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BLOW-UP RESULTS FOR A BOUSSINESQ-TYPE PLATE EQUATION WITH A LOGARITHMIC DAMPING TERM AND VARIABLE-EXPONENT NONLINEARITIES

Dedicated to Professor Stanislav Antontsev
on the occasion of his 80th anniversary

Abstract. We consider the initial boundary value problem for a Boussinesq-type equation with a logarithmic damping term and variable-exponent nonlinearities, which was introduced to describe some physical phenomena such as propagation of small amplitude and long waves on the surface of shallow water. The blow-up of solutions is proved for positive as well as negative initial energy.

1. Introduction. In this paper, we study the variable-exponent fourth-order wave equation with a logarithmic damping term,

$$u_{tt} - \Delta u - a\Delta u_{tt} + \Delta^2 u + \operatorname{div}(\nabla u \cdot \ln|\nabla u|^{p(x)}) - b\Delta u_t = |u|^{q(x)-2}u \tag{1.1}$$

for $(x,t) \in \Omega \times [0,T)$, with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary conditions

$$u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0 \quad \text{or} \quad u(x,t) = \Delta u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T), \tag{1.3}$$

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where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open bounded domain with smooth boundary $\partial\Omega$, $T > 0$ is the maximum existence time of u and n is the unit outward normal on $\partial\Omega$. Here $a, b > 0$ and the exponents $p(\cdot), q(\cdot)$ are given measurable functions on $\bar{\Omega}$ such that

$$0 < p_1 \leq p(x) \leq p_2 < \begin{cases} \infty, & n < 3, \\ \frac{2n}{n-2}, & n \geq 3, \end{cases}$$

$$2 \leq q_1 \leq q(x) \leq q_2 < \begin{cases} \infty, & n < 3, \\ \frac{2n}{n-2}, & n \geq 3, \end{cases}$$

with

$$p_1 := \operatorname*{ess\,inf}_{x \in \overline{\Omega}} p(x), \quad p_2 := \operatorname*{ess\,sup}_{x \in \overline{\Omega}} p(x),$$
$$q_1 := \operatorname*{ess\,inf}_{x \in \overline{\Omega}} q(x), \quad q_2 := \operatorname*{ess\,sup}_{x \in \overline{\Omega}} q(x).$$

In recent years, there has been an increasing activity on models involving nonlinear fourth-order partial differential equations. For example, Di et al. [9] considered the equation

$$u_{tt} + \Delta^2 u = |u|^p u,$$

and obtained the global existence and uniqueness of regular solution and weak solution by using Galerkin approximation and potential well methods. In another study, Yang et al. [29] investigated a fourth-order wave equation with a strong damping term of the form

$$u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = f(u).$$

They proved the finite time blow-up of solutions at three different initial energy levels. Di and Shang [8] studied the following double dispersive-dissipative fourth-order wave equation with nonlinear damping and source terms:

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_{tt} - \Delta u_t + a|u_t|^{m-2} u_t = b|u|^{p-2} u,$$

and proved the global existence and asymptotic behavior of solutions by using the Galerkin and monotonicity compactness methods. Later, Chen and Xu [7] considered the following fourth-order dispersive wave equation with a nonlinear weak damping term, linear strong damping and logarithmic source terms:

$$u_{tt} - \Delta u + \Delta^2 u - \omega(\Delta u_{tt} + \Delta u_t) + |u_t|^{r-1} u_t = u \ln |u|.$$

Under several conditions for initial data, they established the global existence of solutions and infinite time blow-up, by using the potential well method. Also, they compared and discussed the blow-up of solutions from two different strategies. In the first one, the authors assumed that the blow-up result is bound to the original logarithmic source by weakening the dispersive-dissipative structure, while the second one is based on the nonlinear wave equation with complete dispersive-dissipative structure, but the logarithmic source is replaced by an enhanced version. Recently, Zhang and Zhou [30] considered the sixth-order Boussinesq equation with logarithmic nonlinearity:

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u + \Delta(u\log u) = 0,$$

in a bounded domain of $\mathbb{R}^n$. They proved the well-posedness and dynamical behavior of solutions. The main ingredient of their work is the introduction of several conditions on initial data leading to global existence of solutions with finite time blow-up. In another study, Pang et al. [18] studied the global existence and blow-up in infinite time for the following fourth-order wave equation with damping and logarithmic strain terms:

$$u_{tt} + \alpha\Delta^2 u - \beta\Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|u_{x_i}| \ln |u_{x_i}|^p) - \Delta u_t + |u_t|^{r-1}u_t = |u|^{q-1}u,$$

where $\alpha, \beta > 0$ and in a bounded open subset of $\mathbb{R}^n$ $(n \leq 3)$.

On the other hand, wave equations with variable-exponent nonlinearities and nonstandard growth conditions attracted the attention of many researchers in recent years. Ferreira and Messaoudi [12] considered a nonlinear viscoelastic plate equation with a lower-order perturbation of the form

$$\partial_{tt}u + \Delta_x^2 u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i}\right) + \int_0^t \mu(t-s)\Delta_x u(s)\,ds - \epsilon\Delta_x\partial_t u + f(u) = 0. \tag{1.4}$$

They proved a general decay result under suitable conditions on g, f and the variable exponent of the $\vec{p}(x,t)$-Laplacian operator

$$\Delta_{\vec{p}(x,t)}u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i}\right).$$

Later, Antontsev and Ferreira [1] proved a blow-up result for (1.4) with negative initial energy under suitable conditions on g, f and the variable-exponent of the $\vec{p}(x,t)$-Laplacian operator. Recently, Shahrouzi [25] studied the following variable-exponent fourth-order viscoelastic initial boundary value problem:

$$\begin{aligned} &|u_t|^{\rho(x)}u_{tt} + \Delta[(a + b|\Delta u|^{m(x)-2})\Delta u] - \int_0^t g(t-s)\Delta^2 u(s)\,ds \\ &\qquad = |u|^{p(x)-2}u, \qquad x \in \Omega,\ t > 0, \\ &\begin{cases} u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ a\Delta u(x,t) = \int_0^t g(t-s)\Delta u(s)\,ds - b|\Delta u|^{m(x)-2}\Delta u, & x \in \Gamma_1,\ t > 0, \end{cases} \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Under suitable conditions on variable exponents and initial data, he proved that the solutions will grow up as an exponential function with positive initial energy level. See also [2–5, 13, 16, 17, 19, 20, 24, 26–28].

In addition to the introduction, this paper consists of two sections. Firstly, in Section 2, we present the definitions and some properties of the variable-exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ and we introduce the energy functional. In Section 3, we prove the blow-up of solutions for positive initial energy $E(0) < 0$ and negative initial energy $E(0) < 0$.

2. Preliminaries. In this work, we use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^k(\Omega)$ with their usual scalar products and norms. We denote by $\|\cdot\|_q$ the L^q-norm over Ω. In particular, the L^2-norm in Ω is denoted $\|\cdot\|$.

To investigate problem (1.1)–(1.3), some information about the Lebesgue and Sobolev function spaces with variable exponents is required (for more details, see [6, 10]).

Suppose that $p : \Omega \to [1, \infty]$ is a measurable function, where Ω is a subset of $\mathbb{R}^n$. The *variable-exponent Lebesgue space* is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{u \;\Big|\; u \text{ is measurable in } \Omega \text{ and } \int_\Omega |\lambda u(x)|^{p(x)}\,dx < \infty \text{ for some } \lambda > 0\Big\}.$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg-type norm

$$\|u\|_{p(\cdot)} := \inf\Big\{\lambda > 0 \;\Big|\; \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,dx \leq 1\Big\}.$$

Lemma 2.1 ([10]). *Let Ω be a bounded domain in $\mathbb{R}^n$. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, we have the generalized Hölder inequality*

$$\Big|\int_\Omega fg\,dx\Big| \leq \Big(\frac{1}{p_1} + \frac{1}{q_1}\Big)\|f\|_{p(\cdot)}\|g\|_{q(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

The following formula is used to determine the relationship between the modular $\int_\Omega |f|^{p(x)}\,dx$ and the norm:

$$\min(\|f\|_{p(\cdot)}^{p_1}, \|f\|_{p(\cdot)}^{p_2}) \leq \int_\Omega |f|^{p(x)}\,dx \leq \max(\|f\|_{p(\cdot)}^{p_1}, \|f\|_{p(\cdot)}^{p_2}).$$

The *variable-exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla_x u \textit{ exists and } |\nabla_x u| \in L^{p(\cdot)}(\Omega)\}.$$

It is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla_x u\|_{p(\cdot)}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, an equivalent norm is defined as

$$\|u\|_{1,p(\cdot)} = \|\nabla_x u\|_{p(\cdot)}.$$

Let the *log-Hölder continuity condition* be satisfied by the variable component $p(\cdot)$: there are constants $A > 0$ and $0 < \delta < 1$ such that

$$|p(x) - p(y)| \le \frac{-A}{\log|x-y|} \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta.$$

LEMMA 2.2 (Poincaré inequality [6, 10]). *Suppose that Ω is a bounded domain of $\mathbb{R}^n$ and the log-Hölder condition is satisfied by $p(\cdot)$. Then*

$$\|u\|_{p(\cdot)} \le c_p \|\nabla_x u\|_{p(\cdot)} \quad \textit{for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.1}$$

where $c_p = c(p_1, p_2, |\Omega|) > 0$.

LEMMA 2.3 ([6, 10]). *Let $p(\cdot) \in C(\bar{\Omega})$ and $q : \Omega \to [1,\infty)$ be a measurable function that satisfy*

$$\operatorname*{ess\,inf}_{x\in\bar{\Omega}} (p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact, where

$$p^* = \begin{cases} \frac{np_1}{n-p_1} & \textit{if } p_1 < n, \\ \infty & \textit{if } p_1 \ge n. \end{cases}$$

Moreover, if the log-Hölder condition is satisfied by $p(\cdot)$, we have

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)} & \textit{if } p(x) < n, \\ \infty & \textit{if } p(x) \ge n. \end{cases}$$

LEMMA 2.4 (Young inequality [10]). *Let $q, q', s : \Omega \to [1,\infty)$ be measurable functions such that*

$$\frac{1}{s(x)} = \frac{1}{q(x)} + \frac{1}{q'(x)} \quad \textit{for a.e. } x \in \Omega.$$

Then, for all $X, Y \ge 0$,

$$\frac{(XY)^{s(\cdot)}}{s(\cdot)} \le \frac{X^{q(\cdot)}}{q(\cdot)} + \frac{Y^{q'(\cdot)}}{q'(\cdot)}.$$

By taking $s = 1$, it follows that for any $\theta > 0$,

$$XY \le \theta X^{q(\cdot)} + C(\theta, q(\cdot))Y^{q'(\cdot)}, \tag{2.2}$$

where $C(\theta, q(\cdot)) = \frac{1}{q'(\cdot)}(\theta q(\cdot))^{-\frac{q'(\cdot)}{q(\cdot)}}$.

In order to prove the blow-up result with positive initial energy, we use the following lemma that was introduced in [15] and called the modified concavity method.

LEMMA 2.5. *Let $\mu > 0$, $c_1, c_2 \ge 0$ and $c_1 + c_2 > 0$. Assume that $L(t)$ is a twice differentiable positive function such that*

$$L(t)L''(t) - (1+\mu)[L'(t)]^2 \ge -2c_1 L(t)L'(t) - c_2[L(t)]^2$$

for all $t \ge 0$. If

$$L(0) > 0 \quad \text{and} \quad L'(0) + \gamma_2 \mu^{-1} L(0) > 0,$$

then there exists a finite time t^* such that

$$L(t) \to +\infty \qquad \text{as } t \to t^*.$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \mu c_2} \quad \text{and} \quad \gamma_2 = -c_1 - \sqrt{c_1^2 + \mu c_2}.$$

For the sake of completeness, the local existence result for problem (1.1)–(1.3) is stated as follows. This theorem could be proved by the Faedo–Galerkin approximation method. For details we refer the reader to [11, 14, 18, 21–23].

THEOREM 2.6 (Local existence). *Let $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ be given. Assume that variable exponents are bounded. Then problem (1.1)–(1.3) has a weak solution such that*

$$u \in L^\infty(0, T; H_0^2(\Omega) \cap L^{q(\cdot)}(\Omega)), \qquad u_t \in L^\infty(0, T; H_0^1(\Omega)),$$
$$u_{tt} \in L^2(0, T; H_0^1(\Omega)).$$

The energy of the system is defined by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{a}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{4}\int_\Omega p(x)|\nabla u|^2\,dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx. \tag{2.3}$$

LEMMA 2.7 (Monotonicity of energy). *Assume that $u(x,t)$ is a local solution of (1.1)–(1.3). Then, along the solution, we have*

$$E'(t) = -b\|\nabla u_t\|^2 \le 0. \tag{2.4}$$

Proof. Multiplying equation (1.1) by u_t and integrating over Ω, inequality (2.4) can be obtained for any regular solution. By a simple density argument, the inequality is fulfilled for weak solutions. ∎

3. Blow-up results. This section has two parts. First, we assume that problem (1.1)–(1.3) has positive initial energy and prove that under suitable conditions on the data, there exists a finite time such that the solutions blow-up at this time. Next, the blow-up of solutions with negative initial energy under appropriate conditions on the variable exponents is proved.

3.1. Blow-up result for positive initial energy. In order to prove the blow-up result for positive initial energy, we define, for any $\varepsilon > 0$,

$$\psi(t) = \int_{\Omega} u(u_t - a\Delta u_t)\,dx + \frac{b}{2}\|\nabla u\|^2 - \frac{2}{\varepsilon}E(t). \tag{3.1}$$

Regarding the functional $\psi(\cdot)$, we have

LEMMA 3.1. *Assume that $u(x,t)$ is a local solution of* (1.1)–(1.3) *and $a \ge \max\{c_2, 4b^2/p_1\}$ where c_2 satisfies inequality* (2.1) *for $p(x) \equiv 2$. Then, along the solution, the functional $\psi(\cdot)$ satisfies*

$$\psi(t) \ge \psi(0)e^{\frac{2b}{a}t}.$$

Proof. Differentiating (3.1), we obtain

$$\begin{aligned}\psi'(t) &= \|u_t\|^2 + a\|\nabla u_t\|^2 + \int_{\Omega} u(u_{tt} - a\Delta u_{tt})\,dx + b\int_{\Omega} \nabla u \nabla u_t\,dx \\ &\quad - \frac{2}{\varepsilon}E'(t) \\ &= \|u_t\|^2 + \left(a + \frac{2b}{\varepsilon}\right)\|\nabla u_t\|^2 + \int_{\Omega} u(u_{tt} - a\Delta u_{tt})\,dx \\ &\quad + b\int_{\Omega} \nabla u \nabla u_t\,dx,\end{aligned} \tag{3.2}$$

where (2.4) has been used.

Multiplying (1.1) by u to estimate the integral terms on the right hand side of (3.2), we get

$$\begin{aligned}\psi'(t) &= \|u_t\|^2 + \left(a + \frac{2b}{\varepsilon}\right)\|\nabla u_t\|^2 - \|\nabla u\|^2 - \|\Delta u\|^2 \\ &\quad + \int_{\Omega} |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx + \int_{\Omega} |u|^{q(x)}\,dx.\end{aligned} \tag{3.3}$$

At this point, since ε is an arbitrary positive constant, we infer from (3.3) that

(3.4)

$$\begin{aligned}\psi'(t)-\varepsilon\psi(t) &= \|u_t\|^2+\left(a+\frac{2b}{\varepsilon}\right)\|\nabla u_t\|^2-\|\nabla u\|^2-\|\Delta u\|^2\\ &\quad+\int_\Omega |\nabla u|^2\ln|\nabla u|^{p(x)}\,dx+\int_\Omega |u|^{q(x)}\,dx\\ &\quad-\varepsilon\int_\Omega u(u_t-a\Delta u_t)\,dx-\frac{\varepsilon b}{2}\|\nabla u\|^2+2E(t)\\ &= -\varepsilon\int_\Omega uu_t\,dx-\varepsilon a\int_\Omega \nabla u\nabla u_t\,dx-\frac{\varepsilon b}{2}\|\nabla u\|^2+2\|u_t\|^2\\ &\quad+2\left(a+\frac{b}{\varepsilon}\right)\|\nabla u_t\|^2+\int_\Omega |u|^{q(x)}\,dx+\frac{1}{2}\int_\Omega p(x)|\nabla u|^2\,dx\\ &\quad-2\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx.\end{aligned}$$

Thanks to the additional conditions on the variable exponents, we deduce

$$\begin{aligned}\psi'(t)-\varepsilon\psi(t) \geq\ & -\varepsilon\int_\Omega uu_t\,dx-\varepsilon a\int_\Omega \nabla u\nabla u_t\,dx+\left(\frac{p_1}{2}-\frac{\varepsilon b}{2}\right)\|\nabla u\|^2\\ &+2\|u_t\|^2+2\left(a+\frac{b}{\varepsilon}\right)\|\nabla u_t\|^2+\frac{q_1-2}{q_1}\int_\Omega |u|^{q(x)}\,dx. \qquad (3.5)\end{aligned}$$

By using the Young and Poincaré inequalities, we have

$$\begin{aligned}\varepsilon\left|\int_\Omega uu_t\,dx\right| &\leq \varepsilon\left(\frac{b}{4c_2}\|u\|^2+\frac{c_2}{b}\|u_t\|^2\right) \qquad (3.6)\\ &\leq \frac{\varepsilon b}{4}\|\nabla u\|^2+\frac{\varepsilon c_2}{b}\|u_t\|^2,\\ \varepsilon a\left|\int_\Omega \nabla u\nabla u_t\,dx\right| &\leq \varepsilon a\left(\frac{b}{4a}\|\nabla u\|^2+\frac{a}{b}\|\nabla u_t\|^2\right) \qquad (3.7)\\ &\leq \frac{\varepsilon b}{4}\|\nabla u\|^2+\frac{\varepsilon a^2}{b}\|\nabla u_t\|^2.\end{aligned}$$

Combining (3.6) and (3.7) with (3.5), we obtain

$$\begin{aligned}\psi'(t)-\varepsilon\psi(t) \geq &\left(\frac{p_1}{2}-\varepsilon b\right)\|\nabla u\|^2+\left(2-\frac{\varepsilon c_2}{b}\right)\|u_t\|^2 \qquad (3.8)\\ &+2\left(a+\frac{b}{\varepsilon}-\frac{\varepsilon a^2}{2b}\right)\|\nabla u_t\|^2+\frac{q_1-2}{q_1}\int_\Omega |u|^{q(x)}\,dx.\end{aligned}$$

Now, if we set $\varepsilon := 2b/a$ then by using the assumption of Lemma 3.1, we get

$$\psi'(t) - \frac{2b}{a}\psi(t) \geq 0;$$

integrating from 0 to t, we get the desired result. ∎

THEOREM 3.2. *Suppose that the conditions of Lemma* 3.1 *hold. Suppose the initial data* u_0, u_1 *satisfy*

$$0 < E(0) \leq \frac{b}{a}\Big(\int_\Omega u_0(u_1 - a\Delta u_1)\,dx + \frac{b}{2}\|\nabla u_0\|^2\Big). \tag{3.9}$$

Then, along the solutions of problem (1.1)–(1.3), *there exists a finite time* t^* *such that*

$$\lim_{t\to t^*}(\|u\|^2 + a\|\nabla u\|^2) = +\infty.$$

Proof. Define

$$L(t) = \|u\|^2 + a\|\nabla u\|^2, \tag{3.10}$$

and therefore

$$L'(t) = 2\int_\Omega uu_t\,dx + 2a\int_\Omega \nabla u\nabla u_t\,dx, \tag{3.11}$$

$$L''(t) = 2\int_\Omega uu_{tt}\,dx + 2\|u_t\|^2 + 2a\int_\Omega \nabla u\nabla u_{tt}\,dx + 2a\|\nabla u_t\|^2. \tag{3.12}$$

It is easy to see that

$$\begin{aligned} L''(t) &= 2\|u_t\|^2 + 2a\|\nabla u_t\|^2 - 2\|\nabla u\|^2 - 2\|\Delta u\|^2 \\ &\quad + 2\int_\Omega |\nabla u|^2\ln|\nabla u|^{p(x)}\,dx - 2b\int_\Omega \nabla u\nabla u_t\,dx + 2\int_\Omega |u|^{q(x)}\,dx \\ &= \frac{4b}{a}\Big(\psi(t) - \int_\Omega u(u_t - a\Delta u_t)\,dx - \frac{b}{2}\|\nabla u\|^2\Big) \\ &\quad + 4E(t) + 2\|u_t\|^2 + 2a\|\nabla u_t\|^2 - 2\|\nabla u\|^2 - 2\|\Delta u\|^2 \\ &\quad + 2\int_\Omega |\nabla u|^2\ln|\nabla u|^{p(x)}\,dx - 2b\int_\Omega \nabla u\nabla u_t\,dx + 2\int_\Omega |u|^{q(x)}\,dx, \end{aligned} \tag{3.13}$$

where the definition (3.1) of $\psi(t)$ has been used.

Next, by using the additional conditions on the variable exponents and (2.3), we see from (3.13) that

(3.14)

$$\begin{aligned} L''(t) &\geq \frac{4b}{a}\Big(\psi(t) - \int_\Omega uu_t\,dx - a\int_\Omega \nabla u\nabla u_t\,dx\Big) + \Big(p_1 - \frac{2b^2}{a}\Big)\|\nabla u\|^2 \\ &\quad + 4\|u_t\|^2 + 4a\|\nabla u_t\|^2 + \frac{2(q_1-2)}{q_1}\int_\Omega |u|^{q(x)}\,dx - 2b\int_\Omega \nabla u\nabla u_t\,dx. \end{aligned}$$

Thanks to Lemma 3.1, since $q_1 \geq 2$ and $\psi(t) \geq \psi(0)$, we obtain

(3.15)
$$L''(t) \geq \frac{4b}{a}\psi(0) - \frac{4b}{a}\Big(\int_\Omega uu_t\,dx + a\int_\Omega \nabla u\nabla u_t\,dx\Big) + \Big(p_1 - \frac{2b^2}{a}\Big)\|\nabla u\|^2 + 4\|u_t\|^2 + 4a\|\nabla u_t\|^2 - 2b\int_\Omega \nabla u\nabla u_t\,dx.$$

Multiplying both sides of (3.15) by $L(t)$ we get

$$L(t)L''(t) \geq \frac{4b}{a}\psi(0)L(t) - \frac{2b}{a}L(t)L'(t) + \Big(p_1 - \frac{2b^2}{a}\Big)\|\nabla u\|^2 L(t) + 4(\|u_t\|^2 + a\|\nabla u_t\|^2)L(t) - 2bL(t)\int_\Omega \nabla u\nabla u_t\,dx. \tag{3.16}$$

To estimate the last term on the right-hand side of (3.16), by using the Young inequality, we obtain

$$\Big|2b\int_\Omega \nabla u\nabla u_t\,dx\Big| \leq 3a\|\nabla u_t\|^2 + \frac{b^2}{3a}\|\nabla u\|^2. \tag{3.17}$$

By inserting (3.17) and (3.9) into (3.16), we deduce

(3.18)
$$L(t)L''(t) \geq -\frac{2b}{a}L(t)L'(t) + \Big(\|u_t\|^2 + a\|\nabla u_t\|^2 + \Big(p_1 - \frac{7b^2}{3a}\Big)\|\nabla u\|^2\Big)L(t).$$

On the other hand, by using (3.11), and the Hölder, Young and Poincaré inequalities, we have

$$\begin{aligned}(L'(t))^2 &= 4\Big(\int_\Omega uu_t\,dx + a\int_\Omega \nabla u\nabla u_t\,dx\Big)^2 \\ &\leq 4\|u\|^2\|u_t\|^2 + 4a^2\|\nabla u\|^2\|\nabla u_t\|^2 \\ &\quad + 8a\Big(\frac{1}{2a}\Big[\int_\Omega uu_t\,dx\Big]^2 + \frac{a}{2}\Big[\int_\Omega \nabla u\nabla u_t\,dx\Big]^2\Big) \\ &\leq 4\|u\|^2\|u_t\|^2 + 8a^2\|\nabla u\|^2\|\nabla u_t\|^2 + 4c_2^2\|\nabla u\|^2\|u_t\|^2.\end{aligned} \tag{3.19}$$

Now, there exists a positive constant $\gamma < 1$ such that for sufficiently large p_1 and a (see the hypothesis of Lemma 3.1), (3.19) gives

(3.20)
$$\begin{aligned}(L'(t))^2 &\leq \gamma\Big(4\|u_t\|^2 + a\|\nabla u_t\|^2 + \Big(p_1 - \frac{7b^2}{3a}\Big)\|\nabla u\|^2\Big)(\|u\|^2 + a\|\nabla u\|^2) \\ &= \gamma\Big(4\|u_t\|^2 + a\|\nabla u_t\|^2 + \Big(p_1 - \frac{7b^2}{3a}\Big)\|\nabla u\|^2\Big)L(t).\end{aligned}$$

Applying (3.20) in (3.18), we obtain

$$L(t)L''(t) - \gamma^{-1}(L'(t))^2 \geq -\frac{2b}{a}L(t)L'(t), \tag{3.21}$$

Hence, we see that the hypotheses of Lemma 2.5 are fulfilled with

$$\mu = \gamma^{-1} - 1, \quad c_1 = \frac{b}{a}, \quad c_2 = 0.$$

Therefore, the modified concavity argument shows that there exists a finite time t^* such that the solutions blow up at this time, i.e.

$$\lim_{t\to t^*}(\|u\|^2 + a\|\nabla u\|^2) = +\infty,$$

and the proof of Theorem 3.2 is complete. ∎

3.2. Blow-up result for negative initial energy. Our blow-up result for certain solutions with negative initial energy reads as follows:

THEOREM 3.3. *Let the conditions of Theorem* 2.6 *hold and assume that* $E(0) < 0$. *Then there exist positive constants* Λ_0, Λ_1 *and sufficiently small* $\varepsilon > 0$ *such that solutions of problem* (1.1)–(1.3) *blow up at a finite time*

$$t^* \leq \frac{\Lambda_1(1-\sigma)}{\varepsilon\Lambda_0\sigma\phi^{\frac{\sigma}{1-\sigma}}(0)},$$

where $0 < \sigma < 1$ *and* $\phi(t)$ *is given in* (3.23).

Proof. Define $H(t) = -E(t)$. By using monotonicity of energy, i.e. (2.4), we arrive at

$$H'(t) = b\|\nabla u_t\|^2 \geq 0. \tag{3.22}$$

Then negative initial energy and (3.22) give $H(t) \geq H(0) > 0$. Also, by the definition of $H(t)$, it is easy to see that

$$H(t) \leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx.$$

Define, for $0 < \sigma < 1$, and with $\varepsilon > 0$ to be specified later,

$$\phi(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega u(u_t - a\Delta u_t)\,dx. \tag{3.23}$$

By taking the derivative of (3.23) and using (1.1), we get

(3.24)

$$\begin{aligned}\phi'(t) &= (1-\sigma)H'(t)H^{-\sigma}(t) + \varepsilon\|u_t\|^2 + \varepsilon a\|\nabla u_t\|^2 + \varepsilon\int_\Omega u(u_{tt} - a\Delta u_{tt})\,dx\\ &= (1-\sigma)H'(t)H^{-\sigma}(t) + \varepsilon\|u_t\|^2 + \varepsilon a\|\nabla u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\Delta u\|^2\\ &\quad + \varepsilon\int_\Omega |\nabla u|^2\ln|\nabla u|^{p(x)}\,dx - \varepsilon b\int_\Omega \nabla u\nabla u_t\,dx + \varepsilon\int_\Omega |u|^{q(x)}\,dx.\end{aligned}$$

By using the definition of $H(t)$ and for any $\delta > 0$, from (3.24) we get

$$\begin{aligned}\phi'(t) = \delta H(t) &+ \left(\frac{\delta}{2} + \varepsilon\right)\|u_t\|^2 + a\left(\frac{\delta}{2} + \varepsilon\right)\|\nabla u_t\|^2 + \left(\frac{\delta}{2} - \varepsilon\right)\|\nabla u\|^2 \\ &+ \left(\frac{\delta}{2} - \varepsilon\right)\|\Delta u\|^2 + \frac{\delta}{4}\int_\Omega p(x)|\nabla u|^2\,dx - \delta\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx \\ &+ \varepsilon\int_\Omega |u|^{q(x)}\,dx + \left(\varepsilon - \frac{\delta}{2}\right)\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx \\ &+ (1-\sigma)H'(t)H^{-\sigma}(t) - \varepsilon b\int_\Omega \nabla u\nabla u_t\,dx.\end{aligned}$$

Thanks to the boundedness of $p(\cdot)$ and $q(\cdot)$, we deduce

$$\begin{aligned}\phi'(t) \geq \delta H(t) &+ \left(\frac{\delta}{2} + \varepsilon\right)\|u_t\|^2 + a\left(\frac{\delta}{2} + \varepsilon\right)\|\nabla u_t\|^2 + \left(\frac{\delta}{2} - \varepsilon\right)\|\nabla u\|^2 \\ &+ \left(\frac{\delta}{2} - \varepsilon\right)\|\Delta u\|^2 + \frac{\delta p_1}{4}\|\nabla u\|^2 + \left(\varepsilon - \frac{\delta}{q_1}\right)\int_\Omega |u|^{q(x)}\,dx \\ &+ \left(\varepsilon - \frac{\delta}{2}\right)\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx + (1-\sigma)H'(t)H^{-\sigma}(t) \\ &- \varepsilon b\int_\Omega \nabla u\nabla u_t\,dx.\end{aligned}$$

Since $q_1 > 2$, we obtain

$$\begin{aligned}\phi'(t) \geq \delta H(t) &+ \left(\frac{\delta}{2} + \varepsilon\right)\|u_t\|^2 + a\left(\frac{\delta}{2} + \varepsilon\right)\|\nabla u_t\|^2 \\ &+ \left(\frac{\delta}{2} - \varepsilon\right)\|\nabla u\|^2 + \left(\frac{\delta}{2} - \varepsilon\right)\|\Delta u\|^2 + \frac{\delta p_1}{4}\|\nabla u\|^2 \\ &+ \left(\varepsilon - \frac{\delta}{2}\right)\left(\int_\Omega |u|^{q(x)}\,dx + \int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}\,dx\right) \\ &+ (1-\sigma)H'(t)H^{-\sigma}(t) - \varepsilon b\int_\Omega \nabla u\nabla u_t\,dx.\end{aligned} \tag{3.25}$$

Now, if we set $\delta := 2\varepsilon$, then inequality (3.25) takes the form

$$\begin{aligned}\phi'(t) \geq 2\varepsilon H(t) &+ 2\varepsilon\|u_t\|^2 + 2a\varepsilon\|\nabla u_t\|^2 + \frac{\varepsilon p_1}{2}\|\nabla u\|^2 \\ &+ (1-\sigma)H'(t)H^{-\sigma}(t) - \varepsilon b\int_\Omega \nabla u\nabla u_t\,dx.\end{aligned} \tag{3.26}$$

Using the Cauchy–Schwarz and Young inequalities, it is easy to estimate the last term on the right-hand side of (3.26) as follows:

$$\varepsilon b\Big|\int_\Omega \nabla u\nabla u_t\,dx\Big| \le \frac{\varepsilon p_1}{4}\|\nabla u\|^2 + \frac{\varepsilon b^2}{p_1}\|\nabla u_t\|^2 = \frac{\varepsilon p_1}{4}\|\nabla u\|^2 + \frac{\varepsilon b}{p_1}H'(t),$$

where (3.22) has been used.

Since $H(t) \ge H(0) > 0$, there exists a sufficiently large constant K such that

$$\varepsilon b\Big|\int_\Omega \nabla u\nabla u_t\,dx\Big| \le \frac{\varepsilon p_1}{4}\|\nabla u\|^2 + \frac{\varepsilon b}{p_1}KH^{-\sigma}(t)H'(t). \tag{3.27}$$

Applying (3.27) in (3.26) we deduce

$$\begin{aligned}\phi'(t) &\ge 2\varepsilon H(t) + 2\varepsilon\|u_t\|^2 + \frac{\varepsilon p_1}{4}\|\nabla u\|^2 + 2a\varepsilon\|\nabla u_t\|^2 \\ &\quad + \Big(1-\sigma-\frac{\varepsilon bK}{p_1}\Big)H'(t)H^{-\sigma}(t).\end{aligned} \tag{3.28}$$

Now, suppose that ε is sufficiently small and K large enough such that $1-\sigma-\varepsilon bK/p_1 > 0$ and (3.27) holds. Then

$$\phi'(t) \ge \varepsilon \Lambda_0(H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2), \tag{3.29}$$

where $\Lambda_0 = \min\{2, p_1/4, 2a\}$. Therefore we deduce that $\phi(t) \ge \phi(0) > 0$ for all $t \ge 0$.

On the other hand, we have

(3.30)

$$\begin{aligned}\phi^{\frac{1}{1-\sigma}}(t) &:= \Big(H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t\,dx + \varepsilon a\int_\Omega \nabla u\nabla u_t\,dx\Big)^{\frac{1}{1-\sigma}} \\ &\le 2^{\frac{2(1-\sigma)}{\sigma}}\Big(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} + (\varepsilon a)^{\frac{1}{1-\sigma}}\Big|\int_\Omega \nabla u\nabla u_t\,dx\Big|^{\frac{1}{1-\sigma}}\Big) \\ &\le \Lambda_1\Big(H(t) + \Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} + \Big|\int_\Omega \nabla u\nabla u_t\,dx\Big|^{\frac{1}{1-\sigma}}\Big),\end{aligned}$$

where we have used the following fact:

$$\Big(\sum_{i=1}^m a_i\Big)^\lambda \le 2^{\frac{m-1}{\lambda-1}}\sum_{i=1}^m a_i^\lambda.$$

By using the Hölder, Young and Poincaré inequalities, for some positive constants C_1, C_2 we get

$$\left|\int_\Omega uu_t\,dx\right|^{\frac{1}{1-\sigma}} \le \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}} \le C_1(\|\nabla u\|^2 + \|u_t\|^2 + H(t)), \tag{3.31}$$

$$\begin{aligned}\left|\int_\Omega \nabla u\nabla u_t\,dx\right|^{\frac{1}{1-\sigma}} &\le \|\nabla u\|^{\frac{1}{1-\sigma}}\|\nabla u_t\|^{\frac{1}{1-\sigma}}\\ &\le C_2(\|\nabla u\|^2 + \|\nabla u_t\|^2 + H(t)).\end{aligned} \tag{3.32}$$

Thus, using (3.31) and (3.32) we deduce from (3.30) that

$$\phi^{\frac{1}{1-\sigma}}(t) \le \Lambda_2(H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2) \le \frac{\Lambda_2}{\varepsilon\Lambda_0}\phi'(t),$$

where (3.29) has been used. Therefore

$$\phi'(t) \ge \frac{\varepsilon\Lambda_0}{\Lambda_1}\phi^{\frac{1}{1-\sigma}}(t). \tag{3.33}$$

Integrating (3.33) from 0 to t, we deduce

$$\phi^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{\phi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\varepsilon\Lambda_0\sigma t}{\Lambda_1(1-\sigma)}}.$$

This shows that solutions blow-up in finite time $t^* = \frac{\Lambda_1(1-\sigma)}{\varepsilon\Lambda_0\sigma\phi^{\frac{\sigma}{1-\sigma}}(0)}$, and the proof of Theorem 3.3 is complete. ■

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