekind-finite, and* RT *holds in $\mathcal{N}$, then* EDM *holds in $\mathcal{N}$.*
(3) EDM *holds in $\mathcal{N}_3$. Consequently,* EDM *implies none of* WOAM, CS, *and* A *(Antichain Principle) in* ZFA.

Proof. (1) Assume that WOAM + RT is true. Let $G = (V_G, E_G)$ be a graph and $f : [V_G]^2 \to \{0,1\}$ be a coloring such that all sets monochromatic in color 0 are countable, and all sets monochromatic in color 1 are finite. If V_G is well-orderable then we are done by Proposition 3.3 and the fact that WOAM implies "$\aleph_1$ is regular" in ZF (cf. Fact 3.1(2)). Assume V_G is not well-orderable. By WOAM, V_G has an amorphous subset, say A. Define the following partition of $[A]^2$:

$$X = \{\{a,b\} \in [A]^2 : f\{a,b\} = 0\}, \quad Y = \{\{a,b\} \in [A]^2 : f\{a,b\} = 1\}.$$

Since $(A, E_G \restriction A)$ is an infinite graph where all sets monochromatic in color 1 are finite, there is no infinite subset $B' \subseteq A$ such that $[B']^2 \subseteq Y$. By RT, there is an infinite subset $B \subseteq A$ such that $[B]^2 \subseteq X$. So $(A, E_G \restriction A)$ has an infinite set monochromatic in color 0, say C. By assumption, C is a countably infinite subset of A. This contradicts the fact that A is amorphous.

Similarly, $\mathsf{WOAM} + \mathsf{CAC}$ implies $\mathsf{CAC}^{\aleph_0}$ in ZF in view of Proposition 3.3. The rest follows from the fact that $\mathsf{WOAM} + \mathsf{RT}$ + "There exists an amorphous set" is true in the basic Fraenkel model $\mathcal{N}_1$ (cf. [4, 9]).

(2) In $\mathcal{N}$, assume $G = (V_G, E_G)$ and $f : [V_G]^2 \to \{0, 1\}$ as in the proof of (1). If V_G is well-orderable then we are done by Proposition 3.3. Otherwise, by Lemma 2.7, there exists a bijection from an infinite subset A' of A onto some $H \subset V_G$ under the given assumptions. Define the following partition of $[H]^2$ as in the proof of (1): $X = \{\{a, b\} \in [H]^2 : f\{a, b\} = 0\}$, $Y = \{\{a, b\} \in [H]^2 : f\{a, b\} = 1\}$. By RT and following the arguments of (1), there is a countably infinite subset C of H. Thus A' is Dedekind-infinite in $\mathcal{N}$ since $|H| = |A'|$. Consequently, the set A of atoms is Dedekind-infinite in $\mathcal{N}$, which contradicts the fact that A is a Dedekind-finite set in $\mathcal{N}$.

(3) We recall the definition of $\mathcal{N}_3$ from [9]. We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a countably infinite set A of atoms using an ordering $\leq$ on A chosen so that $(A, \leq)$ is order-isomorphic to the set $\mathbb{Q}$ of rational numbers with the usual ordering. Let $\mathcal{G}$ be the group of all order automorphisms of $(A, \leq)$ and $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by the subgroups $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Let $\mathcal{N}_3$ be the Fraenkel–Mostowski model determined by M, $\mathcal{G}$, and $\mathcal{F}$. The rest follows from (2), and the following known facts about $\mathcal{N}_3$:

(i) $\mathcal{N}_3$ satisfies Mostowski's intersection lemma (cf. [13]).
(ii) RT is true in $\mathcal{N}_3$ (cf. [25, Theorem 2.4]).
(iii) The set of atoms A is a Dedekind-finite set in $\mathcal{N}_3$.
(iv) WOAM fails in $\mathcal{N}_3$ (cf. [9]).
(v) CS and LW ("Every linearly ordered set can be well ordered") fail in $\mathcal{N}_3$ [19, Theorem 7] and A implies LW in ZFA [13, Theorem 9.1]. ∎

PROPOSITION 4.3. (ZF) *The statements* EDM^n (*"If $G = (V_G, E_G)$ is a graph such that V_G is uncountable, then for every coloring $f : [V_G]^2 \to n$ there are some distinct $i_1, i_2 \in n$ such that either there is an uncountable set $X_1 \subseteq V_G$ monochromatic in color i_1 or there is a countably infinite set $X_2 \subseteq V_G$ monochromatic in color i_2"*) *are equivalent for all integers $n \geq 2$. Moreover,* EDM^n *implies* RT *for all $n \in \omega \setminus \{0, 1\}$.*

Proof. Since any $f : [V_G]^2 \to n$ maps $[V_G]^2$ to m if $m > n \geq 2$, EDM^m implies EDM^n under these circumstances. We prove that EDM^n implies EDM^{n+1} for $n \geq 2$. The rest follows by induction. Let $f : [V_G]^2 \to n+1$ be a coloring where V_G is uncountable. Let $f_1 : [V_G]^2 \to n$ be given by $f_1(A) = \min(f(A), n-1)$. By EDM^n for some distinct $i_1, i_2 \in n$, either there is an uncountable set $X_1 \subseteq V_G$ that is f_1-monochromatic in color i_1, or there is a countably infinite set $X_2 \subseteq V_G$ that is f_1-monochromatic in color i_2. Fix $k \in \{1,2\}$. If $i_k \leq n-2$, then $f[X_k]^2 = i_k \in n+1$.

Case (i): $i_1 = n-1$. Then $f(A) \in \{n, n-1\}$ for all $A \in [X_1]^2$. Define $f_2 : [X_1]^2 \to 2$ by $f_2(A) = n - f(A)$. By EDM^2 (which follows from EDM^n), for some j, either there is an uncountable set $Y_1 \subseteq X_1$ that is f_2-monochromatic in color j, or there is a countably infinite set $Y_2 \subseteq X_1$ that is f_2-monochromatic in color $1-j$ where $j \in \{0,1\}$. Thus either $f[Y_1]^2 = n-j \in n+1$ or $f[Y_2]^2 = n-1+j \in n+1$.

Case (ii): $i_2 = n-1$. Then $f(A) \in \{n, n-1\}$ for all $A \in [X_2]^2$. Define the following partition of $[X_2]^2$:

$$X = \{\{a,b\} \in [X_2]^2 : f\{a,b\} = n\}, \quad Y = \{\{a,b\} \in [X_2]^2 : f\{a,b\} = n-1\}.$$

By RT (which follows from EDM^2, see the arguments of Theorem 4.1(3)), there is a countably infinite subset $B \subseteq X_2$ such that either $[B]^2 \subseteq X$ or $[B]^2 \subseteq Y$. Thus either $f[B]^2 = n \in n+1$ or $f[B]^2 = n-1 \in n+1$.

This completes the proof of the first assertion. Following Theorem 4.1(3), EDM^2 implies RT. Thus, EDM^n implies RT for all $n \in \omega \setminus \{0,1\}$. ∎

Theorem 4.4. *Fix any $n \in \omega \setminus \{0,1\}$ and $X \in \{\mathsf{CAC}^{\aleph_0}, \mathsf{CAC}_1^{\aleph_0}, \mathsf{CS}\}$. There is a model $\mathcal{M}$ of* ZFA *where X holds but AC_n^- and the statement "There are no amorphous sets" fail. Moreover, the following hold in $\mathcal{M}$:*

(1) $\neg\mathsf{EDM}$ *and* $\neg\mathsf{EDM}^k$ *for each $k \geq 2$.*
(2) *Antichain Principle* A.

Proof. Let $\mathcal{N}_{HT}^1(n)$ be the permutation model constructed by Halbeisen–Tachtsis in the proof of [10, Theorem 8] where AC_n^- fails. Let M be a model of $\mathsf{ZFA} + \mathsf{AC}$ where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i : i \in \omega\}$ where for all $i \in \omega$, $A_i = \{a_{i_1}, \ldots, a_{i_n}\}$ and $|A_i| = n$. The group $\mathcal{G}$ is defined in [10] in such a way that *if $\eta \in \mathcal{G}$, then η only moves finitely many atoms* and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. Let $\mathcal{F}$ be the normal filter generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$ where $\mathcal{I} = [A]^{<\omega}$ is the normal ideal. The model $\mathcal{N}_{HT}^1(n)$ is the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}$. The set of atoms A is amorphous in $\mathcal{N}_{HT}^1(n)$ (cf. the proof of [10, Theorem 8]). Banerjee [1, 3] observed that $\mathsf{CAC}^{\aleph_0}$, CS, and $\mathsf{CAC}_1^{\aleph_0}$ hold in $\mathcal{N}_{HT}^1(n)$ (cf. Proposition 3.5 as well).

(1) follows from Theorem 4.1(3), Proposition 4.3, and the fact that RT fails in $\mathcal{N}_{HT}^1(n)$ (cf. [25]).

(2) follows from Proposition 3.5 since if $\eta \in \mathcal{G}$, then η only moves finitely many atoms. ∎

Remark 4.5. The referee has informed us that neither $\mathsf{CACT}^{\aleph_0}$ nor $\mathsf{CACT}_1^{\aleph_0}$ is provable in ZF. Consider the second Fraenkel model $\mathcal{N}_2$ of [9], in which the set A of atoms is a countable disjoint union of pairs so that $A = \bigcup\{A_n : n \in \omega\}$ where $|A_n| = 2$ for all $n \in \omega$ and, for all $n, m \in \omega$ with $n \neq m$, $A_n \cap A_m = \emptyset$; $\mathcal{G}$ is the group of all permutations of A which fix A_n for every $n \in \omega$, and $\mathcal{F}$ is the normal filter on $\mathcal{G}$ generated by the subgroups $\mathrm{fix}_{\mathcal{G}}(E), E \in [A]^{<\omega}$. Let

$$P = \{\emptyset\} \cup \big\{f : f \text{ is a choice function for } \{A_i : i \leq n\} \text{ for some } n \in \omega\big\}.$$

Define a partial order $\leq$ on P by stipulating, for all $p, q \in P$, $p \leq q$ if and only if $p \subseteq q$. It is clear that $(P, \leq) \in \mathcal{N}_2$ (because $\mathrm{Sym}_{\mathcal{G}}((P, \leq)) = \mathcal{G} \in \mathcal{F}$). Furthermore, $(P, \leq)$ is an ω-tree (with $\emptyset$ as its root), whose chains are all finite since the family $\mathcal{A} = \{A_n : n \in \omega\}$, which is countable in $\mathcal{N}_2$, does not have a partial choice function in $\mathcal{N}_2$. Following the proof of [26, Theorem 2.11], we can see that all antichains in P are finite in $\mathcal{N}_2$. For the sake of contradiction, assume that U is an infinite antichain in P. Let $E \in [A]^{<\omega}$ be a support of U. Without loss of generality assume that $E = \bigcup_{i \leq k} A_k$ for some $k \in \omega$. Let

$$V = \big\{p \in P : \mathrm{dom}(p) = \{A_0, \ldots, A_k\} \wedge (\exists f \in U)(p \subsetneq f)\big\}.$$

Since U is infinite and $\prod_{i \leq k} A_i$ is finite, we see that $V \neq \emptyset$ and there is an element $p_0 \in V$ such that $Y := \{f \in U : p_0 \subsetneq f\}$ is infinite. This yields the existence of at least two elements f and g of Y such that $\mathrm{dom}(f) \subsetneq \mathrm{dom}(g)$. Let

$$M = \{m \in \omega : A_m \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \text{ and } f(A_m) \neq g(A_m)\}.$$

Then for all $m \in M$, we have $m > k$. Let ϕ be the permutation of A which swaps $f(A_m)$ and $g(A_m)$ for all $m \in M$, and fixes all the other atoms. Clearly, $\phi \in \mathrm{fix}_{\mathcal{G}}(E)$. Since E is a support of U, we have $\phi(U) = U$. Thus $\phi(f) \in U$. However, $\phi(f) = g \restriction \mathrm{dom}(f)$, so $\phi(f) \subsetneq g$. This contradicts the fact that U is an antichain in P. Thus, every antichain in P is finite.

However, P is not countable in $\mathcal{N}_2$, and thus $\mathsf{CACT}^{\aleph_0}$ and $\mathsf{CACT}_1^{\aleph_0}$ are both false in $\mathcal{N}_2$. Now, since $\neg\mathsf{CACT}^{\aleph_0}$ and $\neg\mathsf{CACT}_1^{\aleph_0}$ are boundable and have a permutation model, it follows from the Jech–Sochor First Embedding Theorem (see [13, Theorem 6.1]) that they have a symmetric ZF-model.

5. Łoś's theorem, and the uniqueness of algebraic closures. We recall a fact that we need in order to prove Theorem 5.2.

FACT 5.1. *If $\mathcal{K}$ is an algebraically closed field and π is a non-trivial automorphism of $\mathcal{K}$ satisfying $\pi^2 = 1_{\mathcal{K}}$, and if $i = \sqrt{-1} \in \mathcal{K}$, then $\pi(i) = -i \neq i$* (cf. [9, Note 41]).

THEOREM 5.2. (ZFA) *The following hold:*

(1) $\mathsf{AC}^{\mathsf{LO}}$ + Form 233 *implies neither* $\mathsf{ŁT}$ *nor* $\mathsf{W}_{\aleph_1}$.
(2) $\mathsf{AC}^{\mathsf{LO}}$ + Form 233 + EDM *implies neither* $\mathsf{ŁT}$ *nor* $\mathsf{W}_{\aleph_2}$.

Proof. (1) We consider the permutation model $\mathcal{N}$ given in the proof of [21, Theorem 4.7] where $\mathsf{AC}^{\mathsf{LO}}$ holds and $\mathsf{ŁT}$ fails. We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with an $\aleph_1$-sized set A of atoms such that $A = \bigcup\{A_i : i < \aleph_1\}$ where $|A_i| = \aleph_0$ for all $i < \aleph_1$, and $A_i \cap A_j = \emptyset$ for all $i, j < \aleph_1$ with $i \neq j$. Let $\mathcal{G}$ be the group of all permutations ϕ of A such that $(\forall i < \aleph_1)$ $(\exists j < \aleph_1)(\phi(A_i) = A_j)$, and ϕ moves only $\aleph_0$ atoms. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\text{fix}_{\mathcal{G}}(E)$, where $E = \bigcup\{A_i : i \in \mathcal{I}\}$ for some $\mathcal{I} \in [\aleph_1]^{<\aleph_1}$. The model $\mathcal{N}$ is the permutation model determined by M, $\mathcal{G}$ and $\mathcal{F}$. We note that if $x \in \mathcal{N}$, then there exists $E = \bigcup\{A_i : i \in \mathcal{I}\}$ for some $\mathcal{I} \in [\aleph_1]^{<\aleph_1}$ such that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{Sym}_{\mathcal{G}}(x)$. Any such set $E \subseteq A$ is called a *support* of x.

Claim 5.3. Form 233 *holds in* $\mathcal{N}$.

Proof. Fix a field $\mathcal{K}'$ in $\mathcal{N}$. Let $\mathcal{K}$ be an algebraic closure of $\mathcal{K}'$ in $\mathcal{N}$ with support $E = \bigcup\{A_i : i \in K\}$ for some $K \in [\aleph_1]^{<\aleph_1}$. We show that $\mathcal{K}$ is well-orderable in $\mathcal{N}$. Otherwise, there is a $x \in \mathcal{K}$ and a $\phi \in \text{fix}_{\mathcal{G}}(E)$ with $\phi(x) \neq x$. Under such assumptions, Tachtsis constructed a permutation $\psi \in \text{fix}_{\mathcal{G}}(E)$ such that $\psi(x) \neq x$ but ψ^2 is the identity mapping (cf. the proof of LW ("Every linearly ordered set can be well-ordered" [9, Form 90]) in $\mathcal{N}$ from [21, Claim 4.10]). The permutation ψ induces an automorphism of $\mathcal{K}$ and we can therefore apply Fact 5.1 to conclude that $\psi(i) = -i \neq i$ for $i = \sqrt{-1} \in \mathcal{K}$.

In order to obtain a contradiction, we prove that for every $\pi \in \text{fix}_{\mathcal{G}}(E)$, $\pi(i) = i$ for every $i = \sqrt{-1} \in \mathcal{K}$. Fix an $i = \sqrt{-1} \in \mathcal{K}$. It is enough to show that E is a support of i. We note that i is a solution to the equation $x^2 + 1 = 0$ all of whose coefficients are fixed by any $\eta \in \text{fix}_{\mathcal{G}}(E)$. So if $\eta \in \text{fix}_{\mathcal{G}}(E)$, then $\eta(i)$ is also a solution to $x^2+1 = 0$. Suppose E is not a support of i. We follow the subsequent three steps to complete the proof.

Step 1: We follow the ideas due to Tachtsis from [25, proof of Lemma 1] to show that *if* $x \in \mathcal{N}$ *and* E_1, E_2 *are supports of* x*, then* $E_1 \cap E_2$ *is a support of* x*, i.e.,* $\mathcal{N}$ *satisfies Mostowski's intersection lemma* by making the minor, necessary, modifications.

Let $x \in \mathcal{N}$ and $E_1 = \bigcup\{A_i : i \in K_1\}$, $E_2 = \bigcup\{A_i : i \in K_2\}$, where $K_1, K_2 \in [\aleph_1]^{<\aleph_1}$, be two supports of x. Let $\phi \in \text{fix}_{\mathcal{G}}(E_1 \cap E_2)$, where $E_1 \cap E_2 = \bigcup\{A_i : i \in K_1 \cap K_2\}$. We will show that $\phi(x) = x$. In particular, we construct permutations $\rho_1 \in \text{fix}_{\mathcal{G}}(E_1)$, $\rho_2 \in \text{fix}_{\mathcal{G}}(E_2)$, and $\phi' \in \text{fix}_{\mathcal{G}}(E_1 \cup E_2)$ such that $\phi = \rho_1^{-1}\rho_2^{-1}\phi'\rho_2\rho_1$. Then $\phi(x) = \rho_1^{-1}\rho_2^{-1}\phi'\rho_2\rho_1(x) = x$, and we are done.

We first let $\mathcal{A} = \{A_i : i < \aleph_1\}$ and, for a set $x \subseteq A$, we let $\text{tr}(x) = \{i \in \aleph_1 : A_i \cap x \neq \emptyset\}$. Put $W = \{a \in A : \phi(a) \neq a\}$ and $W^* = \bigcup\{A_i : i \in \text{tr}(W)\}$. Note that, from the definition of $\mathcal{G}$, W is countable, and thus so is W^*. Let $\mathcal{U}, \mathcal{V}$ be two disjoint subsets of $\mathcal{A}$ such that each of the sets

$U = \bigcup \mathcal{U}$ and $V = \bigcup \mathcal{V}$ is disjoint from $E_1 \cup E_2 \cup W^*$, $\mathrm{tr}(U)$ has the same order type as $K_2 \setminus K_1$ and $\mathrm{tr}(V)$ has the same order type as $(K_1 \cup \mathrm{tr}(W)) \setminus K_2$. Consider two bijections $f : U \to E_2 \setminus E_1$ and $g : V \to (E_1 \cup W^*) \setminus E_2$ such that if $i \in \mathrm{tr}(U)$ and $j \in \mathrm{tr}(V)$, then $f[A_i] \in \{A_k : k \in K_2 \setminus K_1\}$ and $g[A_j] \in \{A_k : k \in (K_1 \cup \mathrm{tr}(W)) \setminus K_2\}$. We define

$$\rho_1 = \prod_{u \in U} (u, f(u)) \quad \text{and} \quad \rho_2 = \prod_{v \in V} (v, g(v)),$$

i.e., each one of ρ_1 and ρ_2 as a product of disjoint transpositions. Define $\phi' = \rho_2 \rho_1 \phi \rho_1^{-1} \rho_2^{-1}$. We can see that $\rho_1 \in \mathrm{fix}_{\mathcal{G}}(E_1)$ and $\rho_2 \in \mathrm{fix}_{\mathcal{G}}(E_2)$. In order to see that $\phi' \in \mathrm{fix}_{\mathcal{G}}(E_1 \cup E_2)$, let $e \in E_1 \cup E_2$. We consider three cases: $e \in E_2 \setminus E_1$ or $e \in E_1 \cap E_2$ or $e \in E_1 \setminus E_2$. In each of these cases, we can see that $\phi'(e) = e$. This completes the proof of Step 1.

Let E' be a support of i and let $F = E' \setminus E$. Then $F \neq \emptyset$ (since E is not a support of i) and $F \cap E = \emptyset$. Without loss of generality, we may assume that $E \subsetneq E'$.

STEP 2: Using Step 1, we prove that *if* $\phi, \phi' \in \mathrm{fix}_{\mathcal{G}}(E)$ *and* $\phi(F) \cap \phi'(F) = \emptyset$, *then* $\phi(i) \neq \phi'(i)$. For the sake of contradiction assume that ϕ, ϕ' in $\mathrm{fix}_{\mathcal{G}}(E)$ are such that $\phi(F) \cap \phi'(F) = \emptyset$ and $\phi(i) = \phi'(i)$. Then $(\phi')^{-1}\phi(i) = i$. Since E' supports i, it follows that $(\phi')^{-1}\phi(E')$ supports $(\phi')^{-1}\phi(i) = i$. By Step 1, $(\phi')^{-1}\phi(E') \cap E'$ supports i. However, $(\phi')^{-1}\phi(E') \cap E' = E$ since $\phi(F) \cap \phi'(F) = \emptyset$ and $\phi, \phi' \in \mathrm{fix}_{\mathcal{G}}(E)$, which contradicts the assumption that E is not a support of i.

STEP 3: Using Step 2 and the features of $\mathcal{N}$, we can observe that *there is a set* $S = \{\phi_k(i) : k \in \omega\}$ *in* $\mathcal{N}$ *such that for every* $k \in \omega$, $\phi_k \in \mathrm{fix}_{\mathcal{G}}(E)$, *and for all* $k, l \in \omega$, *if* $k \neq l$ *then* $\phi_k(i) \neq \phi_l(i)$.

In particular, we consider a denumerable collection $\{B_i : i \in \omega\}$ of subsets of $A \setminus (E \cup F)$ where $B_i = \bigcup \{A_j : j \in K_i\}$ for some $K_i \in [\aleph_1]^{<\aleph_1}$ such that $|K_i| = |\mathrm{tr}(F)|$ for every $i \in \omega$, and $B_i \cap B_j = \emptyset$ if $i \neq j$. For every $k \in \omega$, let $H_k : F \to B_k$ be a bijection such that, for every $i \in \mathrm{tr}(F)$, $H_k[A_i] = A_j$ for some $j \in K_k$, and also let $\phi_k = \prod_{a \in F} (a, H_k(a))$. Then $\phi_k(F) = B_k$. Fix any $k, l \in \omega$ such that $k \neq l$. By Step 2, $\phi_k(i) \neq \phi_l(i)$ since $B_k \cap B_l = \emptyset$. Since, E' is a support of i, $\phi_k(E')$ is a support of $\phi_k(i)$ for every $k \in \omega$. Then $\bigcup_{k \in \omega} \phi_k(E')$ is a support of $S = \{\phi_k(i) : k \in \omega\}$ since the set of supports is closed under countable unions. Consequently, the set S is in $\mathcal{N}$ and S is infinite (in fact, countably infinite).

So, the equation $x^2 + 1 = 0$ has infinitely many solutions in $\mathcal{K}$, which is a contradiction. Thus, E is a support of i.

Since $\mathcal{K}$ is well-orderable in $\mathcal{N}$, we can use transfinite induction without using any form of choice to finish the proof. ∎

CLAIM 5.4. $W_{\aleph_1}$ *fails in* $\mathcal{N}$.

Proof. First, we show that $\mathrm{Sym}(A) = \aleph_0\mathrm{Sym}(A)$ in $\mathcal{N}$. For the sake of contradiction, assume $f \in \mathrm{Sym}(A) \setminus \aleph_0\mathrm{Sym}(A)$. Let $E = \bigcup\{A_i : i \in \mathcal{I}\}$ with $\mathcal{I} \in [\aleph_1]^{<\aleph_1}$ be a support of f. Then there exists $i \in \aleph_1 \setminus \mathcal{I}$ such that $a \in A_i$, $b \in A \setminus (E \cup \{a\})$ and $b = f(a)$. Let $b \in A_i$. Consider $\phi\restriction A_i$ such that $\phi\restriction A_i$ moves every atom in A_i except b and $\phi\restriction(A \setminus A_i) = 1_{A\setminus A_i}$. Clearly, $\phi \in \mathcal{G}$. Also, $\phi(b) = b$, $\phi \in \mathrm{fix}_{\mathcal{G}}(E)$, and hence $\phi(f) = f$. Thus

$$(a, b) \in f \implies (\phi(a), \phi(b)) \in \phi(f) \implies (\phi(a), b) \in \phi(f) = f.$$

So f is not injective; a contradiction. If $b \in A \setminus (E \cup A_i)$, then consider $\phi\restriction A_i$ such that $\phi\restriction A_i$ moves every atom in A_i and $\phi\restriction(A\setminus A_i) = 1_{A\setminus A_i}$. Again $\phi \in \mathcal{G}$, and we easily obtain a contradiction.

We note that $|A| \not\leq \aleph_1$ in $\mathcal{N}$. In order to show that $W_{\aleph_1}$ fails, we prove that there is no injection $f : \aleph_1 \to A$. Assume there exists such an f and suppose $\{y_n\}_{n\in\aleph_1}$ is an enumeration of the elements of $Y = f(\aleph_1)$. We can use transfinite recursion, without using any form of choice, to construct a bijection $h : Y \to Y$ such that $h(x) \neq x$ for any $x \in Y$. Define $g : A \to A$ as follows: $g(x) = h(x)$ if $x \in Y$, and $g(x) = x$ if $x \in A \setminus Y$. Clearly $g \in \mathrm{Sym}(A)\setminus \aleph_0\mathrm{Sym}(A)$, and hence $\mathrm{Sym}(A) \neq \aleph_0\mathrm{Sym}(A)$; a contradiction. ∎

(2) Consider the permutation model of (1) (say $\mathcal{N}$) by replacing $\aleph_0$ and $\aleph_1$ with $\aleph_1$ and $\aleph_2$ respectively. Following the arguments of [21, Theorem 4.7], we can see that $\mathsf{AC}^{\mathsf{LO}}$ holds in $\mathcal{N}$, but ŁT fails. By the arguments of the previous proof, $\mathsf{W}_{\aleph_2}$ fails and Form 233 holds in $\mathcal{N}$. Moreover, $\mathsf{DC}_{\aleph_1}$ holds since $\mathcal{I}$ is closed under $< \aleph_2$ unions (cf. [13, the arguments in the proof of Theorem 8.3(i)]). Consequently, EDM holds by Theorem 4.1(1). ∎

6. Concluding remarks and questions

6.1. Remarks. (1) Recently, Banerjee [2] and Karagila [14] proved that if V is a model of ZFC, then $\mathsf{DC}_{\aleph_1}$ can be preserved in the symmetric extension $\mathcal{N}$ of V (symmetric submodel of a forcing extension where AC can consistently fail) if the forcing notion $\mathbb{P}$ is either $\aleph_2$-distributive or $\aleph_2$-c.c., $\mathcal{G}$ is any group of automorphisms of $\mathbb{P}$, and the normal filter $\mathcal{F}$ of subgroups over $\mathcal{G}$ is $\aleph_2$-complete. By Theorem 4.1, EDM holds in $\mathcal{N}$.

(2) By Theorems 4.1(1, 3), and the facts that $\mathsf{DC}_{\aleph_1}$ does not imply ŁT and ŁT does not imply RT in ZF (cf. [21, Theorems 4.3, 4.13]), ŁT and EDM are mutually independent in ZF.

(3) Fix $X \in \{\mathsf{BPI}, \mathsf{KW}, \mathsf{AC}_{\mathsf{WO}}$, 2-coloring theorem, "There are no amorphous sets"$\}$. Blass [4] proved that RT is false in the basic Cohen model (Model $\mathcal{M}_1$ in [9]) where X holds. Following Theorem 4.1(3), X does not imply EDM in ZF. On the other hand, X fails in $\mathcal{N}_1$. Thus EDM and X are mutually independent in ZFA by Theorem 4.2(1).

(4) Fix $X \in \{\mathsf{A}, \mathsf{WOAM}, \mathsf{CS}\}$. We can see that X and EDM are mutually independent in ZFA. Following Theorem 4.4 and the fact that X holds in $\mathcal{N}^1_{HT}(2)$ (4), X does not imply EDM in ZFA. The other direction follows from Theorem 4.2(3).

(5) Consider the permutation model $\mathcal{M}$ from Corollary 3.6 where $\mathsf{CAC}^{\aleph_0}$ fails (and thus EDM fails by Theorem 4.1) and DT holds. Secondly, we consider the permutation model $\mathcal{V}$ from [19, Theorem 9(4)] where DT fails and $\mathsf{DC}_{\aleph_1}$ holds, and hence EDM and $\mathsf{CAC}^{\aleph_0}$ hold as well. Consequently, if $X \in \{\mathsf{EDM}, \mathsf{CAC}^{\aleph_0}\}$, then X and DT are mutually independent in ZFA.

(6) Following the arguments of [19, Theorem 4(11)] due to Tachtsis (where he proved that $\mathsf{CAC}^{\aleph_0}_1$ implies CAC in ZF) we can see that $\mathsf{CAC}^{\aleph_0}$ implies CAC in ZF. By Theorem 4.1(4), CAC does not imply $\mathsf{CAC}^{\aleph_0}$ in ZF since $\mathsf{DF} = \mathsf{F}$ implies CAC in ZF.

(7) The referee remarked that $\mathsf{CACT}^{\aleph_0}_1 + \mathsf{CACT}^{\aleph_0} \nrightarrow \mathsf{CAC}$ in ZFA. In particular, by [19, Theorem 6] due to Tachtsis, $\mathsf{AC}_{\mathsf{WO}}$ (i.e., the axiom of choice for families of non-empty, well-orderable sets) does not imply CAC in ZFA, and thus neither does $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ imply CAC in ZFA. This, together with Proposition 3.3(5) yields $\mathsf{CACT}^{\aleph_0}_1 + \mathsf{CACT}^{\aleph_0}$ does not imply CAC in ZFA.

(8) In the second Fraenkel model $\mathcal{N}_2$, $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ fails but MC holds. By Theorem 4.1, if $X \in \{\mathsf{EDM}, \mathsf{CAC}^{\aleph_0}_1, \mathsf{CAC}^{\aleph_0}\}$ then MC does not imply X in ZFA. Since MC fails in $\mathcal{N}_1$, EDM and MC are mutually independent in ZFA by Theorem 4.2.

6.2. Questions

Question 6.1. Does $\mathsf{CAC}^{\aleph_0}$ imply $\mathsf{CAC}^{\aleph_0}_1$ in ZF or in ZFA?

Question 6.2. Does $\mathsf{BPI} + \mathsf{DC}$ imply EDM in ZF or in ZFA?

Question 6.3. Does WOAM imply $\mathsf{CAC}^{\aleph_0}$ in ZF or in ZFA?

Question 6.4. Does $\mathsf{AC}^{\mathsf{LO}}$ imply EDM in ZFA?

Question 6.5. Does EDM hold in Brunner/Pincus's Model ($\mathcal{N}_{26}$ in [9])?

Question 6.6. Does either of $\mathsf{CACT}^{\aleph_0}$ and $\mathsf{CACT}^{\aleph_0}_1$ imply $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ in ZF or in ZFA?

Question 6.7. Does $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ hold in Pincus's Model X ($\mathcal{N}_{34}$ in [9])?

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(4) Tachtsis proved that WOAM holds in $\mathcal{N}^1_{HT}(2)$ [25, Lemma 2].

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