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ON THE (m,t)-EXTENSION DUAL COMPLEX FIBONACCI p-NUMBERS

Abstract. We introduce (m,t)-extension dual complex Fibonacci p-numbers and we establish some of their properties. They are connected to complex Fibonacci numbers, complex Fibonacci p-numbers, m-extension dual complex Fibonacci p-numbers and dual complex Fibonacci p-numbers.

1. Introduction. The *Fibonacci numbers* F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2$$

with initial terms

$$F_1 = F_2 = 1.$$

The Fibonacci numbers F_n and the golden mean (golden ratio)

$$\tau = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$$

have materialized in several areas of art, science, computer science, high energy physics, information and coding theory [3, 4, 6, 7, 17, 18].

The *Fibonacci p-numbers* [17] are defined by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with $n > p+1$, for a given integer $p = 0, 1, 2, \ldots$, and with initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1.$$

The Fibonacci p-numbers coincide with the classical Fibonacci numbers for $p = 1$, e.g. $F_1(n) = F_n$.

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E. G. Kocer et al. [9] introduced the m-extension of Fibonacci p-numbers which satisfy the recurrence relation

$$F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1)$$

with initial terms

$$F_{p,m}(1) = a_1, \quad F_{p,m}(2) = a_2, \quad F_{p,m}(3) = a_3, \; \dots, \; F_{p,m}(p+1) = a_{p+1},$$

where p (≥ 0) is an integer, m (> 0), $n > p+1$ and $a_1, \dots, a_{p+1}$ are arbitrary real or complex numbers.

Prasad et al. [16] introduced the (m,t)-extension of Fibonacci p-numbers which satisfy the recurrence relation

$$F_{p,m,t}(n) = mF_{p,m,t}(n-1) + tF_{p,m,t}(n-p-1)$$

with initial terms

$$F_{p,m,t}(1) = b_1, \quad F_{p,m,t}(2) = b_2, \quad F_{p,m,t}(3) = b_3, \; \dots, \; F_{p,m,t}(p+1) = b_{p+1},$$

where p (≥ 0) is an integer, m (> 0), t (> 0), $n > p+1$ and $b_1, \dots, b_{p+1}$ are arbitrary real or complex numbers.

The *complex Fibonacci numbers* [12] are defined by the recurrence relation

$$F_n^\star = F_{n-1}^\star + F_{n-2}^\star \quad \text{for } n \geqslant 2$$

with initial terms

$$F_0^\star = i, \quad F_1^\star = 1 + i$$

where i is the imaginary unit and

$$F_n^\star = F_n + iF_{n+1}.$$

Clifford algebra is a powerful mathematical tool that offers a natural and direct way to model geometric objects and their transformations. It makes geometric objects (points, lines and planes) into basic elements of computation and defines a few universal operators that are applicable to all types of geometric elements. In the 19th century, Clifford [5] defined a new number system by using the notion $\varepsilon^2 = 0$ and $\varepsilon \neq 0$, i.e. ε is a nilpotent number. This number system is known as the *dual number system* and dual numbers are represented in the form $A = a + \varepsilon a^\star$ where a and $a^\star$ are real numbers and ε is the nilpotent number. Kotelnikov [10] and Study [19] found the first applications of dual numbers in mechanics. They also have applications in algebraic geometry, kinematics, number theory, theory of relativity etc. [2, 11, 14].

Dual complex numbers [11] are defined as follows:

$$W = \{w = z_1 + \varepsilon z_2 : z_1, z_2 \in \mathbb{C}, \, \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0\}.$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then any dual complex number can be written as follows:

$$w = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2 \quad \text{where} \quad i^2 = -1,\ \varepsilon \neq 0,\ \varepsilon^2 = 0,\ (i\varepsilon)^2 = 0.$$

The real and dual quaternions form a division ring but dual complex numbers form a commutative ring with characteristic zero. The multiplication of dual complex numbers gives the structure of the two-dimensional complex Clifford algebra and the four-dimensional real Clifford algebra. The base elements of dual complex numbers satisfy the commutative multiplication scheme shown in Table 1.

Table 1. Multiplication scheme of dual complex numbers

$\times$	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	-1	$i\varepsilon$	$-\varepsilon$
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

Five different conjugation operators work on dual complex numbers [11] in the following manner: if $w = z_1 + \varepsilon z_2 = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2$, $z_2 \neq 0$, then

$$\begin{aligned}
w^{\star 1} &= (x_1 - iy_1) + \varepsilon(x_2 - iy_2) = z_1^\star + \varepsilon z_2^\star && \text{(complex conjugation)},\\
w^{\star 2} &= (x_1 + iy_1) - \varepsilon(x_2 + iy_2) = z_1 - \varepsilon z_2 && \text{(dual conjugation)},\\
w^{\star 3} &= (x_1 - iy_1) - \varepsilon(x_2 - iy_2) = z_1^\star - \varepsilon z_2^\star && \text{(coupled conjugation)},\\
w^{\star 4} &= (x_1 - iy_1)\Big(1 - \frac{\varepsilon(x_2 + iy_2)}{(x_1 + iy_1)}\Big) = z_1^\star\Big(1 - \frac{\varepsilon z_2}{z_1}\Big) && \text{(dual complex conjugation)},\\
w^{\star 5} &= (x_2 + iy_2) - \varepsilon(x_1 + iy_1) = z_2 - \varepsilon z_1 && \text{(antidual conjugation)}.
\end{aligned}$$

The norms of dual complex numbers are defined in the following manner:

$$\begin{aligned}
N_{w^{\star 1}} &= \|w \times w^{\star 1}\| = \sqrt{|z_1^2| + 2\varepsilon \operatorname{Re}(z_1 z_2^\star)},\\
N_{w^{\star 2}} &= \|w \times w^{\star 2}\| = \sqrt{z_1^2},\\
N_{w^{\star 3}} &= \|w \times w^{\star 3}\| = \sqrt{|z_1^2| - 2i\varepsilon \operatorname{Im}(z_1 z_2^\star)},\\
N_{w^{\star 4}} &= \|w \times w^{\star 4}\| = \sqrt{|z_1^2|},\\
N_{w^{\star 5}} &= \|w \times w^{\star 5}\| = \sqrt{z_1 z_2 + \varepsilon(z_2^2 - z_1^2)}.
\end{aligned}$$

In 2017, Güngör and Azak [8] defined the *dual complex Fibonacci numbers* as follows:

$$DF_n^\star = F_n + iF_{n+1} + \varepsilon(F_{n+2} + iF_{n+3}),$$

where the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the conditions

$$i^2 = -1, \quad \varepsilon \neq 0, \quad \varepsilon^2 = 0, \quad (i\varepsilon)^2 = 0.$$

In 2018, Aydın [1] defined the *dual complex k-Fibonacci numbers* as follows:

$$DCF_{k,n} = F_{k,n} + iF_{k,n+1} + \varepsilon(F_{k,n+2} + iF_{k,n+3}),$$

where the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the above condition.

In 2019, Prasad [12] introduced *complex Fibonacci p-numbers* by the following recurrence relation:

$$F_p^\star(n) = F_p^\star(n-1) + F_p^\star(n-p-1)$$

with $n > p+1$ and with initial terms

$$F_p^\star(0) = i, \quad F_p^\star(1) = F_p^\star(2) = \cdots = F_p^\star(p) = 1 + i.$$

In 2021, Prasad [13] introduced *dual complex Fibonacci p-numbers* by the following recurrence relation:

$$\begin{aligned} DF_p^\star(n) &= F_p^\star(n-1) + \varepsilon F_p^\star(n-p-1) \\ &= F_p(n-1) + iF_p(n) + \varepsilon(F_p(n-p-1) + iF_p(n-p)) \end{aligned}$$

with $n > p+1$ and with initial terms

$$F_p^\star(0) = i, \quad F_p^\star(1) = F_p^\star(2) = \cdots = F_p^\star(p) = 1 + i,$$

and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfying the above condition.

In 2022, Prasad [15] introduced *m-extension dual complex Fibonacci p-numbers* by the recurrence relation

$$\begin{aligned} DF_{p,m}^\star(n) &= mF_{p,m}^\star(n-1) + \varepsilon F_{p,m}^\star(n-p-1) \\ &= m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p)) \end{aligned} \tag{1.1}$$

with $n > p+1$ and with initial terms

$$F_{p,m}^\star(n) = m^{n-1}, \quad n = 1, \ldots, p+1,$$

and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfying the same condition.

In this paper, we introduce (m,t)-extension dual complex Fibonacci p-numbers (see Definition 2.1 below) and establish some of their properties.

2. (m,t)-extension dual complex Fibonacci p-numbers

DEFINITION 2.1. *(m,t)-extension dual complex Fibonacci p-numbers* are given by the recurrence relation

$$\begin{aligned} DF_{p,m,t}^\star(n) &= mF_{p,m,t}^\star(n-1) + t\varepsilon F_{p,m,t}^\star(n-p-1) \\ &= m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + t\varepsilon(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) \end{aligned} \tag{2.1}$$

with $n > p+1$ and with initial terms

$$F_{p,m,t}^\star(1) = c_1, \quad F_{p,m,t}^\star(2) = c_2, \ \ldots, \ F_{p,m,t}^\star(p+1) = c_{p+1},$$

where p (≥ 0) is an integer, m (>0), t (>0), $n > p+1$ and $c_1, \ldots, c_{p+1}$ are arbitrary real or complex numbers and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition $i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $(i\varepsilon)^2 = 0$.

Definition 2.2. Addition and subtraction of (m,t)-extension dual complex Fibonacci p-numbers are defined by

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n_1) \pm DF^{\star}_{p,m,t}(n_2)\\
&\quad = mF^{\star}_{p,m,t}(n_1-1) + t\varepsilon F^{\star}_{p,m,t}(n_1-p-1)\\
&\qquad \pm (mF^{\star}_{p,m,t}(n_2-1) + t\varepsilon F^{\star}_{p,m,t}(n_2-p-1))\\
&\quad = m(F_{p,m,t}(n_1-1) + iF_{p,m,t}(n_1))\\
&\qquad + t\varepsilon(F_{p,m,t}(n_1-p-1) + iF_{p,m,t}(n_1-p))\\
&\qquad \pm (m(F_{p,m,t}(n_2-1) + iF_{p,m,t}(n_2))\\
&\qquad + t\varepsilon(F_{p,m,t}(n_2-p-1) + iF_{p,m,t}(n_2-p)))\\
&\quad = m(F_{p,m,t}(n_1-1) \pm F_{p,m,t}(n_2-1)) + im(F_{p,m,t}(n_1) \pm F_{p,m,t}(n_2))\\
&\qquad + t\varepsilon(F_{p,m,t}(n_1-p-1) \pm F_{p,m,t}(n_2-p-1))\\
&\qquad + it\varepsilon(F_{p,m,t}(n_1-p) \pm F_{p,m,t}(n_2-p)).
\end{aligned}$$

Definition 2.3. Multiplication of (m,t)-extension dual complex Fibonacci p-numbers by real scalars λ is defined by

$$\begin{aligned}
&\lambda DF^{\star}_{p,m,t}(n)\\
&= \lambda mF_{p,m,t}(n-1) + i\lambda mF_{p,m,t}(n) + \varepsilon\lambda tF_{p,m,t}(n-p-1) + i\varepsilon\lambda tF_{p,m,t}(n-p).
\end{aligned}$$

Definition 2.4. Multiplication of (m,t)-extension dual complex Fibonacci p-numbers is defined by

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n_1)DF^{\star}_{p,m,t}(n_2)\\
&= [mF_{p,m,t}(n_1-1) + imF_{p,m,t}(n_1) + t\varepsilon F_{p,m,t}(n_1-p-1) + it\varepsilon F_{p,m,t}(n_1-p)]\\
&\quad \times [mF_{p,m,t}(n_2-1) + imF_{p,m,t}(n_2) + t\varepsilon F_{p,m,t}(n_2-p-1) + it\varepsilon F_{p,m,t}(n_2-p)]\\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-1) - F_{p,m,t}(n_1)F_{p,m,t}(n_2)]\\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)]\\
&\quad + m\varepsilon t[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2-1)\\
&\qquad\qquad - F_{p,m,t}(n_1)F_{p,m,t}(n_2-p) - F_{p,m,t}(n_2)F_{p,m,t}(n_1-p)]\\
&\quad + im\varepsilon t[F_{p,m,m}(n_1-1)F_{p,m,m}(n_2-p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1)\\
&\quad + F_{p,m,t}(n_2)F_{p,m,t}(n_1-p-1) + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1)]\\
&= DF^{\star}_{p,m,t}(n_2)DF^{\star}_{p,m,t}(n_1).
\end{aligned}$$

Since five kinds of conjugation can be defined for dual complex numbers [11], conjugation of (m,t)-extension dual complex Fibonacci p-numbers can be defined in five different ways:

$$DF^{\star 1}_{p,m,t}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p)) \quad \text{(complex conjugation)}, \tag{2.2}$$

$$DF^{\star 2}_{p,m,t}(n) = m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) - \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) \quad \text{(dual conjugation)}, \tag{2.3}$$

$$DF^{\star 3}_{p,m,t}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) - \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p)) \quad \text{(coupled conjugation)}, \tag{2.4}$$

$$DF^{\star 4}_{p,m,t}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) \times \left(1 - \varepsilon t \frac{F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1) + iF_{p,m,t}(n))}\right) \quad \text{(dual complex conjugation)}, \tag{2.5}$$

$$DF^{\star 5}_{p,m,t}(n) = m(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) - \varepsilon t(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) \quad \text{(antidual conjugation)}. \tag{2.6}$$

Some properties of these conjugations are given in the following theorems.

THEOREM 2.1. *We have the following relations:*

$$DF^{\star}_{p,m,t}(n)DF^{\star 1}_{p,m,t}(n) = m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)] + 2m\varepsilon t[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + F_{p,m,t}(n)F_{p,m,t}(n-p)], \tag{2.7}$$

$$DF^{\star}_{p,m,t}(n)DF^{\star 2}_{p,m,t}(n) = m^2[F^2_{p,m,t}(n-1) - F^2_{p,m,t}(n)] + i2m^2F_{p,m,t}(n-1)F_{p,m,t}(n), \tag{2.8}$$

$$DF^{\star}_{p,m,t}(n)DF^{\star 3}_{p,m,t}(n) = m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)] + i2m\varepsilon t[F_{p,m,t}(n-p)F_{p,m,t}(n-1) - F_{p,m,t}(n)F_{p,m,t}(n-p-1)], \tag{2.9}$$

$$DF^{\star}_{p,m,t}(n)DF^{\star 4}_{p,m,t}(n) = m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)], \tag{2.10}$$

$$DF^{\star}_{p,m,t}(n)DF^{\star 5}_{p,m,t}(n) = m^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) - F_{p,m,t}(n)F_{p,m,t}(n-p)] + im^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p) + F_{p,m,t}(n)F_{p,m,t}(n-p-1)] + m\varepsilon t[F^2_{p,m,t}(n-p-1) + F^2_{p,m,t}(n) - F^2_{p,m,t}(n-1) - F^2_{p,m,t}(n-p)] + i2m\varepsilon t[F_{p,m,t}(n-p-1)F_{p,m,t}(n-p) - F_{p,m,t}(n-1)F_{p,m,t}(n)], \tag{2.11}$$

$$\text{(2.12)}\quad DF^{\star}_{p,m,t}(n)+DF^{\star 1}_{p,m,t}(n) = 2[mF_{p,m,t}(n-1)+\varepsilon tF_{p,m,t}(n-p-1)],$$

$$\text{(2.13)}\quad DF^{\star}_{p,m,t}(n) + DF^{\star 2}_{p,m,t}(n) = 2m[F_{p,m,t}(n-1) + iF_{p,m,t}(n)],$$

$$\text{(2.14)}\quad DF^{\star}_{p,m,t}(n) + DF^{\star 3}_{p,m,t}(n) = 2[mF_{p,m,t}(n-1) + i\varepsilon tF_{p,m,t}(n-p)],$$

$$\text{(2.15)}\quad (F_{p,m,t}(n-1) + iF_{p,m,t}(n))DF^{\star 4}_{p,m,t}(n) = [F_{p,m,t}(n-1) - iF_{p,m,t}(n)]DF^{\star 2}_{p,m,t}(n),$$

$$\text{(2.16)}\quad \varepsilon tDF^{\star}_{p,m,t}(n)+mDF^{\star 5}_{p,m,t}(n)=m^2[F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)],$$

$$\text{(2.17)}\quad mDF^{\star}_{p,m,t}(n) - \varepsilon tDF^{\star 5}_{p,m,t}(n) = m^2[F_{p,m,t}(n-1) + iF_{p,m,t}(n)],$$

$$\text{(2.18)}\quad DF^{\star}_{p,m,t}(n) + DF^{\star}_{p,m,t}(n+p) = DF^{\star}_{p,m,t}(n+p+1).$$

Proof. We will prove the above relations one by one. By (2.1) and (2.2), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)DF^{\star 1}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad\times [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))]\\
&\quad= m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)]\\
&\qquad+ 2m\varepsilon t[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + F_{p,m,t}(n)F_{p,m,t}(n-p)].
\end{aligned}$$

By (2.1) and (2.3), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)DF^{\star 2}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad\times [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\quad= m^2[F^2_{p,m,t}(n-1) - F^2_{p,m,t}(n)] + i2m^2F_{p,m,t}(n-1)F_{p,m,t}(n).
\end{aligned}$$

By (2.1) and (2.4), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)DF^{\star 3}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad\times [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))]\\
&\quad= m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)]\\
&\qquad+ i2m\varepsilon t[F_{p,m,t}(n-p)F_{p,m,t}(n-1) - F_{p,m,t}(n)F_{p,m,t}(n-p-1)].
\end{aligned}$$

By (2.1) and (2.5), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)DF^{\star 4}_{p,m,t}(n)\\
&= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\quad\times \left[m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))\left(1-\varepsilon t\frac{F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))}\right)\right]\\
&= m^2[F^2_{p,m,t}(n-1) + F^2_{p,m,t}(n)].
\end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)DF^{\star 5}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad\times [m(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))-\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))]\\
&\quad= m^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)-F_{p,m,t}(n)F_{p,m,t}(n-p)]\\
&\qquad+ im^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p)+F_{p,m,t}(n)F_{p,m,t}(n-p-1)]\\
&\qquad+ m\varepsilon t[F^2_{p,m,t}(n-p-1)+F^2_{p,m,t}(n)-F^2_{p,m,t}(n-1)-F^2_{p,m,t}(n-p)]\\
&\qquad+ i2m\varepsilon t[F_{p,m,t}(n-p-1)F_{p,m,t}(n-p)-F_{p,m,t}(n-1)F_{p,m,t}(n)].
\end{aligned}$$

By (2.1) and (2.2), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)+DF^{\star 1}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad+ [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))]\\
&\quad= 2[mF_{p,m,t}(n-1)+\varepsilon tF_{p,m,t}(n-p-1)].
\end{aligned}$$

By (2.1) and (2.3), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)+DF^{\star 2}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad+ [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad= 2m[F_{p,m,t}(n-1)+iF_{p,m,t}(n)].
\end{aligned}$$

By (2.1) and (2.4), we get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n)+DF^{\star 3}_{p,m,t}(n)\\
&\quad= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\qquad+ [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))]\\
&\quad= 2[mF_{p,m,t}(n-1)+i\varepsilon tF_{p,m,t}(n-p)].
\end{aligned}$$

By (2.3) and (2.5), we get

$$\begin{aligned}
&(F_{p,m,t}(n-1)+iF_{p,m,t}(n))DF^{\star 4}_{p,m,t}(n)\\
&\ = (F_{p,m,t}(n-1)+iF_{p,m,t}(n))\\
&\quad\times \left[m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))\left(1-\varepsilon t\frac{F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))}\right)\right]\\
&\ = [F_{p,m,t}(n-1)-iF_{p,m,t}(n)]\\
&\quad\times [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\ = [F_{p,m,t}(n-1)-iF_{p,m,t}(n)]DF^{\star 2}_{p,m,t}(n).
\end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned}
&\varepsilon tDF^{\star}_{p,m,t}(n)+mDF^{\star 5}_{p,m,t}(n)\\
&\quad= m\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+m^2(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))\\
&\qquad- m\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))\\
&\quad= m^2[F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)].
\end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned}
mDF^{\star}_{p,m,t}(n) &- \varepsilon t DF^{\star 5}_{p,m,t}(n)\\
&= m[m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]\\
&\quad - \varepsilon t[m(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))-\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))]\\
&= m^2[F_{p,m,t}(n-1)+iF_{p,m,t}(n)].
\end{aligned}$$

We know that $DF^{\star}_{p,m,t}(n) = DF^{\star}_{p,m,t}(n-1) + DF^{\star}_{p,m,t}(n-p-1)$. Putting $n = n+p+1$, we get

$$DF^{\star}_{p,m,t}(n+p+1) = DF^{\star}_{p,m,t}(n+p) + DF^{\star}_{p,m,t}(n) = DF^{\star}_{p,m,t}(n) + DF^{\star}_{p,m,t}(n+p).$$

Therefore, the norm of (m,t)-extension dual complex Fibonacci p-numbers can be defined in five different ways as follows:

$$\begin{aligned}
NDF^{\star 1}_{p,m,t}(n) &= \|DF^{\star}_{p,m,t}(n)DF^{\star 1}_{p,m,t}(n)\|\\
&= m^2(F^2_{p,m,t}(n-1)+F^2_{p,m,t}(n))\\
&\quad + 2m\varepsilon t(F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)+F_{p,m,t}(n)F_{p,m,t}(n-p))
\end{aligned}$$

by (2.1) and (2.2);

$$\begin{aligned}
NDF^{\star 2}_{p,m,t}(n) &= \|DF^{\star}_{p,m,t}(n)DF^{\star 2}_{p,m,t}(n)\|\\
&= [(m^2(F^2_{p,m,t}(n-1)-F^2_{p,m,t}(n)))^2+4m^4F^2_{p,m,t}(n)F^2_{p,m,t}(n-1)]^{1/2}\\
&= m^2(F^2_{p,m,t}(n-1)+F^2_{p,m,t}(n))
\end{aligned}$$

by (2.1) and (2.3);

$$NDF^{\star 3}_{p,m,t}(n) = \|DF^{\star}_{p,m,t}(n)DF^{\star 3}_{p,m,t}(n)\| = m^2(F^2_{p,m,t}(n-1)+F^2_{p,m,t}(n))$$

by (2.1) and (2.4);

$$NDF^{\star 4}_{p,m,t}(n) = \|DF^{\star}_{p,m,t}(n)DF^{\star 4}_{p,m,t}(n)\| = m^2(F^2_{p,m,t}(n-1)+F^2_{p,m,t}(n))$$

by (2.1) and (2.5);

$$\begin{aligned}
(NDF^{\star 5}_{p,m,t}(n))^2 &= (\|DF^{\star}_{p,m,t}(n)DF^{\star 5}_{p,m,t}(n)\|)^2\\
&= m^4[F^2_{p,m,t}(n-1)F^2_{p,m,t}(n-p-1)+F^2_{p,m,t}(n)F^2_{p,m,t}(n-p)\\
&\qquad + F^2_{p,m,t}(n-1)F^2_{p,m,t}(n-p)+F^2_{p,m,t}(n)F^2_{p,m,t}(n-p-1)]\\
&\quad + 2m^3\varepsilon t[F_{p,m,t}(n-1)F^3_{p,m,t}(n-p-1)-F^3_{p,m,t}(n-1)F_{p,m,t}(n-p-1)\\
&\quad + F_{p,m,t}(n)F^3_{p,m,t}(n-p)-F^3_{p,m,t}(n)F_{p,m,t}(n-p)\\
&\quad + F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)F^2_{p,m,t}(n-p)\\
&\quad - F^2_{p,m,t}(n)F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)\\
&\quad + F_{p,m,t}(n)F_{p,m,t}(n-p)F^2_{p,m,t}(n-p-1)\\
&\quad - F^2_{p,m,t}(n-1)F_{p,m,t}(n)F_{p,m,t}(n-p)]
\end{aligned}$$

by (2.1) and (2.6). ∎

THEOREM 2.2. *Let $n_1, n_2 \geq 0$. The Honsberger identity for (m,t)-extension dual complex Fibonacci p-numbers is given by*

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n_1)DF^{\star}_{p,m,t}(n_2) + DF^{\star}_{p,m,t}(n_1+1)DF^{\star}_{p,m,t}(n_2+1)\\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2+1)]\\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)\\
&\qquad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)]\\
&\quad + \varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2-1)\\
&\qquad - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p+1) - F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2+1)]\\
&\quad + i\varepsilon mt[F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p)\\
&\qquad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)\\
&\qquad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1)\\
&\qquad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1) + F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)].
\end{aligned}$$

Proof. We get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n_1)DF^{\star}_{p,m,t}(n_2) + DF^{\star}_{p,m,t}(n_1+1)DF^{\star}_{p,m,t}(n_2+1)\\
&= [m(F_{p,m,t}(n_1-1)+iF_{p,m,t}(n_1))+\varepsilon tF_{p,m,t}(n_1-p-1)+i\varepsilon tF_{p,m,t}(n_1-p)]\\
&\quad \times [m(F_{p,m,t}(n_2-1)+iF_{p,m,t}(n_2))+\varepsilon tF_{p,m,t}(n_2-p-1)+i\varepsilon tF_{p,m,t}(n_2-p)]\\
&\quad + [m(F_{p,m,t}(n_1)+iF_{p,m,t}(n_1+1))+\varepsilon tF_{p,m,t}(n_1-p)+i\varepsilon tF_{p,m,t}(n_1-p+1)]\\
&\quad \times [m(F_{p,m,t}(n_2)+iF_{p,m,t}(n_2+1))+\varepsilon tF_{p,m,t}(n_2-p)+i\varepsilon tF_{p,m,t}(n_2-p+1)]\\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2+1)]\\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)\\
&\qquad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)]\\
&\quad + \varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2-1)\\
&\qquad - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p+1) - F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2+1)]\\
&\quad + i\varepsilon mt[F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p)\\
&\qquad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)\\
&\qquad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1)\\
&\qquad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1) + F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)]
\end{aligned}$$

in view of (2.1) and the properties of (m,t)-extension Fibonacci p-numbers. ∎

THEOREM 2.3. *Let $n_1, n_2 \geq 0$. D'Ocagne's identity for (m,t)-extension dual complex Fibonacci p-numbers is given by*

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n_1)DF^{\star}_{p,m,t}(n_2+1) - DF^{\star}_{p,m,t}(n_1+1)DF^{\star}_{p,m,t}(n_2)\\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)\\
&\qquad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)]\\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2+1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-1)]
\end{aligned}$$

$$+\varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p)-F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1)$$
$$+F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)-F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1)$$
$$-F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1)+F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p)$$
$$-F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1)+F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)]$$
$$+i\varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p+1)+F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2+1)$$
$$-F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p-1)-F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2-1)].$$

Proof. We get

$$DF^{\star}_{p,m,t}(n_1)DF^{\star}_{p,m,t}(n_2+1)-DF^{\star}_{p,m,t}(n_1+1)DF^{\star}_{p,m,t}(n_2)$$
$$=[m(F_{p,m,t}(n_1-1)+iF_{p,m,t}(n_1))+\varepsilon tF_{p,m,t}(n_1-p-1)+i\varepsilon tF_{p,m,t}(n_1-p)]$$
$$\times[m(F_{p,m,t}(n_2)+iF_{p,m,t}(n_2+1))+\varepsilon tF_{p,m,t}(n_2-p)+i\varepsilon tF_{p,m,t}(n_2-p+1)]$$
$$-[m(F_{p,m,t}(n_1)+iF_{p,m,t}(n_1+1))+\varepsilon tF_{p,m,t}(n_1-p)+i\varepsilon tF_{p,m,t}(n_1-p+1)]$$
$$\times[m(F_{p,m,t}(n_2-1)+iF_{p,m,t}(n_2))+\varepsilon tF_{p,m,t}(n_2-p-1)+i\varepsilon tF_{p,m}(n_2-p)]$$
$$=m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2)-F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)$$
$$+F_{p,m,t}(n_1+1)F_{p,m,t}(n_2)-F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)]$$
$$+im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2+1)-F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-1)]$$
$$+\varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p)-F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1)$$
$$+F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)-F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1)$$
$$-F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1)+F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p)$$
$$-F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1)+F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)]$$
$$+i\varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p+1)+F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2+1)$$
$$-F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p-1)-F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2-1)]$$

in view of (2.1) and the properties of (m,t)-extension Fibonacci p-numbers. ∎

Theorem 2.4. *For $n\geq 1$, Cassini's identity for $DF^{\star}_{p,m,t}(n)$ is given by*

$$DF^{\star}_{p,m,t}(n-1)DF^{\star}_{p,m,t}(n+1)-D^2F^{\star}_{p,m,t}(n)$$
$$=m^2[F^2_{p,m,t}(n)-F^2_{p,m,t}(n-1)+F_{p,m,t}(n-2)F_{p,m,t}(n)$$
$$-F_{p,m,t}(n+1)F_{p,m,t}(n-1)]$$
$$+im^2[F_{p,m,t}(n+1)F_{p,m,t}(n-2)-F_{p,m,t}(n)F_{p,m,t}(n-1)]$$
$$+\varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p)-F_{p,m,t}(n-1)F_{p,m,t}(n-p+1)$$
$$+F_{p,m,t}(n)F_{p,m,t}(n-p-2)-F_{p,m,t}(n-p-1)F_{p,m,t}(n+1)$$
$$-2F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)+2F_{p,m,t}(n)F_{p,m,t}(n-p)]$$
$$+i\varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p+1)-F_{p,m,t}(n-1)F_{p,m,t}(n-p)$$
$$+F_{p,m,t}(n-p-2)F_{p,m,t}(n+1)-F_{p,m,t}(n-p-1)F_{p,m,t}(n)].$$

Proof. We get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n-1)DF^{\star}_{p,m,t}(n+1)-D^2F^{\star}_{p,m,t}(n)\\
&=[m(F_{p,m,t}(n-2)+iF_{p,m,t}(n-1))+\varepsilon t(F_{p,m,t}(n-p-2)+iF_{p,m,t}(n-p-1))]\\
&\quad\times[m(F_{p,m,t}(n)+iF_{p,m,t}(n+1))+\varepsilon t(F_{p,m,t}(n-p)+iF_{p,m,t}(n-p+1))]\\
&\quad-[m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))]^2\\
&=m^2[F^2_{p,m,t}(n)-F^2_{p,m,t}(n-1)+F_{p,m,t}(n-2)F_{p,m,t}(n)\\
&\qquad-F_{p,m,t}(n+1)F_{p,m,t}(n-1)]\\
&\quad+im^2[F_{p,m,t}(n+1)F_{p,m,t}(n-2)-F_{p,m,t}(n)F_{p,m,t}(n-1)]\\
&\quad+\varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p)-F_{p,m,t}(n-1)F_{p,m,t}(n-p+1)\\
&\qquad+F_{p,m,t}(n)F_{p,m,t}(n-p-2)-F_{p,m,t}(n-p-1)F_{p,m,t}(n+1)\\
&\qquad-2F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)+2F_{p,m,t}(n)F_{p,m,t}(n-p)]\\
&\quad+i\varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p+1)-F_{p,m,t}(n-1)F_{p,m,t}(n-p)\\
&\qquad+F_{p,m,t}(n-p-2)F_{p,m,t}(n+1)-F_{p,m,t}(n-p-1)F_{p,m,t}(n)]
\end{aligned}$$

in view of (2.1) and the properties of (m,t)-extension Fibonacci p-numbers. ∎

THEOREM 2.5. *For $n\geq 1$, Catalan's identity for $DF^{\star}_{p,m,t}(n+r)$ is given by*

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n+r-1)DF^{\star}_{p,m,t}(n+r+1)-D^2F^{\star}_{p,m,t}(n+r)\\
&=m^2[F^2_{p,m,t}(n+r)-F^2_{p,m,t}(n+r-1)+F_{p,m,t}(n+r-2)F_{p,m,t}(n+r)\\
&\qquad-F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-1)]\\
&\quad+im^2[F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-2)-F_{p,m,t}(n+r)F_{p,m,t}(n+r-1)]\\
&\quad+\varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p)-F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p+1)\\
&\qquad+F_{p,m,t}(n+r)F_{p,m,t}(n+r-p-2)-F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r+1)\\
&\qquad-2F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p-1)+2F_{p,m,t}(n+r)F_{p,m,t}(n+r-p)]\\
&\quad+i\varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p+1)-F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p)\\
&\qquad+F_{p,m,t}(n+r-p-2)F_{p,m,t}(n+r+1)-F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r)].
\end{aligned}$$

Proof. We get

$$\begin{aligned}
&DF^{\star}_{p,m,t}(n+r-1)DF^{\star}_{p,m,t}(n+r+1)-D^2F^{\star}_{p,m,t}(n+r)\\
&=[m(F_{p,m,t}(n+r-2)+iF_{p,m,t}(n+r-1))\\
&\qquad+\varepsilon t(F_{p,m,t}(n+r-p-2)+iF_{p,m,t}(n+r-p-1))]\\
&\quad\times[m(F_{p,m,t}(n+r)+iF_{p,m,t}(n+r+1))\\
&\qquad+\varepsilon t(F_{p,m,t}(n+r-p)+iF_{p,m,t}(n+r-p+1))]\\
&\quad-[m(F_{p,m,t}(n+r-1)+iF_{p,m,t}(n+r))\\
&\qquad+\varepsilon t(F_{p,m,t}(n+r-p-1)+iF_{p,m,t}(n+r-p))]^2
\end{aligned}$$

$$= m^2[F^2_{p,m,t}(n+r) - F^2_{p,m,t}(n+r-1) + F_{p,m,t}(n+r-2)F_{p,m,t}(n+r) - F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-1)]$$
$$+ im^2[F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-2) - F_{p,m,t}(n+r)F_{p,m,t}(n+r-1)]$$
$$+ \varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p+1) + F_{p,m,t}(n+r)F_{p,m,t}(n+r-p-2) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r+1) - 2F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p-1) + 2F_{p,m,t}(n+r)F_{p,m,t}(n+r-p)]$$
$$+ i\varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p+1) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p) + F_{p,m,t}(n+r-p-2)F_{p,m,t}(n+r+1) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r)],$$

in view of (2.1) and properties of (m, t)-extension Fibonacci p-numbers. ■

3. Conclusion. In this paper, we introduced (m, t)-extension dual complex Fibonacci p-numbers and established some of their properties, including the Honsberger identity, D'Ocagne's identity, Cassini's identity and Catalan's identity. I hope that these results will be useful in applied mathematics, quantum mechanics, quantum physics, Lie groups, number theory, kinematics and differential equations, just as are dual complex k-Fibonacci numbers [1], dual complex Fibonacci p-numbers [13], dual-complex numbers and their holomorphic functions [11] and dual-complex Jacobsthal quaternions [2] etc.

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