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SOME NEW DEVELOPMENTS ON VARIABLE-POWER COPULAS

Abstract. This article aims to contribute to the theory of variable-power copulas. In the first part, we discuss and study two unexplored variable-power copulas based on a modification of the function "$x^{1/y}y^{1/x}$". Their originality in definition offers interesting alternative options to the existing variable-power copulas. However, these copulas have a strong limitation: they are free of any parameters, making them rigid in the functional sense. In light of this, the second part is devoted to some parametric versions of them, still belonging to the variable-power copulas family. They have the feature of being original and of recovering the independence copula for some values of the parameter. Their properties are investigated.

1. Introduction. Copula theory, initially introduced by Sklar [14], has become a vital tool for analyzing multivariate data and characterizing the dependence structure between random variables. The main functions, called copulas, provide a means to separate the marginal distributions from the dependence structure, allowing practitioners to model and estimate the joint distribution independently. This flexibility has enabled copulas to find applications in a wide range of fields, including finance (see [9]), insurance (see [7]), environmental sciences (see [1]), and many others. The main theoretical and applied background on copulas can be found in [12, 5], and recent advancements are described in [15, 16, 3, 4, 2, 11, 13, 6, 18].

Nowadays, the need for new copula constructions arises due to several reasons. Firstly, existing copula families may lack the flexibility to model dependence structures with asymmetric tail behavior or extreme values, which

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are common in various domains. Secondly, the dependence structure in real-world data often exhibits non-linear or non-monotonic patterns, which cannot be effectively captured by traditional copula models. Furthermore, the growing availability of high-dimensional data requires copula models that can handle large-scale dependence structures.

Developing new copula families that address these limitations is of paramount importance to enhance the accuracy and applicability of copula theory in practice. Among the recent copula families is the variable-power copula family. In the two-dimensional case, a variable-power copula is characterized by the following general form:

$$C_{op}(x,y) = x^{P(x,y)}y^{Q(x,y)}, \qquad (x,y) \in [0,1]^2,$$

where $P(x,y)$ and $Q(x,y)$ are two-dimensional functions satisfying specific conditions guaranteeing in particular the following values: $C_{op}(0,0) = 0$, $C_{op}(0,1) = 0$, $C_{op}(1,0) = 0$ and $C_{op}(1,1) = 1$. Clearly, if $P(x,y) = Q(x,y) = 1$, the independence copula is obtained, i.e., $C_{op}(x,y) = xy = \Pi(x,y)$. The case where $P(x,y) = 1$ and $Q(x,y)$ is a function only depending on x is explored in [4]. A collection of new two-dimensional variable-power copulas was obtained, providing a perspective on dependence models beyond the standard scheme. In [3], a more sophisticated direction was studied; it is proved that the variable-power function $F(x,y) = x^y y^x$ for $(x,y) \in (0,1]^2$ and $F(x,y) = 0$ for $x = 0$ or $y = 0$ is not a valid copula. However, based on this function and the construction of the Gumbel–Barnett copula, a variable-power copula extension is provided and examined. These theoretical studies are the foundations for new perspectives on dependence modeling.

In this article, we make a contribution to the subject by developing innovative constructions of variable-power copulas. In a first part, we prove that $F(x,y) = x^{1/y}y^{1/x}$ for $(x,y) \in (0,1]^2$ and $F(x,y) = 0$ for $x = 0$ or $y = 0$ is not a valid copula. Then we offer two different modified versions of $F(x,y)$ that are valid variable-power copulas. They are novel because of the originality of these expressions as well as their symmetric and negative correlation properties. They have, however, a limitation: they are free of tuning parameters, making them unpractical from the statistical viewpoint. In light of this, in the second part, some parametric versions of them are developed. They have the feature of being original and of recovering the independence copula for some values of the parameters. Their properties are investigated by means of analytical, graphical, and numerical tools. Overall, our findings will contribute to the advancement of copula theory and provide valuable insights for researchers, practitioners, and decision-makers in various fields.

The article is organized as follows: Section 2 provides a preliminary result and two new variable-power copulas based on "$x^{1/y}y^{1/x}$". Some parametric versions of them are examined in Section 3. Finally, Section 4 outlines future directions for research in this area.

2. New variable-power copulas. Before presenting our new variable-power copulas, some definition and preliminary results need to be presented.

2.1. Preliminary result. In this article, we consider the notion of copula in the standard two-dimensional absolutely continuous (TDAC) context. A precise definition is recalled below (see [12]).

DEFINITION 2.1. In the TDAC context, we define a *copula* to be a function that is continuous on $[0,1]^2$ and twice continuously differentiable on $(0,1)^2$, say $F(x,y)$, $(x,y) \in [0,1]^2$, satisfying:

H1: $F(x,1) = x$, $F(1,y) = y$, $F(x,0) = 0$ and $F(0,y) = 0$,

H2: $\partial_{x,y} F(x,y) = \dfrac{\partial^2}{\partial x \partial y} F(x,y) \geq 0$.

From the mathematical viewpoint, assumption H2 is often less direct to prove than H1; it can demand tedious differentiations and algebraic calculations.

In [3], it is proved that the simple variable-power function $F(x,y) = x^y y^x$ for $(x,y) \in (0,1]^2$ and $F(x,y) = 0$ for $x = 0$ or $y = 0$ is not a valid copula. More specifically, a point $(x_\circ, y_\circ) \in [0,1]^2$ is found such that $\partial_{x,y} F(x,y)|_{(x,y)=(x_\circ,y_\circ)} < 0$, thus refuting assumption H2.

In the next proposition, a similar result is established for another simple variable-power function.

PROPOSITION 2.2. *The two-dimensional function defined by*

$$F(x,y) = x^{1/y} y^{1/x}, \qquad (x,y) \in (0,1]^2, \tag{1}$$

and $F(x,y) = 0$ *for* $x = 0$ *or* $y = 0$, *is not a valid copula.*

Proof. Let us investigate assumptions H1 and H2 of Definition 2.1. To begin, we note that assumption H1 is satisfied. Indeed, for any $x \in (0,1]$, we have

$$F(x,1) = x^{1/1} \times 1^{1/x} = x$$

and $F(0,1) = 0$. Similarly, for any $y \in [0,1]$, we have $F(1,y) = y$. For any $x \in (0,1]$, since $\lim_{y\to 0} \log(x)/y$ is $-\infty$ (if $x > 0$) or 0 (if $x = 1$), and $\lim_{y\to 0} \log(y) = -\infty$, we have

$$\lim_{y\to 0} F(x,y) = \lim_{y\to 0} x^{1/y} y^{1/x} = \lim_{y\to 0} \exp\left[\frac{1}{y}\log(x) + \frac{1}{x}\log(y)\right] = 0$$
$$(= F(x,0)).$$

Using the same arguments, for any $y \in (0,1]$, we have $\lim_{x\to 0} F(x,y) = 0$. Finally, $F(0,0) = 0$ by definition.

Let us now investigate assumption H2. Using the standard differentiation rules and a suitable arearrangement, we get

$$\partial_{x,y}F(x,y) \\ = x^{1/y-3}y^{1/x-3}\{x\log(x)[y\log(y)-x]-y[x(x+y-1)+y\log(y)]\}.$$

Hence, by evaluating this function at $(x,y)=(1,1)$, we obtain

$$\partial_{x,y}F(x,y)|_{(x,y)=(1,1)} = 1^{1/1-3}1^{1/1-3} \\ \times\{1\times\log(1)\times[1\times\log(1)-1]-1\times[1\times(1+1-1)+1\times\log(1)]\} \\ =-1<0.$$

As a result, H2 is not satisfied, so $F(x,y)$ is not a valid copula. ■

Figure 1 illustrates Proposition 2.2; the function $\partial_{x,y}F(x,y)$ is plotted for $(x,y)\in[0,1]^2$, and the light gray zone clearly indicates the domain in $[0,1]^2$ where $\partial_{x,y}F(x,y)$ is negative (including the point $(1,1)$ considered in the proof). The refutation of assumption H2 is clear.

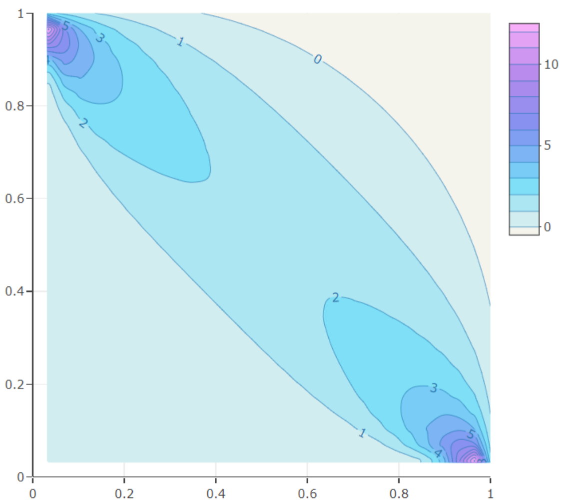

Fig. 1. Plot of $\partial_{x,y}F(x,y)$ for $(x,y)\in[0,1]^2$; the light gray zone indicates the values of (x,y) such that $\partial_{x,y}F(x,y)\le 0$.

To the best of our knowledge, there is no established copula involving the term $x^{1/y}y^{1/x}$. This direction is, however, of interest from the mathematical viewpoint and for the development of new dependence models.

2.2. First variable-power copula. In the next result, we propose a first variable-power copula based on $x^{1/y}y^{1/x}$.

PROPOSITION 2.3. *The two-dimensional function defined by*

$$C_{op}(x,y) = x^{1/y+\log(y)} y^{1/x}, \qquad (x,y) \in (0,1]^2, \tag{2}$$

and $C_{op}(x,y) = 0$ for $x = 0$ or $y = 0$, is a valid copula.

Proof. Let us investigate assumptions H1 and H2. For H1, for any $x \in (0,1]$, we have

$$C_{op}(x,1) = x^{1/1+\log(1)} \times 1^{1/x} = x$$

and $C_{op}(0,1) = 0$. Similarly, for any $y \in [0,1]$, we have $C_{op}(1,y) = y$. For any $x \in (0,1]$, since $\lim_{y\to 0} 1/y + \log(y) = \infty$, $\log(x) \leq 0$ and $\lim_{y\to 0}\log(y) = -\infty$, we have

$$\begin{aligned}\lim_{y\to 0} C_{op}(x,y) &= \lim_{y\to 0} x^{1/y+\log(y)} y^{1/x} \\ &= \lim_{y\to 0} \exp\left\{\left[\frac{1}{y} + \log(y)\right]\log(x) + \frac{1}{x}\log(y)\right\} = 0.\end{aligned}$$

Similarly, for any $y \in (0,1]$, we have $\lim_{x\to 0} C_{op}(x,y) = 0$. Since $C_{op}(0,0) = 0$ by definition, assumption H1 is satisfied.

Concerning H2, by applying differentiation rules and factoring in a suitable manner, we have

$$\begin{aligned}&\partial_{x,y} C_{op}(x,y) = y^{1/x-3} x^{1/y+\log(y)-3} \\ &\quad\times \{(1-x)y[x(1-y) + y\log_*(y)] + x(1-y)\log_*(x)[(1-x)y\log_*(y) + x]\},\end{aligned}$$

where $\log_*(x) = -\log(x)$ and $\log_*(y) = -\log(y)$. For any $(x,y) \in (0,1)^2$, it is clear that $1-x \geq 0$, $1-y \geq 0$, $\log_*(x) \geq 0$ and $\log_*(y) \geq 0$. As a result, the main terms in $\partial_{x,y} C_{op}(x,y)$ are non-negative, which implies that $\partial_{x,y} C_{op}(x,y) \geq 0$, so H2 is satisfied and $C_{op}(x,y)$ is a valid copula. ∎

Let us call the copula in (2) the *variable-power 1* (VP1) *copula*. To the best of our knowledge, it provides a new member of the variable-power copulas family (see [3, 4]). It is worth noting that the VP1 copula and the function in (1) are related as follows:

$$C_{op}(x,y) = F(x,y) x^{\log(y)}.$$

Furthermore, since $x^{\log(y)} = e^{\log(x)\log(y)} = y^{\log(x)}$, we can write

$$C_{op}(x,y) = x^{1/y} y^{1/x+\log(x)}, \tag{3}$$

which corresponds to $C_{op}(y,x)$. As sketched in the proof of Proposition 2.3, we can write the VP1 copula in the following exponential-logarithmic form:

$$C_{op}(x,y) = \exp\left\{\left[\frac{1}{y} + \log(y)\right]\log(x) + \frac{1}{x}\log(y)\right\}.$$

Despite a resemblance in form, the VP1 copula does not belong to the extreme-value copulas family due to the presence of the ratio terms $1/x$ and $1/y$.

Let us now study it in a more detailed manner.

Figure 2 illustrates Proposition 2.3 by plotting the VP1 copula and the associated VP1 copula density given as

$$c_{op}(x,y) = \partial_{x,y} C_{op}(x,y) = y^{1/x-3} x^{1/y+\log(y)-3}$$
$$\times \{(1-x)y[x(1-y) + y\log_*(y)] + x(1-y)\log_*(x)[(1-x)y\log_*(y) + x]\},$$
$$(x,y) \in (0,1)^2.$$

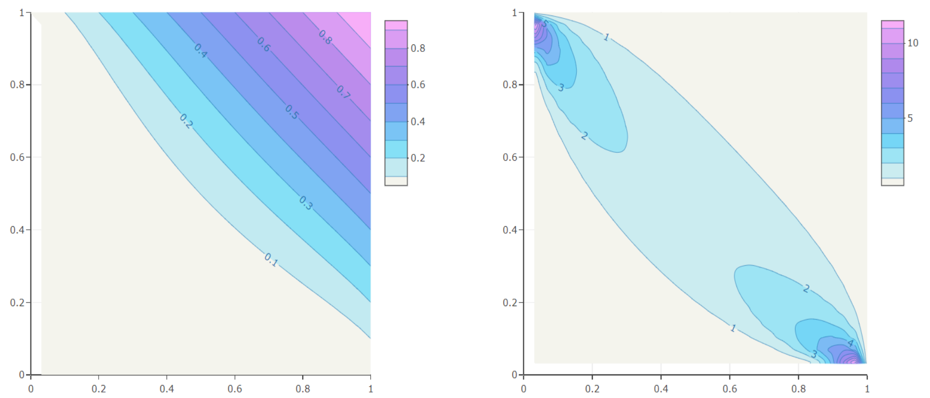

Fig. 2. Plots of the VP1 copula (left) and its density (right)

In this figure, the light gray zone indicates small values, sometimes close to 0, but never negative. We see that the higher values of the VP1 copula density are concentrated in the neighborhood of the extreme corner points $(0,1)$ and $(1,0)$. This indicates a tail dependence in these regions, as shown later with appropriate dependence parameters.

Among its main properties, based on (3), the VP1 copula is diagonally symmetric. This copula is not associative, as demonstrated by

$$C_{op}(0.5, C_{op}(0.6, 0.7)) \approx 0.02562692 \neq 0.01558517 \approx C_{op}(C_{op}(0.5, 0.6), 0.7).$$

In particular, it is not Archimedean.

The VP1 survival copula is given as

$$\hat{C}_{op}(x,y) = x + y - 1 + C_{op}(1-x, 1-y)$$
$$= x + y - 1 + (1-x)^{1/(1-y)+\log(1-y)}(1-y)^{1/(1-x)}, \quad (x,y) \in [0,1)^2,$$

and $\hat{C}_{op}(x,y) = 1$ for $x = 1$ or $y = 1$. Based on this expression, it is clear that the VP1 copula is not radially symmetric: there exists an $(x_\circ, y_\circ) \in [0,1]^2$ such that $\hat{C}_{op}(x_\circ, y_\circ) \neq C_{op}(x_\circ, y_\circ)$.

By the well-known copula theory, the Fréchet–Hoeffding bounds hold (see [12]). They imply $\max(x+y-1, 0) \leq C_{op}(x,y) \leq \min(x,y)$, i.e.,

$$\max(x+y-1,0) \leq x^{1/y+\log(y)} y^{1/x} \leq \min(x,y).$$

The lower left (LL), lower right (LR), upper left (UL), and upper right (UR) tail dependence parameters can be computed based on the formulas in [12] and [8]. Using typical limit methods, one finds that

$$\lambda_{\mathrm{LL}} = \lim_{x\to 0} \frac{C_{op}(x,x)}{x} = \lim_{x\to 0} x^{2/x+\log(x)-1} = 0$$

(the limit is taken for 0^+),

$$\begin{aligned}
\lambda_{\mathrm{LR}} &= \lim_{x\to 0} \frac{x - C_{op}(1-x,x)}{x} = \lim_{x\to 0} \frac{x-(1-x)^{1/x+\log(x)}x^{1/(1-x)}}{x} \\
&= \frac{e-1}{e} \approx 0.632121, \\
\lambda_{\mathrm{UL}} &= \lim_{x\to 0} \frac{x - C_{op}(x,1-x)}{x} = \lim_{x\to 0} \frac{x-x^{1/(1-x)+\log(1-x)}(1-x)^{1/x}}{x} \\
&= \frac{e-1}{e} \approx 0.632121, \\
\lambda_{UR} &= \lim_{x\to 1} \frac{1-2x+C_{op}(x,x)}{1-x} = \lim_{x\to 1} \frac{1-2x+x^{2/x+\log(x)}}{1-x} \\
&= -\frac{3}{4}\lim_{x\to 1}(x-1)^3 = 0.
\end{aligned}$$

The LL and UR tail independence properties are satisfied by the VP1 copula. It has, however, the LR and UL tail dependences at the same numerical level, i.e., $(e-1)/e$.

With reference to [12], the medial correlation coefficient of the VP1 copula is

$$M_{ed} = 4C_{op}(0.5,0.5) - 1 = 4\times 2^{-2+\log(2)}2^{-2} - 1 = 2^{\log(2)-2} - 1 \approx -0.59579.$$

This negative value is not so far from -1 and highlights the negative dependence feature of the VP1 copula.

Also, based on the theory in [12], Spearman's rho related to the VP1 copula is

$$\begin{aligned}
\rho_{\mathrm{Spear}} &= 12 \int_{[0,1]^2} [C_{op}(x,y) - \Pi(x,y)]\,dx\,dy \\
&= 12 \int_{[0,1]^2} [x^{1/y+\log(y)}y^{1/x} - xy]\,dx\,dy \approx -0.8062.
\end{aligned}$$

This corresponds to a moderate negative correlation.

One of the main objectives of the VP1 copula is to generate two-dimensional distributions. The basic scheme is as follows: Let $F(x)$ and $G(y)$ be (one-dimensional) cumulative distribution functions of absolutely continuous distributions. Then we define a new two-dimensional cumulative distribution function by

$$H(x,y)=C_{op}(F(x),G(y))=F(x)^{1/G(y)+\log[G(y)]}G(y)^{1/F(x)}, \quad (x,y)\in\mathbb{R}^2.$$

A new two-dimensional distribution defined by this two-dimensional cumulative distribution function is thus created. For the choices of $F(x)$ and $G(y)$ in a lifetime setting, we may refer to the review of [17].

As a last comment, we can mention that some copula schemes allow for the construction of diagonally asymmetric copulas based on symmetric copulas. For instance, by using [10], an asymmetric version is given by $C_{op}(x,y;a,b)=x^{1-a}y^{1-b}C_{op}(x^a,y^b)$, with $a\in[0,1]$ and $b\in[0,1]$, i.e.,

$$C_{op}(x,y;a,b)=x^{a/y^b+ab\log(y)+1-a}y^{b/x^a+1-b}, \quad (x,y)\in(0,1]^2,$$

and $C_{op}(x,y;a,b)=0$ for $x=0$ and $y=0$. A one-parameter symmetric version of the VP1 copula is obtained by taking $a=b$, i.e.,

$$C_{op}(x,y;a)=x^{a/y^a+a^2\log(y)+1-a}y^{a/x^a+1-a}, \quad (x,y)\in(0,1]^2,$$

and $C_{op}(x,y;a)=0$ for $x=0$ and $y=0$. Clearly, the VP1 copula is obtained by taking $a=1$, and the independence copula follows by taking $a=0$. However, the parameter a also affects the power terms, and so the tradeoff between the VP1 copula and the independence copula is a bit artificial. A more suitable solution will be presented in Subsection 3.1.

2.3. Second variable-power copula. In the next result, we propose another variable-power copula based on $x^{1/y}y^{1/x}$.

PROPOSITION 2.4. *The two-dimensional function defined by*

$$C_{op}(x,y)=x^{(1/2)(1/y+1)}y^{(1/2)(1/x+1)}, \quad (x,y)\in(0,1]^2, \tag{4}$$

and $C_{op}(x,y)=0$ *for* $x=0$ *or* $y=0$, *is a valid copula.*

Proof. Let us check assumption H1. For any $x\in(0,1]$, we have

$$C_{op}(x,1)=x^{(1/2)(1/1+1)}\times 1^{(1/2)(1/x+1)}=x^{2/2}\times 1=x$$

and $C_{op}(0,1)=0$. Similarly, for any $y\in[0,1]$, we have $C_{op}(1,y)=y$. For any $x\in(0,1]$, since $\lim_{y\to 0}\log(x)/y$ is $-\infty$ (for $x>0$) or 0 (for $x=1$), and $\lim_{y\to 0}\log(y)=-\infty$ and $\lim_{y\to 0}\log(xy)=-\infty$, we have

$$\begin{aligned}\lim_{y\to 0}C_{op}(x,y)&=\lim_{y\to 0}x^{(1/2)(1/y+1)}y^{(1/2)(1/x+1)}\\&=\lim_{y\to 0}\exp\left\{\frac{1}{2}\left[\frac{1}{y}\log(x)+\frac{1}{x}\log(y)+\log(xy)\right]\right\}=0.\end{aligned}$$

In addition, for any $y\in(0,1]$, we have $\lim_{x\to 0}C_{op}(x,y)=0$. Since $C_{op}(0,0)$ $=0$ by definition, assumption H1 is satisfied.

Let us now examine H2. Upon differentiation and algebraic manipulations, we get

$$\partial_{x,y} C_{op}(x,y) = \tfrac{1}{4} x^{(1/2)(1/y-5)} y^{(1/2)(1/x-5)} \times \{y[x(1-x)(1-y) + (x+1)y \log_*(y)] + x\log_*(x)(xy + x + y\log_*(y))\}$$

(we recall that $\log_*(x) = -\log(x)$ and $\log_*(y) = -\log(y)$). For any $(x,y) \in (0,1)^2$, it is clear that $(1-x)(1-y) \geq 0$, $\log_*(x) \geq 0$ and $\log_*(y) \geq 0$. As a result, the main terms in $\partial_{x,y} C_{op}(x,y)$ being non-negative, we have $\partial_{x,y} C_{op}(x,y) \geq 0$, so H2 is satisfied and $C_{op}(x,y)$ is a valid copula. ∎

Let us call the copula in (4) the *variable-power 2* (VP2) *copula*. To the best of our knowledge, it is also a new member of the variable-power copulas family. The VP2 copula and the function in (1) are related as follows:

$$C_{op}(x,y) = [xyF(x,y)]^{1/2}.$$

Alternatively, from a geometric power viewpoint, we can write

$$C_{op}(x,y) = \Pi(x,y)^a F(x,y)^{1-a},$$

where $a = 1/2$. As sketched in the proof of Proposition 2.4, we can write the VP2 copula in the following exponential-logarithmic form:

$$C_{op}(x,y) = \exp\left\{\frac{1}{2}\left[\frac{1}{y}\log(x) + \frac{1}{x}\log(y) + \log(xy)\right]\right\}.$$

However, like the VP1 copula, it does not belong to the extreme-value copulas family due to the presence of the ratio terms $1/x$ and $1/y$.

Another remark is that we can write

$$C_{op}(x,y) = \phi(x,y)\phi(y,x),$$

where $\phi(x,y) = x^{(1/2)(1/y+1)}$ is a non-symmetric function. Such separable copulas do not seem to be common in the literature.

The above expressions underline the original definition of the VP2 copula. Let us study it in a more detailed manner.

In Figure 3 we illustrate Proposition 2.4 by plotting the VP2 copula and the associated VP2 copula density given as

$$c_{op}(x,y) = \partial_{x,y} C_{op}(x,y) = \frac{1}{4} x^{(1/2)(1/y-5)} y^{(1/2)(1/x-5)} \times \{y[x(1-x)(1-y) + (x+1)y\log_*(y)] + x\log_*(x)[xy + x + y\log_*(y)]\}, \quad (x,y) \in (0,1)^2.$$

In this figure, the light gray zone indicates small values, sometimes close to 0, but never negative. We see that the higher values of the VP2 copula density are concentrated in the neighborhood of the points $(0,1)$ and $(1,0)$. In comparison to the VP1 copula, we see a kind of contraction phenomenon for the values in the neighborhood of the diagonal line $x = y$.

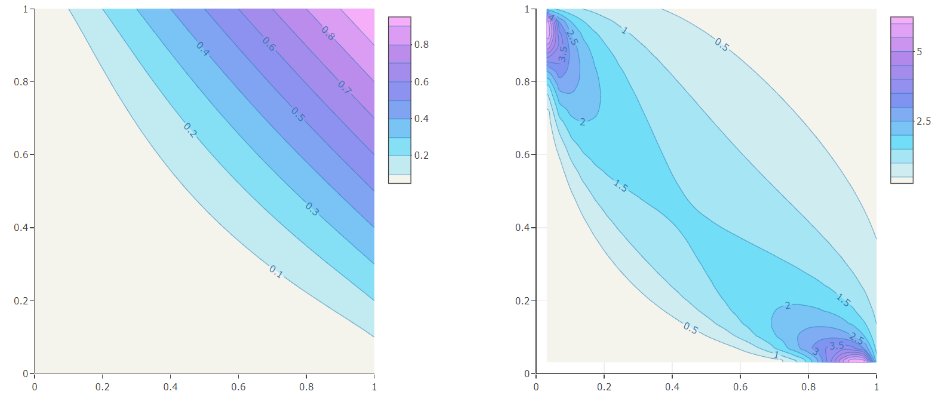

Fig. 3. Plots of the VP2 copula (left) and its density (right)

Among its main properties, the VP2 copula is diagonally symmetric since $C_{op}(x, y) = C_{op}(y, x)$ for any $(x, y) \in [0, 1]^2$. We can calculate that

$$\begin{aligned} C_{op}(0.4, C_{op}(0.5, 0.6)) &\approx 0.002738362 \\ &\neq 0.0008240825 \approx C_{op}(C_{op}(0.4, 0.5), 0.6), \end{aligned}$$

demonstrating that the VP2 copula is not associative. As a result, it is not Archimedean.

The VP2 survival copula is given as

$$\begin{aligned} \hat{C}_{op}(x, y) &= x + y - 1 + C_{op}(1 - x, 1 - y) \\ &= x + y - 1 + (1 - x)^{(1/2)[1/(1-y)+1]}(1 - y)^{(1/2)[1/(1-x)+1]}, \\ &\qquad (x, y) \in [0, 1)^2, \end{aligned}$$

and $\hat{C}_{op}(x, y) = 1$ for $x = 1$ or $y = 1$. From this expression, it is clear that the VP2 copula is not radially symmetric: there exists an $(x_\circ, y_\circ) \in [0, 1]^2$ such that $\hat{C}_{op}(x_\circ, y_\circ) \neq C_{op}(x_\circ, y_\circ)$. The Fréchet–Hoeffding bounds imply that $\max(x + y - 1, 0) \leq C_{op}(x, y) \leq \min(x, y)$, i.e.,

$$\max(x + y - 1, 0) \leq x^{(1/2)(1/y+1)} y^{(1/2)(1/x+1)} \leq \min(x, y).$$

The tail dependence parameters are computed using typical limit methods as follows:

$$\lambda_{\mathrm{LL}} = \lim_{x \to 0} \frac{C_{op}(x, x)}{x} = \lim_{x \to 0} x^{1/x} = 0,$$

$$\lambda_{\mathrm{LR}} = \lim_{x\to 0} \frac{x - C_{op}(1-x, x)}{x} = \lim_{x\to 0} \frac{x - (1-x)^{(1/2)(1/x+1)}x^{(1/2)[1/(1-x)+1]}}{x}$$
$$= \frac{\sqrt{e}-1}{\sqrt{e}} \approx 0.393469,$$
$$\lambda_{\mathrm{UL}} = \lim_{x\to 0} \frac{x - C_{op}(x, 1-x)}{x} = \lim_{x\to 0} \frac{x - x^{(1/2)[1/(1-x)+1]}(1-x)^{(1/2)(1/x+1)}}{x}$$
$$= \frac{\sqrt{e}-1}{\sqrt{e}} \approx 0.393469,$$
$$\lambda_{\mathrm{UR}} = \lim_{x\to 1} \frac{1-2x+C_{op}(x,x)}{1-x} = \lim_{x\to 1} \frac{1-2x+x^{1/x+1}}{1-x} = \frac{1}{2}\lim_{x\to 1}(x-1)^2 = 0.$$

The LL and UR tail independence properties are satisfied by the VP2 copula. It has, however, the LR and UL tail dependences at the same numerical level, i.e., $(\sqrt{e}-1)/\sqrt{e}$.

The medial correlation coefficient of the VP2 copula is

$$M_{ed} = 4C_{op}(0.5, 0.5) - 1 = 4 \times 2^{-(1/2)(2+1)} \times 2^{-(1/2)(2+1)} - 1 = -0.5.$$

This negative value highlights the negative dependence feature of the VP2 copula.

Spearman's rho related to the VP2 copula is

$$\rho_{\mathrm{Spear}} = 12 \int_{[0,1]^2} [C_{op}(x,y) - \Pi(x,y)]\,dx\,dy$$
$$= 12 \int_{[0,1]^2} [x^{(1/2)(1/y+1)}y^{(1/2)(1/x+1)} - xy]\,dx\,dy \approx -0.6932.$$

This corresponds to a moderate negative correlation. We can see that it is greater than the one of the VP1 copula.

The VP2 copula can also be used to generate two-dimensional distributions: if $F(x)$ and $G(y)$ are (one-dimensional) cumulative distribution functions of absolutely continuous distributions, then we define a two-dimensional cumulative distribution function by

$$H(x,y) = C_{op}(F(x), G(y)) = F(x)^{(1/2)[1/G(y)+1]}G(y)^{(1/2)[1/F(x)+1]}, \quad (x,y) \in \mathbb{R}^2.$$

For an asymmetric parametric version of the VP2 copula, we can again use [10] by considering $C_{op}(x,y;a,b) = x^{1-a}y^{1-b}C_{op}(x^a, y^b)$ with $a \in [0,1]$ and $b \in [0,1]$, i.e.,

$$C_{op}(x,y;a,b) = x^{(a/2)(1/y^b+1)+1-a}y^{(b/2)(1/x^a+1)+1-b}, \qquad (x,y) \in (0,1]^2,$$

and $C_{op}(x,y;a,b) = 0$ for $x = 0$ or $y = 0$. A one-parameter symmetric version of the VP2 copula is obtained by taking $a = b$, i.e.,

$$C_{op}(x, y; a) = x^{(a/2)(1/y^a+1)+1-a}y^{(a/2)(1/x^a+1)+1-a}, \qquad (x, y) \in (0, 1]^2,$$

and $C_{op}(x, y; a) = 0$ for $x = 0$ and $y = 0$. The VP2 copula is obtained by taking $a = 1$, and the independence copula is obtained by taking $a = 0$. Here again, the parameter a also affects the power terms, and so the tradeoff between the VP2 copula and the independence copula is somewhat artificial. In Section 3.2, a better approach will be offered.

3. Parametric extensions. As sketched above, one of the main drawbacks of the VP1 and VP2 copulas is their rigidity in the functional sense (no tuning parameters are involved, the independence copula case is not covered, etc.). Some possible solutions have been presented, but they can be unsatisfying from the statistical viewpoint, with the tuning parameters affecting both exponents and constants.

In this section, we fill this gap by proposing parametric generalizations based on geometric power transformations.

3.1. Parametric extension of the VP1 copula. A parametric generalization of the VP1 copula is presented in the result below.

PROPOSITION 3.1. *Let $a \in \mathbb{R}$. The two-dimensional function defined by*

$$C_{op}(x, y; a) = x^{a[1/y+\log(y)]+1-a}y^{a/x+1-a}, \qquad (x, y) \in (0, 1]^2, \tag{5}$$

and $C_{op}(x, y; a) = 0$ for $x = 0$ or $y = 0$, is a valid copula for $a \in [0, 1]$.

Proof. To check assumption H1, for any $x \in (0, 1]$, we have

$$C_{op}(x, 1; a) = x^{a[1/1+\log(1)]+1-a}1^{a/x+1-a} = x^{a+1-a} = x$$

and $C_{op}(0, 1; a) = 0$. Similarly, for any $y \in [0, 1]$, we have $C_{op}(1, y; a) = y$. For any $x \in (0, 1]$, we have $\lim_{y\to0}[1/y+\log(y)] = \infty$, $\log(x) \le 0$, $\lim_{y\to0} \log(y) = -\infty$ and $\lim_{y\to0} \log(xy) = -\infty$. Furthermore, since $a \in [0, 1]$ we obviously have $a \ge 0$ and $1 - a \ge 0$. We thus obtain

$$\begin{aligned}\lim_{y\to0} C_{op}(x, y; a) &= \lim_{y\to0} x^{a[1/y+\log(y)]+1-a}y^{a/x+1-a}\\ &= \lim_{y\to0} \exp\left\{a\left[\frac{1}{y} + \log(y)\right]\log(x) + \frac{a}{x}\log(y) + (1 - a)\log(xy)\right\} = 0.\end{aligned}$$

Similarly, for any $y \in (0, 1]$, we have $\lim_{x\to0} C_{op}(x, y; a) = 0$. Since $C_{op}(0, 0; a) = 0$, assumption H1 is fulfilled.

Concerning H2, with appropriate differentiation and factoring we obtain

$$\begin{aligned}\partial_{x,y}C_{op}(x, y; a) &= y^{a(1/x-1)-2}x^{a(1/y-1)+a\log(y)-2}\\ &\times \big[ax(1 - y)\log_*(x)\{x[a(1 - y) + y] + a(1 - x)y\log_*(y)\}\\ &+ y\{x[a^2(1 - x)(1 - y) + (1 - a)xy] + a(1 - x)y[a(1 - x) + x]\log_*(y)\}\big]\end{aligned}$$

(we recall that $\log_*(x) = -\log(x)$ and $\log_*(y) = -\log(y)$). For any $(x, y) \in (0, 1)^2$, it is clear that $(1-x)(1-y) \geq 0$, $\log_*(x) \geq 0$ and $\log_*(y) \geq 0$. Since $a \in [0, 1]$, we have $a \geq 0$ and $1 - a \geq 0$. Hence, the main terms in $\partial_{x,y} C_{op}(x, y; a)$ are non-negative. Therefore, $\partial_{x,y} C_{op}(x, y; a) \geq 0$, so H2 is satisfied and $C_{op}(x, y; a)$ is a valid copula. ∎

Let us call the copula in (5) the *parametric VP1* (PVP1) *copula*. The parameter a adds a new degree of flexibility since it realizes a tradeoff between the VP1 copula obtained for $a = 1$ and the independence copula for $a = 0$. Furthermore, we can show that the condition $a \in [0, 1]$ is optimal. Globally, the PVP1 copula has the same properties as the VP1 copula, except for the versatility in the shape and the correlation properties, which are significantly nuanced by the parameter a.

We emphasize these aspects in what follows. The PVP1 copula density is given by

$$\begin{aligned} c_{op}(x, y; a) &= \partial_{x,y} C_{op}(x, y; a) = y^{a(1/x-1)-2} x^{a(1/y-1)+a\log(y)-2} \\ &\times \big[ax(1-y)\log_*(x)\{x[a(1-y)+y] + a(1-x)y\log_*(y)\} \\ &+ y\{x[a^2(1-x)(1-y) + (1-a)xy] + a(1-x)y[a(1-x)+x]\log_*(y)\}\big], \\ &\qquad (x, y) \in (0, 1)^2. \end{aligned}$$

Figures 4, 5 and 6 display the PVP1 copula and its density for $a = 0.2$, $a = 0.5$ and $a = 0.8$, respectively.

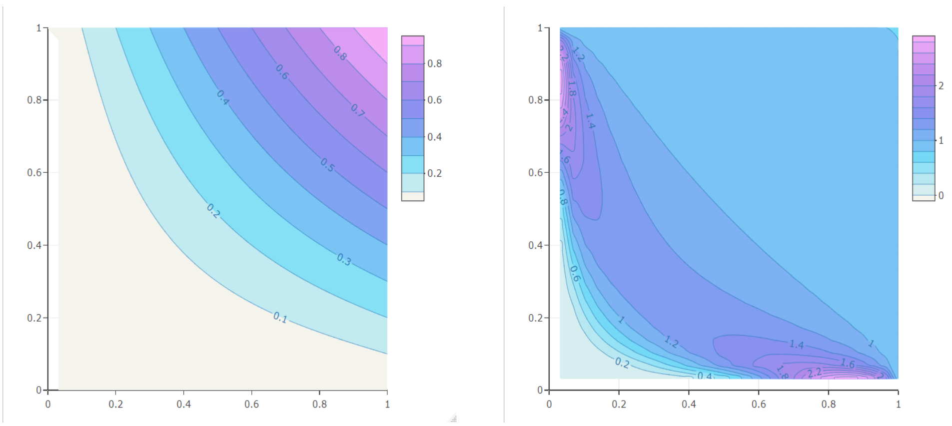

Fig. 4. Plots of the PVP1 copula (left) and its density (right) for $a = 0.2$

From these figures, the effect of the parameter a on the shapes of the PVP1 copula and its density is clear; it modulates the skewness of the shapes in a complex manner, and the highest values are concentrated in the neighborhood of $(0, 1)$ and $(1, 0)$. The main changes are observed in the triangle

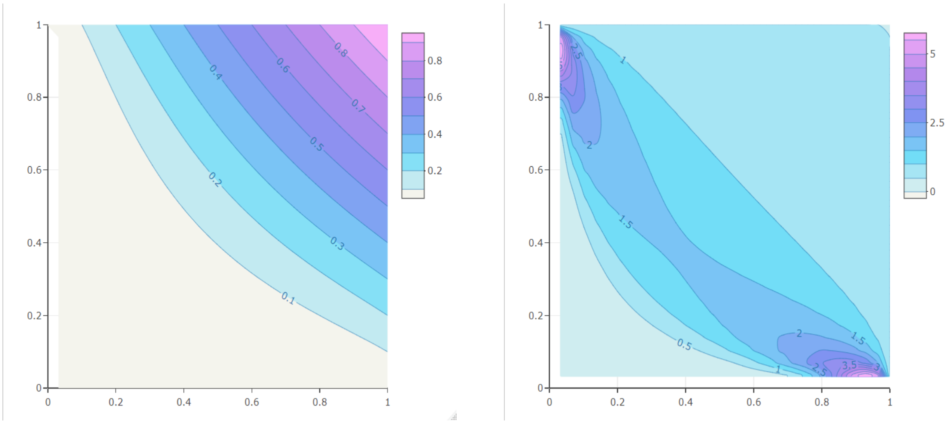

Fig. 5. Plots of the PVP1 copula (left) and its density (right) for $a = 0.5$

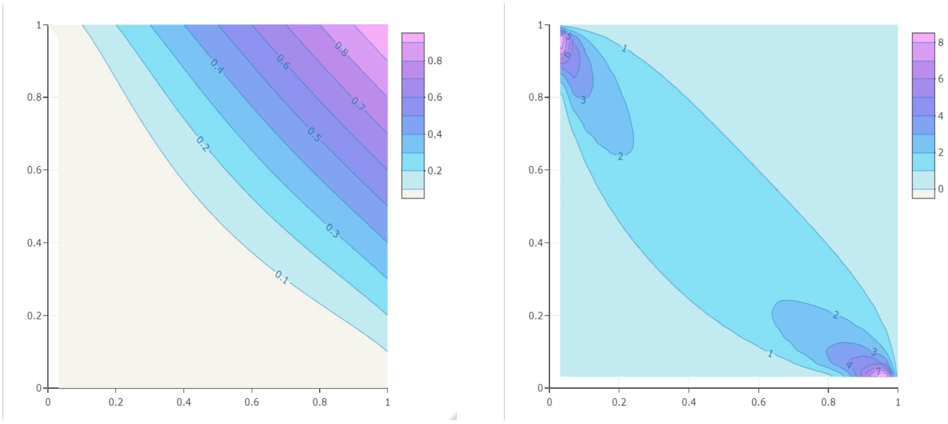

Fig. 6. Plots of the PVP1 copula (left) and its density (right) for $a = 0.8$

$\Delta = \{(x, y) \in [0, 1]^2;\, x + y \leq 1\}$. These are attractive properties for some extreme dependence modeling.

The medial correlation coefficient of the PVP1 copula is

$$M_{ed} = 4C_{op}(0.5, 0.5; a) - 1 = 2^{-a[2-\log(2)]} - 1.$$

Since $2 - \log(2) < 0$, this coefficient is non-positive, highlighting the negative dependence feature of the PVP1 copula.

Spearman's rho related to the PVP1 copula is

$$\begin{aligned}\rho_{\text{Spear}} &= 12 \int_{[0,1]^2} [C_{op}(x, y; a) - \Pi(x, y)]\, dx\, dy \\ &= 12 \int_{[0,1]^2} [x^{a[1/y+\log(y)]+1-a} y^{a/x+1-a} - xy]\, dx\, dy.\end{aligned}$$

Due to the complexity of the integrand, there is no simple expression for this measure. In order to have a numerical assessment, Table 1 presents some of its values for selected values of $a \in [0,1]$.

Table 1. Some values of ρ_{Spear} for $a = 0, 0.1, \ldots, 1$

a	0.0	0.1	0.2	0.3	0.4	0.5
ρ_{Spear}	0	−0.1669	−0.2866	−0.3836	−0.4661	−0.5382

a	0.6	0.7	0.8	0.9	1.0
ρ_{Spear}	−0.6023	−0.6602	−0.7129	−0.7614	−0.8062

From this table, it is clear that the PVP1 copula can reach different levels of negative dependence in the range $[-0.8062, 0]$. This range is close to the optimal negative one for Spearman's rho, i.e., $[-1, 0]$.

To end this subsection, let us mention that several techniques can be applied to introduce asymmetry into the PVP1 copula. Among them, the technique in [10] suggests considering $C_{op}(x, y; a, b, c) = x^{1-b}y^{1-c}C_{op}(x^b, y^c; a)$, with $b \in [0,1]$ and $c \in [0,1]$, i.e.,

$$C_{op}(x, y; a, b, c) = x^{ab[1/y^c + c\log(y)]+1-ab} y^{ca/x^b+1-ac}, \qquad (x, y) \in (0,1]^2,$$

and $C_{op}(x, y; a, b, c) = 0$ for $x = 0$ or $y = 0$. Other techniques, like the flipping technique, are possible (see [12]).

3.2. Parametric extension of the VP2 copula. A parametric generalization of the VP2 copula is presented in the result below.

Proposition 3.2. *Let $a \in \mathbb{R}$. The two-dimensional function defined by*

$$C_{op}(x, y; a) = x^{a/y+1-a} y^{a/x+1-a}, \qquad (x, y) \in (0,1]^2, \tag{6}$$

and $C_{op}(x, y; a) = 0$ for $x = 0$ and $y = 0$, is a valid copula for $a \in [0, 1/2]$.

Proof. Let us check assumption H1. For any $x \in (0,1]$, we have

$$C_{op}(x, 1; a) = x^{(a/1+1-a)} \times 1^{(a/x+1-a)} = x^{a+1-a} \times 1 = x$$

and $C_{op}(0, 1; a) = 0$. Similarly, for any $y \in [0,1]$, we have $C_{op}(1, y; a) = y$. For any $x \in (0,1]$, we have $\lim_{y\to 0} \log(x)/y = -\infty$ (if $x > 0$) or 0 if $x = 1$, and $\lim_{y\to 0} \log(y) = -\infty$ and $\lim_{y\to 0} \log(xy) = -\infty$. Furthermore, since $a \in [0, 1/2]$ we obviously have $a \geq 0$ and $1 - 2a \geq 0$. We thus obtain

$$\begin{aligned} \lim_{y\to 0} C_{op}(x, y; a) &= \lim_{y\to 0} x^{a/y+1-a} y^{a/x+1-a} \\ &= \lim_{y\to 0} \exp\left[\frac{a}{y}\log(x) + \frac{a}{x}\log(y) + (1-a)\log(xy)\right] = 0. \end{aligned}$$

By using the same arguments, for any $y \in (0,1]$, we have $\lim_{x\to 0} C_{op}(x, y; a) = 0$. Since $C_{op}(0, 0; a) = 0$ by definition, H1 is satisfied.

For H2, standard differentiation techniques and rearrangements give

$$\partial_{x,y}C_{op}(x,y;a) = x^{a(1/y-1)-2}y^{a(1/x-1)-2} \times \big(y\{x[a^2(1-x)(1-y)+xy(1-2a)]+ay[a(1-x)+x]\log_*(y)\} + ax\log_*(x)[ax(1-y)+ay\log_*(y)+xy]\big)$$

(we recall that $\log_*(x) = -\log(x)$ and $\log_*(y) = -\log(y)$). For any $(x, y) \in (0,1)^2$, it is clear that $(1-x)(1-y) \geq 0$, $\log_*(x) \geq 0$ and $\log_*(y) \geq 0$. Since $a \in [0,1/2]$, we have $a \geq 0$ and $1-2a \geq 0$. Hence, the main terms in $\partial_{x,y}C_{op}(x,y;a)$ are non-negative. Therefore, $\partial_{x,y}C_{op}(x,y;a) \geq 0$, so assumption H2 is satisfied and $C_{op}(x,y;a)$ is a valid copula. ■

Analogously to the previous subsection, let us call the copula in (6) the *parametric VP2* (PVP2) *copula*. The parameter a adds a new degree of flexibility since it realizes a tradeoff between the VP2 copula defined with $a = 1/2$ and the independence copula defined with $a = 0$. Furthermore, since the condition $a \geq 0$ is necessary for assumption H1 to hold, and $\partial_{x,y}C_{op}(1,1;a) = 1-2a$ must be non-negative, the condition $a \in [0,1/2]$ is optimal.

Globally, the PVP2 copula has the same properties as the VP2 copula, except for the versatility in the shape and the correlation properties, which are significantly nuanced by the parameter a.

We emphasize these aspects in what follows. The PVP2 copula density is given by

$$c_{op}(x,y;a) = \partial_{x,y}C_{op}(x,y;a) = x^{a(1/y-1)-2}y^{a(1/x-1)-2} \times \big(y\{x[a^2(1-x)(1-y)+xy(1-2a)]+ay[a(1-x)+x]\log_*(y)\} + ax\log_*(x)[ax(1-y)+ay\log_*(y)+xy]\big), \qquad (x,y) \in (0,1)^2.$$

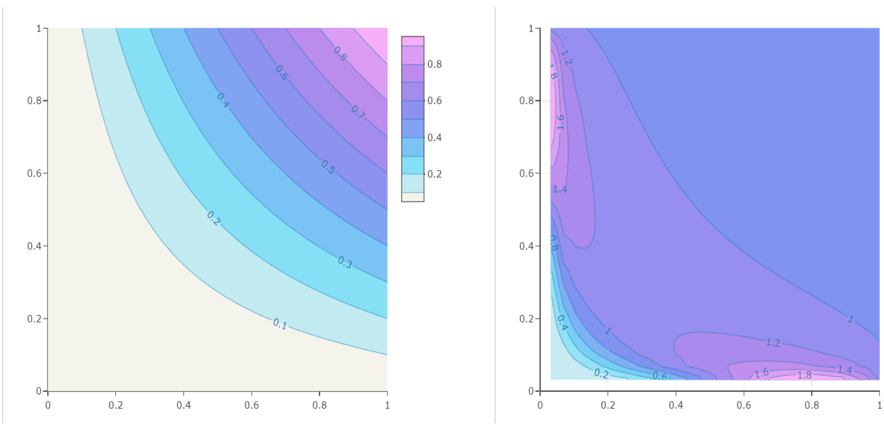

Fig. 7. Plots of the PVP2 copula (left) and its density (right) for $a = 0.1$

Figures 7, 8 and 9 show the PVP2 copula and its density for $a = 0.1$, $a = 0.25$ and $a = 0.4$, respectively.

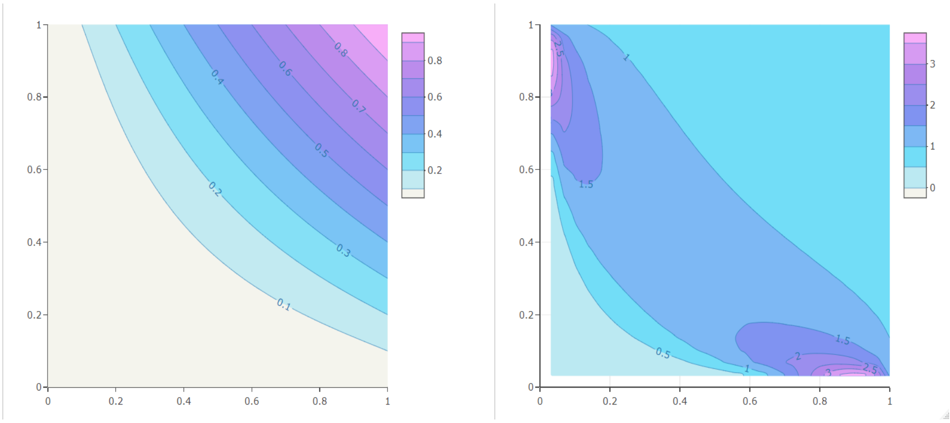

Fig. 8. Plots of the PVP2 copula (left) and its density (right) for $a = 0.25$

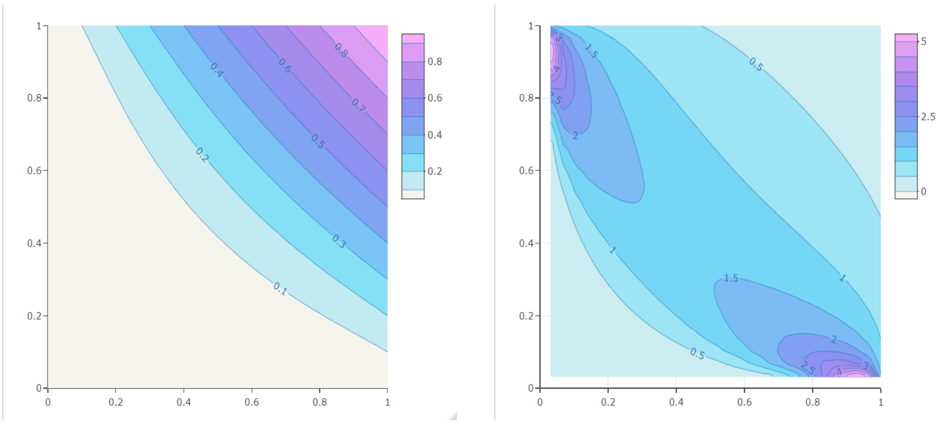

Fig. 9. Plots of the PVP2 copula (left) and its density (right) for $a = 0.4$

From these figures, the effect of the parameter a on the shapes of the PVP2 copula and its density is clear; it modulates the skewness of the shapes in a complex manner. This is particularly the case in the triangle $\varDelta$.

The medial correlation coefficient of the PVP2 copula is

$$M_{ed} = 4C_{op}(0.5, 0.5; a) - 1 = 2^{-2a} - 1.$$

This coefficient is obviously non-positive, highlighting the negative dependence feature of the PVP2 copula.

Spearman's rho related to the PVP2 copula is

$$\rho_{\text{Spear}} = 12 \int_{[0,1]^2} [C_{op}(x, y; a) - \Pi(x, y)]\, dx\, dy$$
$$= 12 \int_{[0,1]^2} [x^{a/y+1-a} y^{a/x+1-a} - xy]\, dx\, dy.$$

There is no simple form for this measure due to the complexity of the integrand. In order to have a numerical assessment, Table 2 presents some of its values for selected values of $a \in [0, 1/2]$.

Table 2. Some values of ρ_{Spear} for $a = 0, 0.05, 0.1, 0.15, \ldots, 0.5$

a	0.00	0.05	0.10	0.15	0.20	0.25
ρ_{Spear}	0	−0.1242	−0.2219	−0.3048	−0.3775	−0.4423

a	0.30	0.35	0.40	0.45	0.50
ρ_{Spear}	−0.501	−0.5548	−0.6043	−0.6503	−0.6932

From this table, it is clear that the PVP2 copula can reach different levels of negative dependence, with the range $[-0.7, 0]$, which is close to the optimal negative one for Spearman's rho, i.e., $[-1, 0]$.

Just as for the PVP1 copula, we can use the technique in [10] to propose an asymmetric version, i.e., $C_{op}(x, y; a, b, c) = x^{1-b} y^{1-c} C_{op}(x^b, y^c; a)$ with $b \in [0, 1]$ and $c \in [0, 1]$. It can be expressed as

$$C_{op}(x, y; a, b, c) = x^{ba/y^c+1-ab} y^{ca/x^b+1-ac}, \qquad (x, y) \in (0, 1]^2,$$

and $C_{op}(x, y; a, b, c) = 0$ for $x = 0$ or $y = 0$.

4. Conclusion. In conclusion, this article introduces and studies new variable-power copulas that offer valuable dependence modeling capabilities. As a summary, they are presented in Table 3.

Table 3. The main variable-power copulas of the article.

Name	Definition	Interval of values for a
VP1	$x^{1/y+\log(y)} y^{1/x}$	✗
VP2	$x^{(1/2)(1/y+1)} y^{(1/2)(1/x+1)}$	✗
PVP1	$x^{a[1/y+\log(y)]+1-a} y^{a/x+1-a}$	$[0, 1]$
PVP2	$x^{a/y+1-a} y^{a/x+1-a}$	$[0, 1/2]$

In comparison to the standard schemes, the utilization of variable powers enables an alternative representation of dependence structures, accommodating a wider range of real-world scenarios. By incorporating these copulas into statistical analyses, researchers and practitioners can improve the accuracy and effectiveness of various risk management, financial modeling, and actuarial applications. These novel copulas pave the way for more robust and accurate modeling techniques in the future.

Some possible perspectives of research include two-dimensional data analyses, the development of three-dimensional variable-power copulas, and the construction of copula regression models.

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