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CONTINUITY OF THE OPERATION IN A SEMILATTICE

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1. Introduction. By a *semilattice* we shall mean a set X together with an associative, commutative, idempotent operation $\wedge$ defined on X. In a natural way $\wedge$ induces a partial order on X, i.e., $x \leqslant y$ iff $x \wedge y = x$.

A *topological semilattice* is a semilattice X, where X is a Hausdorff space and $\wedge$ is continuous. Topological semilattices are similar to topological lattices [6] and have attracted the attention of various authors, e.g., [1] and [2]. An example of a lattice in which $\wedge$ is continuous but not $\vee$ is given in [1].

In [1] and [7] various continuity properties of $\leqslant$ in a topological semilattice were obtained. (Continuous relations have been studied in [4], [5], and [7].) In [3] sufficient conditions on the continuity of $\leqslant$ to insure the continuity of $\wedge$ in a compact Hausdorff space were given. The purpose of this note is to extend the results of [3] by giving necessary and sufficient conditions on the continuity of $\leqslant$ to insure the continuity of $\wedge$ in a compact Hausdorff space. The previously mentioned example in [1] shows that these conditions cannot be self-dual. However, it is clear that the results given here may be extended to the lattice case by using the conditions given here and the dual of these conditions.

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2. Main theorems.

THEOREM 1. *Let X be a compact Hausdorff space and let $\wedge$ be a semilattice operation on X such that the induced partial order $\leqslant$ is order dense. Then $\wedge$ is continuous if and only if*

(i) *the graph of $\leqslant$ is closed*

and

(ii) *$x, y \in X$, $x \leqslant y$, $\{y_a\}$ and $\{y'_a\}$ are nets which converge to y implies that there exists a net $\{x_a\}$ which converges to x such that $x_a \leqslant y_a \wedge y'_a$.*

Proof. Necessity. The fact that the graph of $\leqslant$ is closed if $\wedge$ is continuous is well known. It remains only to show that (ii) holds. Let $x_a = x \wedge y_a \wedge y'_a$. Since $\wedge$ is continuous, $\lim x_a = x \wedge \lim y_a \wedge \lim y'_a = x \wedge y = x$.

Sufficiency. Let $x, y \in X$, $\{x_a\}$ be a net which converges to x, $\{y_a\}$ be a net which converges to y, and let z be a cluster point of $\{x_a \wedge y_a\}$. To show that $\wedge$ is continuous, it suffices to show $z = x \wedge y$. By selection of subnets, we may assume that $\{x_a \wedge y_a\}$ converges to z. Notice that since the graph of $\leqslant$ is closed, $z \leqslant x \wedge y$. We shall assume that $z < x \wedge y$ and arrive at a contradiction. We shall divide the remainder of the proof into two cases.

Case 1: $x = y$. Thus, $z < x \wedge y = x$. From the order dense assumption there exists $z' \in X$ such that $z < z' < x$. By (ii) there exists a net $\{z_a\}$ converging to z' such that $z_a \leqslant x_a \wedge y_a$. Since the graph of $\leqslant$ is closed, it follows that $z' \leqslant z$ which is a contradiction.

Case 2: $x \neq y$. Since $z < x \wedge y \leqslant y$ and $z < x \wedge y \leqslant x$, it follows from (ii) that there exist nets $\{t_a\}$ and $\{k_a\}$ converging to $x \wedge y$ such that $t_a \leqslant x_a$ and $k_a \leqslant y_a$. From Case 1 we see that $\{t_a \wedge k_a\}$ converges to $x \wedge y$. But since $t_a \wedge k_a \leqslant x_a \wedge y_a$ and since the graph of $\leqslant$ is closed, $x \wedge y \leqslant z$, which is a contradiction.

Let X be a semilattice and a compact Hausdorff space and let P $P = \{(x, y) | x < y\}$. Then

$$P^{(-1)} = \{(y, x) | x \leqslant y\}.$$

In Theorem 2 we shall use the work done in [4] and [5] on the continuity of relations.We shall say that $P^{(-1)}$ is *upper semicontinuous* and *point closed* iff P is closed in $X \times X$ in the case when X is compact [5]. The relation $P^{(-1)}$ is lower semicontinuous iff U, an open subset of X, implies that $\{x | \text{there exists } w \in U \text{ such that } x \leqslant w\}$ is open [5]. The relation $P^{(-1)}$ is continuous iff it is point-closed, upper semicontinuous and lower semicontinuous [4].

The following remark, due to Strother [4] will be useful in Theorem 2:

Remark. The following are equivalent:

(a) $P^{(-1)}$ is lower semicontinuous.

(b) If $x \leqslant y$ and $\{y_a\}$ is a net converging to y, then there exists a net $\{x_a\}$ converging to x such that $x_a \leqslant y_a$ for all a.

THEOREM 2. *Let X be a compact Hausdorff space and let $\wedge$ be a semilattice operation on X. Then $\wedge$ is continuous if and only if $\wedge$ is continuous on the diagonal of $X \times X$ and $P^{(-1)}$ is continuous, where P is the graph of the induced partial order.*

Proof. Necessity. The only part of the necessity which is not clear is the fact that $P^{(-1)}$ is lower semicontinuous. Let $(x, y) \epsilon P$ and $\{y_a\}$ be a net converging to y. Define $x_a = x \wedge y_a$. It follows from the continuity of $\wedge$ that $\{x_a\}$ converges to x and thus, by Strother's remark, it follows that $P^{(-1)}$ is lower semicontinuous.

The sufficiency follows from Strother's remark in the same manner as Case 2 of Theorem 1.

The following is an example of a semilattice X in which $P^{(-1)}$ is continuous but $\wedge$ is not continuous on the diagonal of $X \times X$.

EXAMPLE. Let X be the part of the plane bounded by $x = -1$, $x = 1$, $y = 1$, and $y = 0$ together with its boundary. Let $A = \{(x, y) \epsilon X \mid x < 0\}$, $B = \{(x, y) \epsilon X \mid x = 0\}$, and $C = \{(x, y) \epsilon X \mid x > 0\}$. Define $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if one of the following holds:

(1) $(x_1, y_1), (x_2, y_2) \epsilon A$, $x_2 \leqslant x_1$ and $y_1 \leqslant y_2$.

(2) $(x_1, y_1), (x_2, y_2) \epsilon B$ and $y_1 \leqslant y_2$.

(3) $(x_1, y_1), (x_2, y_2) \epsilon C$, $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

(4) $(x_1, y_1) \epsilon B$, $(x_2, y_2) \epsilon A \cup C$ and $y_1 \leqslant |x_2|$.

Then $\wedge$ is not continuous at $\big((0, 1), (0, 1)\big)$.

Let X be a Hausdorff space with a semilattice operation $\wedge$ and let $P = \{(x, y) | x \leqslant y\}$. In [3] P is said to be *strongly continuous* iff the boundary of P is contained in the diagonal of $X \times X$. Another formulation of this definition given in [3] is that P is strongly continuous iff, when $x < y$, there are open neighborhoods U and V of x and y, respectively, such that every point of U is less than every point of V. The next proposition established a connection between strongly continuity and lower semicontinuity.

PROPOSITION 3. *Strong continuity of P implies $P^{(-1)}$ is lower semicontinuous.*

Proof. Suppose U is an open subset of X. Let $V = \{x |$ there exists $y \epsilon U$ with $x \leqslant y\}$. It suffices to show that V is open. If $x \epsilon V$, then there is $y \epsilon U$ such that $x \leqslant y$. If $x = y$, then $x \epsilon U \subset V$. If $x < y$, then by the definition of strong continuity there are open neighborhoods V' and U' of x and y, respectively, such that $U' \subset U$ and every element of V' is less than every element of U'. Thus $x \epsilon V' \subset U$.

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