APPLICATIONES MATHEMATICAE
50,1 (2023), pp. 1–14

A. Deshpande (London)

BOUNDS ON MEAN VARIANCE HEDGING IN JUMP DIFFUSION

Abstract. We compare the maximum principle and the linear quadratic regulator approach (LQR)/well-posedness criterion to mean variance hedging (MVH) when the wealth process follows a jump diffusion. The comparison is made possible via a measurability assumption on the coefficients of the process. Its application to determine an interval range for the MVH is explained. More precisely, in the MVH setup we show that

$$0 \le \inf_{u\in U} \tfrac{1}{2} E\Big[\int_0^T y_s' y_s\, ds + y_T' y_T\Big] = \tfrac{1}{2} y' P_0^0 y + f(P_0^0) \le \tfrac{1}{2} y' P_0 y + f(P_0),$$

where P^0 and P satisfy a backward stochastic differential equation (BSDE) and f is a measurable function affine in its only argument. The upper bound holds under the measurability assumption that all coefficients including the intensity of the jumps that drive P are in fact predictable with respect to the filtration generated only by the Brownian motion. The lower bound is achieved expectedly under perfect hedging when the Föllmer–Schweizer minimal martingale probability measure is equivalent to the physical measure.

1. Introduction. It is well-known that the MVH problem is quantified as a quadratic optimization criterion over a set of allocations of portfolio wealth V between u risky assets and $1-u$ riskless asset(s) to minimize the overall variance of the wealth of the portfolio. A stochastic control approach called Linear-Quadratic Stochastic control (LQSC) is used to solve the MVH.

2020 *Mathematics Subject Classification*: Primary 93E20.
Key words and phrases: mean variance hedging, backward stochastic differential equation.
Received 16 September 2022; revised 31 March 2023.
Published online 19 October 2023.

DOI: 10.4064/am2462-6-2023

For its generic illustration, consider a typical LQSC problem of the form

$$dV_t = (A_tV_t + B_tu_t)dt + (V_tC_t + u_tD_t)dW_t, \qquad V_0 = v, \tag{1}$$

with random coefficients and with objective function

$$F(v,u) = E\Big[\int_0^T (V_t'Q_tV_t + V_t'P_tu_t + u_t'N_tu_t)dt + R(T)|V_T - O(T)|^2\Big] \tag{2}$$

We can immediately see that for $A \equiv 0$, $B = b = \mu - r\mathbf{e}$, $C \equiv 0$, $D = \sigma$, $Q \equiv 0$, $P \equiv 0$, $N \equiv 0$, $R(T) = 1$, $O(T) = \xi$, the objective function becomes $E[(V_T - \xi)^2]$. If ξ is an option payoff, the above optimization criterion is called *mean variance hedging* of the option. This option will be of a European type that is priced at the terminal time T. American type option hedging alludes to the presence of a running cost in the objective function. This makes the resulting optimization MVH problem complicated due to the presence of nonhomogeneous terms as would be encountered later.

When the value of option hedging is positive, it is called market incompleteness. Market incompleteness results in a nonunique option valuation. Then the problem broadens to determining a valid range of option valuation. In the jump-diffusion context, this problem was solved by Bellamy and Jeanblanc (2000) in a special case when the intensity of the jumps occurring is deterministic. As with valuation, we guess that a concept similar to a range of mean variance hedging values exists. This financial motivation is studied here. To the best of our knowledge this is the first work on that subject. While the lower bound is evidently zero (corresponds to a complete market), the upper bound likely needs to be the largest of the MVH objective function values generated between the well-posedness and the maximum principle method. The well-posedness related governing equation is deduced via assuming that all the coefficients in (5) below (including the jump intensity) are predictable with respect to the filtration generated by the standard Brownian motion. This measurability assumption is what we henceforth collectively refer to as *Lim's measurability assumption*. This assumption is also important in another respect. In a relatively recent application, Goutte and Ngoupeyou (2015) found out that when the MVH is considered of the form $\theta_t(Q[V]) + \xi_t$, Lim's assumption makes it possible to interpret θ as a variance optimal martingale probability measure and hence the objective function as a quadratic function of V. In our context, we understand here the influence of Lim's assumption in the context of finding bounds to the MVH problem as made formal in the following result.

THEOREM 1.0.1. *Let y be the residual risk in hedging a claim using a stochastic portfolio in a jump diffusion market. In the mean variance hedging*

(MVH) setup, for some measurable function f defined in (23), *we have* [1]

$$0 \le \inf_{u\in U} \frac{1}{2} E\Big[\int_{s=0}^{T} y_s' y_s \, ds + y_T' y_T \Big] = \frac{1}{2} y' P_0^0 y + f(P_0^0) \le \frac{1}{2} y' P_0 y + f(P_0),$$

where P and P^0 satisfy the backward stochastic differential equations (BSDE) (8) *and* (13) *respectively. The upper bounds holds under the Lim's measurability assumption. The lower bound corresponds to market completeness and is achieved via the equivalence of the physical measure and the Föllmer–Schweizer minimal martingale measure.*

Proof. The deduction of the lower bound is given in Corollary 3.1.1, while the upper bound can be deduced from (21), (24) and (26). ■

In the next section we discuss the LQSC framework to MVH, attributed in the jump diffusion context to Lim (2005). We make the upper bound precise, by discussing the gap (if any) between the notions of the maximum principle and the well-posedness criterion. Some related yet unexplained conceptual measurability related assumptions made in Lim's work cited by Jeanblanc et al. (2012) are addressed at the end. The standard formal proof for the lower bound on the MVH is provided in the Appendix.

2. An LQSC approach to the MVH. Motivated by (2), we consider minimizing the following quadratic objective function:

$$J(v, u(\cdot)) = \frac{1}{2} E\Big[\int_0^T |V_t - E[\xi|\mathcal{F}_t]|^2 \, dt + |V_T - \xi|^2 \Big] \quad \text{with } u \in U \text{ a.e.,} \tag{3}$$

where the $\mathbb{R}^n$-valued wealth process V satisfies (5) below subject to an admissible process u defined on a closed convex set $U \subset \mathbb{R}^n$ consisting of all $(\mathcal{F}_t)$-predictable processes. Here we define $\mathcal{F}_t$ to be generated by the Brownian motion $\mathcal{F}^{\mathcal{W}}$ (say) and the Poisson random measure $\mathcal{F}^N$ (say). The quadratic payoff has a special interpretation when we regard $E[\xi|\mathcal{F}_t]$ as the expected value of an American option payoff. In this context, as seen from (3), the wealth process V is to be kept as close as possible to $E[\xi|\mathcal{F}_t]$ at any time $T \ge t \ge 0$. Hence the above optimization does not lead to hedging of a plain vanilla American option without imposing any additional constraints on the wealth process. Based on the assumption made by Tang and Li (1994), we assume that u satisfies

$$\|u(\cdot)\| := \sup_{0\le t\le T} [E|u(t)|^8]^{1/2} < \infty. \tag{4}$$

To solve the problem of minimizing (3), we need to determine the necessary conditions (that every control should satisfy in order to be an op-

[1] For an array M, we denote by M' its transpose.

timal control), also called the maximum principle, and/or well-posedness conditions (that guarantee finiteness of the objective function) for the MVH problem. Let us first discuss the maximum principle approach.

2.1. The maximum principle. Deduction of the necessary constraints in the optimal control problem comes originally from the work of Pontryagin and his students (Gamkrelidze, 1999). Pontryagin's maximum principle states that any optimal control along with the optimal state trajectory must solve the so called (extended) Hamiltonian system, which is a two-point boundary value problem plus a maximum condition for a function called the Hamiltonian. Its mathematical significance lies in that maximizing the Hamiltonian is much easier than solving the original control problem, which is infinite-dimensional. The idea of Pontryagin's maximum principle is to slightly perturb the optimal control (i.e. spike variation) and then consider the first order term in a sort of Taylor expansion with respect to this perturbation. By sending this perturbation to zero, one obtains a kind of variational inequality which is a two-point boundary value problem.

For the diffusion case, due to presence of a control in the diffusion coefficient, one uses the second-order term in the Taylor expansion of the spike variation. This results in a forward backward SDE and a maximum condition. Tang and Li (1994) extend the necessary conditions for an optimal control problem with jump diffusion state process given by

$$
\begin{aligned}
dV_t &= (A_{t-}V_{t-} + B_{t-}u_{t-})dt + (C_{t-}V_{t-} + D_{t-}u_{t-})dW_t \\
&\quad + \int\limits_{[0,1]} (E_{t-}(y)V_{t-} + F_{t-}(y)u_{t-})\,\tilde{\nu}(dt,dy), \qquad (5)\\
V_0 &= v,
\end{aligned}
$$

where $A, C, E \in \mathbb{R}^{n\times n}$ and $B, D, F \in \mathbb{R}^{n\times m}$ are bounded and satisfy Lim's measurability assumption and $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$. The compensator of the martingale jump process $\tilde{\nu}$ is $\lambda(dy)ds$. Hence the Poisson random measure quantifying the jump term is $\nu(ds,dy) := \tilde{\nu}(ds,dy) + \lambda(dy)ds$.

Let $F_p^2([0,T],\mathbb{R}^n)$ be the space of $\mathbb{R}^{1\times n}$-valued $\mathcal{F}_t$-predictable square integrable vector processess $\hat{f}(\cdot,\cdot,\cdot)$ defined on $[0,1]\times[0,T]\times\Omega$ such that $\iint_{[0,1]\times[0,T]} E[|\hat{f}(z,t,\cdot)|^2]\,\lambda(dz)\,dt < \infty$. We write $\langle\,,\rangle$ for the scalar product of Euclidean spaces. The Hamiltonian as defined by Tang and Li (1994) is

$$
\begin{aligned}
(6)\quad H(t,v,u,p,q,r(\cdot,\cdot)) &= \frac{1}{2}(v - E[\xi|\mathcal{F}_t])'(v - E[\xi|\mathcal{F}_t]) + \langle p, Av + Bu\rangle \\
&\quad + \langle q, Cv + Du\rangle + \int\limits_{[0,1]} \langle r(\cdot,z), (Ev + Fu)\rangle\,\lambda(dz).
\end{aligned}
$$

This map acts from $[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{R}^n\times F_p^2((0,T),\mathbb{R}^n)$ into $\mathbb{R}$.

Expectedly, the stochastic maximum principle yields two backward stochastic differential equations, viz. the dynamics of $p(T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ and $P(T) \in L^2(\Omega, \mathcal{F}, \mathbb{R}^{n\times n})$ satisfy

$$
\begin{aligned}
(7)\qquad p_t &= p_T + \int\limits_{(t,T]} H_v(s, v_s, u_s, p_s, q_s, r(s,\cdot))\, ds - \int\limits_{(t,T]} q_s\, dW_s \\
&\qquad - \int\limits_{(t,T]} \int\limits_{[0,1]} r(s,z)\, \tilde{\nu}(ds, dz) \\
&= V_T - \xi + \int\limits_{(t,T]} \Big((V_s - E[\xi|\mathcal{F}_s]) + A_s p_s + C_s q_s + \int\limits_{[0,1]} \langle E, r(s,z)\rangle\, \lambda(dz) \Big)\, ds \\
&\quad - \int\limits_{(t,T]} q_s\, dW_s - \int\limits_{[0,1]} \int\limits_{(t,T]} r(s,z)\, \tilde{\nu}(ds, dz) \qquad \text{a.s.},
\end{aligned}
$$

and, suppressing the z-dependencies in E, we get

$$
\begin{aligned}
(8)\qquad P_t &= P_T + \int\limits_{(t,T]} \Big(A_s P_s + P_s' A_s' + C_s P_s C_s + C_s Q_s + Q_s' C_s' \\
&+ \Big(\int\limits_{[0,1]} \big(E_s P_s E_s + E_s R(s,z) E_s + E_s R(s,z) + R'(s,z) E_s' \big)\, \lambda(dz) + I \Big) \Big)\, ds \\
&- \int\limits_{(t,T]} Q_s dW_s - \int\limits_{[0,1]} \int\limits_{(t,T]} R(s,z)\, \tilde{\nu}(dz, ds) \qquad \text{a.s.},
\end{aligned}
$$

for $P_T = I$. Both these backward stochastic differential equations admit unique solutions that are càdlàg processes. We refer to Tang and Li (1994) for the technical proofs of the existence and uniqueness.

For completeness, we mention the necessary and sufficient condition for the MVH problem solved via the stochastic maximum principle.

THEOREM 2.1.1. *Let the drift, diffusion and jump coefficient in* (5) *satisfy Assumption* 1 *of Tang and Li (1994). Then a necessary and sufficient condition for an admissible pair* (u, v) *to be optimal for the MVH problem* (3) *is*

$$
(9)\qquad H_u(t, v_{t-}, u_t, p_{t-}, q_t, r_t) = 0 \qquad \textit{a.e. a.s.}
$$

Proof. See Theorem 4.1 of Meng (2013) for a similar result. ■

Now we elaborate on the other tool, the LQR method, where a BSDE is deduced from the perspective of having the cost criterion (3) to be well-posed, i.e. finite.

2.2. The LQR method. Problem (3) in terms of $y_t = V_t - E[\xi|\mathcal{F}_t] \in \mathbb{R}^n$ is

$$J(s,y,u(\cdot)) = \frac{1}{2}E\Big[\int_0^T y_s' y_s\, ds + y_T' y_T\Big]. \tag{10}$$

Since $E[\xi|\mathcal{F}_t]$ is an $\mathcal{F}_t$-martingale, by the martingale representation theorem there exist $Z \in L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ and $U(\cdot,\cdot) \in F_p^2([0,T];\mathbb{R}^n)$ such that

$$E[\xi|\mathcal{F}_t] = E\xi + \int_0^t Z_s\, dW_s + \int_0^t \int_{(0,1]} U(s,a)\, \tilde{\nu}(ds,da). \tag{11}$$

Hence,

$$\begin{aligned} dy_t &= (A_t y_t + B_t u_t + A^{\xi})dt + (C_t y_t + D_t u_t + C^{\xi,z})dW_t \\ &\quad + \int_{[0,1]} (E_t(a) y_{t-} + F_t(a) u_t + E^{\xi,u}(a))\, \tilde{\nu}(dt,da), \\ y_0 &= y, \end{aligned} \tag{12}$$

where $A^{\xi} := A_t E[\xi|\mathcal{F}_t]$, $C^{\xi,z} := C_t E[\xi|\mathcal{F}_t] - Z_t$, $E_t^{\xi,u}(a) := E_t(a)E[\xi|\mathcal{F}_t] - U(t,a)$. Let S^n be the space of $n\times n$ symmetric matrices, and $L^p_{\mathcal{F}}(a,b;X) = \{\phi(t,\omega) : a \le t \le b\}$ be the space of $\mathcal{F}_t$-adapted X-valued measurable processes on $[a,b]$ with $E\int_a^b \|\phi(t,\omega)\|^p\, dt < \infty$, $\|\cdot\|$ being the uniform norm on S^n. Further, $L^2(\Omega;X)$ is the Hilbert space of X-valued square integrable functions on Ω with values in a given Hilbert space X. Let $C(0,T;X)$ be the Banach space of X-valued continuous functions on $[0,T]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space X. Introduce the stochastic Riccati equation (SRE) given by

$$\begin{aligned} dP_t^0 &= \Big\{-\Big(P_t^0 A_t + A_t' P_t^0 + C_t P_t^0 C_t + \Lambda_t C_t + C_t' \Lambda_t' + I - L_t' K_t^{-1} L_t \\ &\quad + \int_{[0,1]} E_t(a) P_t^0 E_t(a)\, \lambda(da)\Big)\Big\} dt + \Lambda_t dW_t \\ P_T^0 &= I, \qquad \mathbb{P}\text{-a.s.}, \qquad \forall t \in [0,T], \end{aligned} \tag{13}$$

for $K_t = D_t P_t^0 D_t + \int_{[0,1]} F_t(a) P_t^0 F_t(a)\, \lambda(da) > 0$ and $L_t = B_t P_t^0 + D_t P_t^0 C_t + D_t \Lambda_t$. Further, let some ϕ solve the BSDE

$$d\phi_t = \tilde{\Gamma}_t dt + \tilde{\Lambda}_t dW_t, \qquad \phi_T = 0. \tag{14}$$

The coefficients $\tilde{\Gamma}$ and $\tilde{\Lambda}$ will be made precise soon. Define the following non-homogeneous (NH) terms:

$$\begin{aligned} NH_t^1 &:= 2P_t^0 A_t^{\xi} + 2C_t^{\xi,z} P_t^0 C_t^{\xi,z} + \int_{[0,1]} (2E_t(a) P_t^0 E_t^{\xi,u}(a))\, \lambda(da) \\ &\quad + 2\Lambda_t C_t^{\xi,z}, \end{aligned} \tag{15}$$

$$NH_t^2 := 2D_t'P_t^0C_t^{\xi,z} + \int_{[0,1]} 2F_t'(a)P_t^0E_t^{\xi,u}(a)\,\lambda(da), \tag{16}$$

$$NH_t^3 := \int_{[0,1]} E_t'^{\xi,u}(a)P_t^0E_t^{\xi,u}(a)\,\lambda(da) + C_t'^{\xi,z}P_t^0C_t^{\xi,z}. \tag{17}$$

An $\mathcal{F}^W$-predictable pair $(P^0, \Lambda) \in |L^2_{\mathcal{F}^W}(0,T;S^n) \cap L^2(\Omega, C(0,T;S^n))| \otimes L^2_{\mathcal{F}^W}(0,T;S^n)$ is called a solution to the Riccati equation (13) if it satisfies all constraints in (13). Since the SRE is driven by a standard Brownian motion, all of its coefficients including the intensity of jumps are assumed to be predictable with respect to the filtration generated only by the standard Brownian motion. That is, Lim's measurability condition readily applies. We have the following explicit deduction of the stochastic control via well-posedness.

THEOREM 2.2.1. *Under Lim's measurability condition, if the SRE given by* (13) *and BSDE* (14) *admits a solution, then the stochastic LQR problem is well-posed.*

Proof. Under Lim's measurability condition, $P^0 \in L^2_{\mathcal{F}^W}(0,T;S^n) \cap L^2(\Omega;C(0,T,S^n))$ can for simplicity be assumed to satisfy

$$dP_t^0 = \Gamma_t dt + \Lambda_t dW_t, \quad t \in [0,T]. \tag{18}$$

Let (y,u) be any admissible pair. Apply Ito's formula to get

$$\begin{aligned} d(y_t'P_t^0y_t) = \Big\{ & y_t'\Big(P_t^0A_t + A_t'P_t^0 + \Lambda_tC_t + C_t'\Lambda_t' + \Gamma_t \\ & + C_tP_t^0C_t + \int_{[0,1]} E_t(a)P_t^0E_t(a)\,\lambda(da)\Big)y_t \\ & + y_t'\Big(P_t^0B_t + B_t'P_t^0 + C_tP_t^0D_t + D_t'P_t^0C_t \\ & + \int_{[0,1]} [F_t'(a)P_t^0E_t(a) + E_t'(a)P_t^0F_t(a)]\,\lambda(da) + \Lambda_tD_t + D_t'\Lambda_t'\Big)u_t\Big\}dt \\ & + \Big[u_t'\Big(D_t'P_t^0D_t + \int_{[0,1]} F_t'(a)P_t^0F_t(a)\,\lambda(da)\Big)u_t\Big]dt \\ & + [y_t'NH_t^1 + u_t'NH_t^2 + NH_t^3]dt + \{\ldots\}dW_t + \{\ldots\}\tilde{\nu}(ds,da). \end{aligned} \tag{19}$$

and

$$\begin{aligned} d(y_t'\phi_t) = {} & \big(\phi_t'A_ty_t + \phi_t'A_t^\xi + y_t'\tilde{\Gamma}_t + y_t'C_t'\tilde{\Lambda}_t + u_t'D_t'\tilde{\Lambda}_t + C_t^{\xi,z}\tilde{\Lambda}_t\big)dt \\ & + \text{martingale terms}. \end{aligned} \tag{20}$$

Integrate (19)–(20) from 0 to T, taking expectations $E^s := [\cdot|\mathcal{F}_s]$, and add them. Via a completion of squares argument we get the final form

$$J(u^*, y) = \tfrac{1}{2}y_0'P_0^0y_0 + f(P^0), \tag{21}$$

by setting

$$
\begin{aligned}
\tilde{\Gamma}_t &= \frac{1}{2}(D_t D_t')^{-1} D_t (N H_t^2 + 2B_t' \phi_t), \\
\tilde{\Lambda}_t &= -\frac{1}{2}(N H_t^1 + 2C_t' \tilde{\Gamma}_t + 2A_t' \phi_t), \\
\Gamma_t &= -\Big(P_t^0 A_t + A_t' P_t^0 + C_t' P_t^0 C_t + \Lambda_t C_t + C_t' \Lambda_t' \\
&\quad + I - L_t' K_t^{-1} L_t + \int_{[0,1]} E_t'(a) P_t^0 E_t(a)\, \lambda(da)\Big),
\end{aligned} \tag{22}
$$

$$
f(P^0) := y_0' \phi_0 + E^s\Big[\int_0^T (\phi_s' A_s^\xi + C^{\xi,z} \tilde{\Lambda}_s + N H_s^3)\, ds\Big] \tag{23}
$$

and by choosing the optimal control

$$
\begin{aligned}
u^* &= -K^{-1} L y \\
&= -\Big(D_t' P_t^0 D_t + \int_{[0,1]} F_t'(a) P_t^0 F_t(a)\, \lambda(da)\Big)^{-1} \\
&\quad \times \Big(B_t P_t^0 + \int_{[0,1]} F_t'(a) P_t^0 E_t(a)\, \lambda(da) + D_t' P_t^0 C_t + D_t \Lambda_t\Big) y_t. \quad \blacksquare
\end{aligned} \tag{24}
$$

Of course u^* in (24) is unique, since it can be rewritten in terms of y and this formula for y substituted back to the objective function would then be a quadratic function in u.

If we can connect the LQR deduced SRE (P^0, Λ) with the stochastic maximum principle deduced BSDE's (p, q, r) then the sufficiency result (i.e. Theorem 2.1.1) applied to the latter will hence be applicable to the former, i.e. the LQR. We now formalize this connection.

LEMMA 2.2.1. *Let the drift, diffusion and jump coefficient in* (12) *satisfy Assumption* 1 *in Tang and Li (1994). Let* (p, q, r) *satisfy* (7) *and* u *satisfy* (9). *Then the* u *so deduced is also sufficient for the LQR problem provided that*

$$
\begin{aligned}
p_t &= P_t^0 y_t, \\
q_t &= (P_t^0 C_t + \Lambda_t) y_t + P_t^0 D_t u_t + P_t^0 C_t^{\xi,z}, \\
r_t(a) &= P_t^0 E_t(a) y_t + P_t^0 F_t(a) u_t + P_t^0 F_t^{\xi,u}(a) \qquad \forall t \in [0,T], a \in [0,1].
\end{aligned} \tag{25}
$$

Proof. Apply Ito's formula to $(P^0 y)$ and compare it with (p, q, r). $\blacksquare$

2.3. On the gap between the maximum principle and the LQR criterion. In the deterministic LQR problem, it is well-known that the Hamiltonian system completely characterizes the optimal control. In a sense, the maximum principle and the well-posedness of the LQR problem are equivalent to each other. It is then natural to expect that in the stochastic case, the solvability of the Hamiltonian system would yield the well-posedness of the LQR. This is unfortunately not true in general. However, if we use Lim's assumptions on measurability, then this fact could at least be tested. In that respect we have the following result.

PROPOSITION 2.3.1. *Under Lim's measurability condition, P_t from* (8) *and P_t^0 from* (13) *satisfy*

$$P_t \geq P_t^0, \qquad \mathbb{P}\text{-}a.s., \ \forall t \in [0,T]. \tag{26}$$

In other words, the solvability of the Hamiltonian system would not yield the well-posedness of the LQR when the inequality in (26) *is strict.*

Proof. Since Lim's measurability condition holds true via assumption (5) on our dynamics, for comparison's sake, we proceed from (5) and equation (2.4) of Lim (2005) to obtain $C \equiv E = 0$. Hence the generators of the BSDEs (13) and (8) henceforth referred to as $f_1(s,P,Q,R) = A'P^0 + P^0A + I - LK^{-1}L$ and $f_2(s,P,Q,R) = A'P + P'A + I$ satisfy $f_1(s,P,Q,R) \leq f_2(s,P,Q,R)$ for all $s \in [0,T]$. To use the comparison result of Theorem 2.5 of Royer (2006) it remains to prove that f_1 satisfies all conditions enumerated in H_{ex} and f_2 satisfies $\mathrm{H}_{\mathrm{comp}}$ in that article. In that respect, via (24), f_1 is Lipschitz with respect to the diffusion component Λ and trivially Lipschitz with respect to the jump component. Further f_1 driven by P^0 is further driven by a diffusion on the compact interval $[0,T]$. Thus, it is continuous and bounded. There exists an $\alpha \in \mathbb{R}$ such that for all $t \geq 0$ and $p^0, \hat{p}^0 \in \mathbb{R}$, $(p^0 - \hat{p}^0)(f_1(s,p^0,\Lambda,u) - f_1(s,\hat{p}^0,\Lambda,u)) \leq \alpha|p^0 - \hat{p}^0|^2$ $\mathbb{P}$-a.s., the thus satisfying the monotonicity assumption of H_{ex}. Now it remains to show that f_2 satisfies $\mathrm{H}_{\mathrm{comp}}$. By definition we have $f_2(s,0,0,0) = I$. Thus $E[\int_0^T f_2(s,0,0,0)^2\,ds] < \infty$. Secondly, f_2, being linear in P and independent of the diffusion component, is Lipschitz in P and Q. Thirdly, f_2, having zero jump component, obviously satisfies the (A_γ) condition of Royer. Thus f_2 is now shown to satisfy $\mathrm{H}_{\mathrm{comp}}$. Hence by positive definiteness of K we have $f_1(P,Q,R) \leq f_2(P,Q,R)$, while $P_T^0 = P_T = I$, implying $P^0 \leq P$ for all $t \in [0,T]$. ∎

Therefore for the same residual risk y, the well-posedness principle provides an upper bound for the optimization criterion, viz. $\frac{1}{2}y'P_0y + f(P_0)$.

In the work of Lim (2005), in addition to the coefficients of V, it is assumed that the filtration used for the payoffs ξ and the strategy u is generated by $\mathcal{F}_t$. These assumptions were also noted to be conceptually unclear by Jeanblanc et al. (2012). We clarify those here. Unlike the BSDE P^0, BSDE p is driven by jump diffusion. Since $p_T = V_T - \xi$, ξ is generated by $\mathcal{F}$. Further, from its definition, y and therefore u in (24) will also be generated by $\mathcal{F}$.

3. Appendix

3.1. The lower bound. The analysis is not new but tailored to the problem as stated in the abstract. The work presented here is originally due to Colwell and Elliott (1993) who in particular contextualize the results of Schweizer (1991) for jump diffusion asset dynamics.

Consider the following general risky asset(s) value process under a physical measure (or EMM) $\mathbb{P}$ (say):

$$dX_t = \alpha_t dt + \sigma_t dW_t + \int_{[0,1]} \phi_t(z)\,\tilde{\nu}(dt,dz), \qquad 0 \le t \le T < \infty. \tag{27}$$

We want to determine the conditions that guarantee the martingale property, $E_{\mathbb{Q}^p}[V(T, D_{T-}X_{T-}, u)|\mathcal{F}_t] = V(t, D_{t-}X_{t-}, u)$ a.s. for portfolio wealth V and where the Radon–Nikodym density is informally defined as $\frac{d\mathbb{Q}^p}{d\mathbb{P}}|_{\mathcal{F}_t} = D_t$ for an EMM $\mathbb{Q}^p$. The claim to be hedged, $g := \xi$, is assumed attainable, implying that the martingale expression will lead to perfect hedging under the assumed existence of $\mathbb{Q}^p$. Let $\mathcal{L}(X^c)$ represent the class of predictable processes for which the stochastic integral with respect to the continuous part of the semi-martingale X exists. On the set $\Omega \times [0,T]$ we define $\mathcal{P}$ (resp. $\tilde{\mathcal{P}}$) as the sigma-field generated by adapted and continuous processes (resp. predictable processes). $G_{\mathrm{loc}}(\tilde{\nu})$ is assumed to contain $\tilde{\mathcal{P}}$ functions. We restate the equivalent probability measures $\mathbb{Q}^p$ by the Girsanov density for processes $j \in \mathcal{L}(X^c)$ and $h - 1 \in G_{\mathrm{loc}}(\tilde{\nu})$:

$$\begin{aligned} D_t = 1 &+ \int_0^t D_{s-} j(s, X_{s-})\,dW_s \\ &+ \int_0^t \int_{[0,1]} D_{s-}[h(s, X_{s-}, y) - 1]\,(\nu(ds,dy) - \lambda(dy)ds). \end{aligned} \tag{28}$$

We assume that $\psi := j$ and $\gamma + 1 = h = p$ are such that $DX = (D_t X_t)_{t\ge 0}$ is a semi-martingale under $\mathbb{P}$. Under $\mathbb{Q}^p$, $W_t - \int_0^t j(s, D_{s-}X_{s-})\,ds$ is a standard Brownian motion, and the compensator of $\nu(dt,dy)$ is $h(t, D_{t-}X_{t-}, y)\lambda(dy)dt$. Under $\mathbb{Q}^p$, X can be expounded as

$$\frac{dX_t}{X_{t-}} = r(t)dt + \sigma_t dW_t^p + \int_{[0,1]} \phi_t(y)\,(\nu(dt,dy) - p_t \lambda_t dy dt), \tag{29}$$

where $W_t^p = W_t - \int_0^t \psi_u\,du$ is a $\mathbb{Q}^p$ standard Brownian motion and the intensity of jumps is $\lambda(dy) = \lambda_t dy$.

We recall that under the original measure $\mathbb{P}$ the process is given by

$$d(D_t X_t) = (D_{s-}\sigma_s - X_{s-}\psi_s D_{s-})(dW_s - j(s, D_{s-}X_{s-})ds) \tag{30}$$

$$
\begin{aligned}
&+ \int_{[0,1]} (D_{s-}\phi_s(z) - X_{s-}D_{s-}\gamma_s(z) - D_{s-}\phi_s(z)\gamma_s(z)) \\
&\times (\nu(ds,dz) - h(s, D_{s-}X_{s-}, z)\lambda(dz)ds) \\
&+ \Big\{ D_{s-}\Big(\alpha_s - \Big(\sigma_s\psi_s + \int_{[0,1]} \phi_s(z)\gamma_s(z)\,\lambda(dz)\Big)\Big) \\
&+ (D_{s-}\sigma_s - X_{s-}\psi_s D_{s-}) j(s, D_{s-}X_{s-}) \\
&+ \int_{[0,1]} (D_{s-}\phi_s(z) - X_{s-}D_{s-}\gamma_s(z) - D_{s-}\phi_s(z)\gamma_s(z)) \\
&\times [h(s, D_{s-}X_{s-}, z) - 1]\,\lambda(dz)\Big\}\, ds.
\end{aligned}
$$

We know that $\tilde{\nu}(ds,dz) := \nu(ds,dz) - h(s, D_{s-}X_{s-}, z)\lambda(dz)ds$ and $dW_s - j(s, D_{s-}X_{s-})ds$ are local martingales under the measure $\mathbb{Q}^p$. This entails under certain $\mathbb{Q}^p$ that DX is a $\tilde{\mathcal{P}}$ local martingale iff

$$
d^s(z) + v^s j(s, D_{s-}X_{s-}) + \int_{[0,1]} j^s(z)[h(s, D_{s-}X_{s-}, z) - 1]\,\lambda(dz) = 0 \tag{31}
$$

where $d^s(z) := \alpha_s - (\sigma_s\psi_s + \int_{[0,1]} \phi_s(z)\gamma_s(z)\,\lambda(dz))$, $v^s := \sigma_s - X_{s-}\psi_s$, $j^s := \phi_s(z) - X_{s-}\gamma_s(z) - \phi_s(z)\gamma_s(z)$. We have the following result on the increments of V. We state it without proof as it is similar to the one deduced in the thesis of Vandaele (2010).

THEOREM 3.1.1. *The $\mathbb{Q}^p$-local martingale*

$$
V(t,u) = E[e^{-\int_u^T r_s\,ds} g(s, D_sX_s)|\mathcal{F}_t] = e^{-\int_t^T r_s\,ds} F(s, D_sX_s, u)
$$

for $0 \le t < u \le T$ with an arbitrary smooth payoff function F is given by

$$
\begin{aligned}
V(t,u) = V(0,u) &+ \int_0^t F_x(s, D_{s-}X_{s-}, u)\sigma_s(dW_s - j(s, D_{s-}X_{s-}))\,ds \\
&+ \int_0^t \int_{[0,1]} F(s, D_{s-}X_{s-} + \phi_s(z), u) - F(s, D_{s-}X_{s-}, u) \\
&\qquad \times [\nu(ds,dz) - h(s, D_{s-}X_{s-}, z)\,\lambda(dz)\,ds].
\end{aligned} \tag{32}
$$

We know that the Föllmer–Schweizer (FS) decomposition gives a locally risk minimizing strategy; see Schweizer (1991). Hence it is vital to find this FS decomposition. In order to do so we impose the following conditions:

(i) $V(u,u)(\phi) = g(u, D_uX_u)$,
(ii) $V(t,u)(\phi) = V(0,u)(\phi) + \int_0^t \phi^{\mathrm{FS}}(s,u)\,d(D_sX_s) + \Gamma^{\mathrm{FS}}(t,u)$.
(iii) $\Gamma^{\mathrm{FS}}(t,u)$ is a martingale under $\mathbb{P}$ and is orthogonal to the martingale part of the discounted risky asset under $\mathbb{P}$. The functions $\phi^{\mathrm{FS}}(t,u)$ and $\Gamma^{\mathrm{FS}}(t,u)$ in the FS decomposition (ii) of the portfolio are unknown.

The earlier made requirement

(iv) the chosen EMM $\mathbb{Q}^p$ is such that DX is a local martingale under $\mathbb{Q}^p$

as well as the condition

(v) $V(t,u)(\phi)$ is a local martingale under $\mathbb{Q}^p$

further help us to find the FS decomposition (ii) explicitly. We state this formally.

THEOREM 3.1.2. *The locally risk minimizing hedging strategy for the claim $g(u, D_u X_u)$ can be found by performing a change of measure as described in* (28). *The FS decomposition under $\mathbb{P}$ of the associated portfolio, satisfying conditions* (i)–(v), *is*

$$V(t,u)(\phi) = V(0,u)(\phi) + \int_0^t \phi^{\mathrm{FS}}(s,u)\,d(D_s X_s) + \int_0^t \mathcal{G}^{(a)}(s,u)\,dW_s + \int_0^t \int_{[0,1]} \mathcal{G}^{(b)}(s,y,u)\,\tilde{\nu}(ds,dy) \tag{33}$$

with

$$\phi^{\mathrm{FS}}(s,u) = \frac{F_x \sigma_s(v^s) + \int_{[0,1]} j^s(z) J(s,x,z)\,\lambda(dz)}{D_{s-}(v^s)^2 + \int_{[0,1]} (j^s(z))^2\,\lambda(dz)}, \tag{34}$$

$$\mathcal{G}^{(a)}(s,u) := F_x(s, D_{s-}X_{s-}, u)\sigma_s - \phi^{\mathrm{FS}}(s,u) D_{s-}(\sigma_S - X_{s-}\psi_s), \tag{35}$$

$$\mathcal{G}^{(b)}(s,y,u) := J(s,y,u) - \phi^{\mathrm{FS}}(s,u) D_{s-}(\phi_s(y) - X_{s-}\gamma_s(y) - \phi_s(y)\gamma_s(y)), \tag{36}$$

where

$$J(s,y,u) := F(s, Dx + \phi(y), u) - F(s, Dx, u). \tag{37}$$

The strategy at time t with $0 \le t \le u$ is $(\phi(t,u), \eta(t,u))$ with $\eta(t,u) = V(t,u)(\phi) - \phi(t,u) X_t$.

Proof. From conditions (v), (i) and the definition of $V(t,u)(\phi)$, we find that

$$V(t,u)(\phi) = E_{\mathbb{Q}^p}[V(u,u)(\phi) | \mathcal{F}_t] = V(t,u), \tag{38}$$

in particular

$$V(0,u)(\phi) = V(0,u). \tag{39}$$

Using the required form described in (ii) and the facts that $\Gamma^{\mathrm{FS}}(t,u)$ is a local martingale under $\mathbb{P}$, orthogonal to the martingale part DM of the risky asset under $\mathbb{P}$, yields

$$\Gamma^{\mathrm{FS}}(t,u) = \int_0^t \mathcal{G}^{(a)}(s,u)\,dW_s + \int_0^t \int_{[0,1]} \mathcal{G}^{(b)}(s,z,u)\,(\nu(ds,dz) - \lambda(dz)ds) \tag{40}$$

where $\mathcal{G}^{(a)}(t,u)$ and $\mathcal{G}^{(b)}(t,z,u)$ are as in (35)–(36), and

$$\phi^{\mathrm{FS}}(s,u) = \frac{F_x(s,D_{s-}X_{s-},u)\sigma_s v^s + \int_{[0,1]} j^s(z)J(s,x,z)\,\lambda(dz)}{D_{s-}v^{s2} + \int_{[0,1]} (j^s(z))^2\,\lambda(dz)}. \ \blacksquare \tag{41}$$

Substitute ϕ^{FS} using condition (iii) to get

$$\begin{aligned} &- F_x(s,D_{s-}X_{s-},u)j(s,D_{s-}X_{s-}) \\ &\quad + \int_{[0,1]} J(s,x,z)(1-h(s-,D_{s-}X_{s-},z))\lambda(dz) \\ &= \left\{\frac{F_x(s,D_{s-}X_{s-},u)\sigma_s v^s + \int_{[0,1]} j^s(z)J(s,x,z)\,\lambda(dz)}{D_{s-}(v^s)^2 + \int_{[0,1]} (j^s(z))^2\,\lambda(dz)}\right\} \\ &\quad \times \Big\{D_{s-}\Big(\alpha_s - \Big(\sigma_s\psi_s + \int_{[0,1]} \phi_s(z)\gamma(z)\,\lambda(dz)\Big)\Big)\Big\}. \end{aligned} \tag{42}$$

Multiply (31) by F_x and (42) by v^t, and add them to obtain

$$\begin{aligned} &F_x(t,D_{t-}X_{t-},u)d^t + \int_{[0,1]} F_x(t,D_{t-}X_{t-},u)j^t(z)(h-1)\,\lambda(dz) \\ &\quad + v^t \int_{[0,1]} J(1-h)\,\lambda(dz) \\ &= \left\{\frac{F_x(t,D_{t-}X_{t-},u)\sigma_t v^t + \int_{[0,1]} j^t(z)J(t,x,z)\,\lambda(dz)}{D_{t-}(v^t)^2 + \int_{[0,1]} (j^t(z))^2\,\lambda(dz)}\right\} \\ &\quad \times \Big\{D_{t-}\Big(\alpha_t - \Big(\sigma_t\psi_t + \int_{[0,1]} \phi_t(z)\gamma(z)\,\lambda(dz)\Big)\Big)\Big\}v^t. \end{aligned} \tag{43}$$

A possible solution herein which is independent of the claim to be hedged is $h \equiv 1$ and $d^t = 0$. This observation yields the following result.

COROLLARY 3.1.1. *Assume the option claim is attainable. The condition $d^t \equiv 0$ yields a minimal martingale measure $\widetilde{\mathbb{P}}$ (say). Additionally, the condition $h \equiv 1$ when applied must lead to the minimal martingale measure being equivalent to the physical probability measure. Perfect hedging is now realized.*

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A. Deshpande
London Metropolitan University
London, UK
E-mail: addeshpa@gmail.com