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An abstract version of Sierpiński's theorem and the algebra generated by A and CA functions

by

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Abstract. We give an abstract version of Sierpiński's theorem which says that the closure in the uniform convergence topology of the algebra spanned by the sums of lower and upper semicontinuous functions is the class of all Baire 1 functions. Later we show that a natural generalization of Sierpiński's result for the uniform closure of the space of all sums of **A** and **CA** functions is not true. Namely we show that the uniform closure of the space of all sums of **A** and **CA** functions is a proper subclass of the space of all functions measurable with respect to the least class containing intersections of analytic and coanalytic sets and which is closed under countable unions (**A** and **CA** functions are analogues of lower and upper semicontinuous functions, respectively, when measurability with respect to open sets is replaced by that with respect to analytic sets).

Let $\mathbb{N}$ denote the set of all natural numbers, $\mathbb{Z}$ the set of all integers and $\mathbb{R}$ the set of all real numbers. Let also $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. $\mathcal{C}$ will stand for the classical Cantor set $2^{\mathbb{N}}$.

The family of all subsets of a set X will be denoted by $P(X)$. For $\mathcal{A} \subseteq P(X)$ by $\mathcal{A}_\sigma$, $\mathcal{A}_\delta$, $\mathcal{A}_s$, $\mathcal{A}_d$, $\mathcal{A}^c$ we denote, respectively, the families of all countable unions, countable intersections, finite unions, finite intersections and complements with respect to X of elements of $\mathcal{A}$. Let $r(\mathcal{A})$ stand for the algebra of sets generated by $\mathcal{A}$. A family $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_{ds}$ is a *σ-class* if $\mathcal{A}_\sigma = \mathcal{A}$, and a *δ-class* if $\mathcal{A}_\delta = \mathcal{A}$. For $\mathcal{A} \subset P(Y)$ and $X \subseteq Y$ we define $\mathcal{A}|X = \{S \cap X : S \in \mathcal{A}\}$. Let now X and Y be any sets. For $A \subseteq X \times Y$ and $x \in X$ let $A_x = \{y \in Y : (x, y) \in A\}$. If $\mathcal{A}$ is a family of subsets of Y a set $A \subseteq X \times Y$ is called a *universal set* for $\mathcal{A}$ if $\mathcal{A} = \{A_x : x \in X\}$.

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Let X be any Polish space. We use standard notation of Descriptive Set Theory: $\Sigma^0_\alpha(X)$ ($\Pi^0_\alpha(X)$) denotes the αth additive (multiplicative, resp.) class in the hierarchy of Borel sets, and $\Sigma^1_n(X)$ stands for the nth projective class of sets on X.

Assume now that for each Polish space Z we have defined a certain family $\mathcal{A}(Z)$ of subsets of Z. By $\mathcal{A}$ we shall denote the collection of these families. We say that $\mathcal{A}$ is *closed under continuous substitution* if for any Polish spaces X and Y and for every continuous function f from X into Y the set $f^{-1}(A)$ belongs to $\mathcal{A}(X)$ if A belongs to $\mathcal{A}(Y)$. We shall call $\mathcal{A}$ a *hereditary σ-class* (*δ-class*, resp.) if $\mathcal{A}$ is closed under continuous substitution and if for each Polish space Z the following two conditions are satisfied:

(I) $\mathcal{A}(Z)$ is a σ-class (δ-class, resp.);

(II) $\mathcal{A}(Z)|X = \mathcal{A}(X)$ for each closed $X \subseteq Z$.

Obviously Σ^0_α (Π^0_α), $\alpha < \omega_1$, serve as examples of hereditary σ-classes (δ-classes, resp.) and Σ^1_n is a hereditary δ- and σ-class for $n \in \mathbb{N}$.

For any family of functions $\mathcal{F} \subseteq {}^X\mathbb{R}$ let $\mathrm{cl}(\mathcal{F})$ denote the closure of $\mathcal{F}$ in the uniform convergence topology on ${}^X\mathbb{R}$. If $\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$ then by $\mathcal{F} + \mathcal{G}$ we denote the family $\{f + g : f \in \mathcal{F},\ g \in \mathcal{G}\}$. Let $\mathcal{A} \subseteq P(X)$. Let $\underline{M}(\mathcal{A}) = \{f \in {}^X\mathbb{R}^* : f^{-1}(a, \infty] \in \mathcal{A}$ for every $a \in \mathbb{R}^*\}$. Similarly let $\overline{M}(\mathcal{A}) = \{f \in {}^X\mathbb{R}^* : -f \in \underline{M}(\mathcal{A})\}$. Let also $M(\mathcal{A}) = \underline{M}(\mathcal{A}) \cap \overline{M}(\mathcal{A})$. We shall also use the following families of functions: $\underline{M}(\mathcal{A}, L) = \underline{M}(\mathcal{A}) \cap {}^XL$, $\overline{M}(\mathcal{A}, L) = \overline{M}(\mathcal{A}) \cap {}^XL$, $M(\mathcal{A}, L) = M(\mathcal{A}) \cap {}^XL$, where L is any subset of $\mathbb{R}^*$. $\underline{M}(\Sigma^1_1(X), \mathbb{R})$ and $\overline{M}(\Sigma^1_1(X), \mathbb{R})$ are called in [4] the families of **A** and **CA** functions on X, respectively. Let $\mathcal{F}$ be a family of functions included in ${}^Y\mathbb{R}^*$. A function $F \in {}^{X\times Y}\mathbb{R}^*$ is called *universal* for $\mathcal{F}$ if $\mathcal{F} \subseteq \{F_x : x \in X\}$, where F_x denotes the one variable function $F(x, \cdot)$.

We prove the following abstract version of Sierpiński's theorem.

Theorem 1. *If $\mathcal{A}$ is a σ-class of subsets of X and $(r(\mathcal{A}))_\sigma = (\mathcal{A}^c)_\sigma$ (or, equivalently, $\mathcal{A} \subseteq (\mathcal{A}^c)_\sigma$) then*

$$M((r(\mathcal{A}))_\sigma, \mathbb{R}) = \mathrm{cl}(\underline{M}(\mathcal{A}, \mathbb{R}) + \overline{M}(\mathcal{A}, \mathbb{R}))\,.$$

Proof. Let $\mathcal{A}_1 = (r(\mathcal{A}))_\sigma$ and let $f \in M(\mathcal{A}_1, \mathbb{R})$. Let $n \in \mathbb{N}$, $i \in \mathbb{Z}$ and

$$B^n_i = f^{-1}\left(\left(\frac{i-1}{2^n}, \frac{i+1}{2^n}\right)\right).$$

As $\bigcup\{B^n_i : i \in \mathbb{Z}\} = X$ and the class $\mathcal{A}_1$ has the σ-reduction property ([5, Theorem 4.5.1]) there exist disjoint sets C^n_i such that $C^n_i \subseteq B^n_i$, $C^n_i \in \mathcal{A}_1$, $i \in \mathbb{Z}$, and $\bigcup\{C^n_i : i \in \mathbb{Z}\} = X$, for every $n \in \mathbb{N}$.

Every set C_i^n can be expressed as the union

$$C_i^n = \bigcup_{j=1}^{\infty} D_{i,j}^n$$

of sets $D_{i,1}^n \subseteq D_{i,2}^n \subseteq \ldots$ belonging to $\mathcal{A}^c$. For convenience we define $D_{i,0}^n = \emptyset$ for every $i \in \mathbb{Z}$ and $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we define two functions φ_n and ψ_n as follows:

$$\varphi_n(x) = \begin{cases} \dfrac{2|i|+j}{2^n}, & x \in D_{i,j+1}^n \setminus D_{i,j}^n,\ i \geq 0, \\ \dfrac{|i|+j}{2^n}, & x \in D_{i,j+1}^n \setminus D_{i,j}^n,\ i < 0, \end{cases}$$

$$\psi_n(x) = \begin{cases} \dfrac{|i|+j}{2^n}, & x \in D_{i,j+1}^n \setminus D_{i,j}^n,\ i \geq 0, \\ \dfrac{2|i|+j}{2^n}, & x \in D_{i,j+1}^n \setminus D_{i,j}^n,\ i < 0. \end{cases}$$

It is not difficult to check that both functions are in $\underline{M}(\mathcal{A}, \mathbb{R})$ and

$$|f - (\varphi_n - \psi_n)| < \frac{1}{2^n}\,.$$

Thus we have proved that

$$M((r(\mathcal{A}))_\sigma, \mathbb{R}) \subseteq \mathrm{cl}(\underline{M}(\mathcal{A}, \mathbb{R}) + \overline{M}(\mathcal{A}, \mathbb{R}))\,.$$

The reverse inclusion is straightforward since $M((r(\mathcal{A}))_\sigma, \mathbb{R})$ is closed under the uniform convergence. ■

Let us observe that all Σ_α^0 classes satisfy the hypothesis of the above theorem. As a corollary we obtain, for instance, the following classical theorem of Sierpiński (the more general result in [3, IX, 41, VI] also follows from Theorem 1).

COROLLARY (Sierpiński [7]). *Any function of Baire class* 1 *is the uniform limit of a sequence of differences of lower semicontinuous functions.*

In this paper we show that the analogous result for the class Σ_1^1 does not hold.

THEOREM 2. *Let $\mathcal{A}$ be a hereditary σ- and δ-class, let Z be any uncountable Polish space and let $\mathcal{A}(Z)$ have a universal set in $\mathcal{A}(\mathcal{C} \times Z)$. Then there exists a function $h \in M(r(\mathcal{A}(X))_\sigma, \mathbb{R})$ such that there is no countable partition $\{Z_n : n \in \mathbb{N}\}$ of Z such that for each $n \in \mathbb{N}$ the function $h|Z_n$ is in*

$$\mathrm{cl}(\underline{M}(\mathcal{A}|Z_n, \mathbb{R}) + \overline{M}(\mathcal{A}|Z_n, \mathbb{R}))\,.$$

Proof. Let Ψ be any bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$. For any $c\in\mathcal{C}$ let $(c)_n$ be defined by $(c)_n(k)=c(\Psi(n,k))$. Note that for every sequence $\{c_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ there exists $c\in\mathcal{C}$ such that $(c)_n=c_n$ for every $n\in\mathbb{N}$.

The assumption about $\mathcal{A}$ implies (see e.g. [1, the proof of Theorem 2.1]) that there exists a function $F\in\underline{M}(\mathcal{A}(\mathcal{C}\times Z),\mathbb{R}^*)$ which is universal for $\underline{M}(\mathcal{A}(Z),\mathbb{R}^*)$. Let

$$T=\{(x,y)\in\mathcal{C}\times Z: F(x,y)\in\mathbb{R}\}\,.$$

It can be easily verified that $T\in r(\mathcal{A}(\mathcal{C}\times Z))$. Let us define F^* to be equal to F on T and to zero elsewhere.

It is easy to check that

$$F^*\in M(r(\mathcal{A}(\mathcal{C}\times Z)),\mathbb{R})$$

and that F^* is universal for $\underline{M}(\mathcal{A}(Z),\mathbb{R})$. Next let us define

$$H(c,x)=F^*((c)_0,x)-F^*((c)_1,x)\,.$$

Then $H\in M((r(\mathcal{A}(\mathcal{C}\times Z)))_\sigma,\mathbb{R})$ and H is universal for $\underline{M}(\mathcal{A}(Z),\mathbb{R})+\overline{M}(\mathcal{A}(Z),\mathbb{R})$. Let us put

$$\varphi(x)=\begin{cases}1, & x>1,\\ x, & -1\le x\le 1,\\ -1, & x<-1,\end{cases}$$

and $\widetilde{H}=\varphi\circ H$. Finally, we put

$$H^*(c,x)=H((c)_0,x)+\sum_{n=1}^{\infty}\frac{\widetilde{H}((c)_n,x)}{2^n}\,.$$

It is almost trivial to see that H^* is universal for

$$\mathrm{cl}(\underline{M}(\mathcal{A}(Z),\mathbb{R})+\overline{M}(\mathcal{A}(Z),\mathbb{R}))$$

and $H^*\in M((r(\mathcal{A}(\mathcal{C}\times Z)))_\sigma,\mathbb{R})$.

Suppose now $X\subseteq Z$ and $f\in\mathrm{cl}(\underline{M}(\mathcal{A}(Z)|X,\mathbb{R})+\overline{M}(\mathcal{A}(Z)|X,\mathbb{R}))$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\underline{M}(\mathcal{A}(Z)|X)+\overline{M}(\mathcal{A}(Z)|X)$ such that

$$\sum_{n=0}^{\infty}f_n=f\quad\text{and}\quad\sup\{|f_n(z)|:z\in Z\}|\le\frac{1}{2^n}$$

for every $n\ge 1$. Let $d_0,g_0\in\underline{M}(\mathcal{A}(Z)|X,\mathbb{R})$ be such that $f_0=d_0-g_0$. Then there are d_0^* and g_0^* in $\underline{M}(\mathcal{A}(Z),\mathbb{R}^*)$ (see [1, Proposition 1.1]) such that $d_0\subseteq d_0^*$ and $g_0\subseteq g_0^*$. Let $c_0\in\mathcal{C}$ be such that $F^*_{(c_0)_0}=d_0^*$ and $F^*_{(c_0)_1}=g_0^*$. Then $f_0\subseteq H_{c_0}$. Let now $n\ge 1$. We choose d_n, g_n in $\underline{M}(\mathcal{A}(Z)|X,\mathbb{R})$ such that

$$f_n=\frac{1}{2^n}(d_n-g_n)\,.$$

As before we find d_n^* and g_n^* in $\underline{M}(\mathcal{A}(Z),\mathbb{R}^*)$ such that $d_n \subseteq d_n^*$ and $g_n \subseteq g_n^*$ and we find c_n such that $2^n f_n \subseteq \widetilde{H}_{c_n}$. Hence we have shown that for every $X \subseteq Z$ and

$$f \in \mathrm{cl}(\underline{M}(\mathcal{A}(Z)|X) + \overline{M}(\mathcal{A}(Z)|X))$$

there exists $c \in \mathcal{C}$ such that $H_c^* \supseteq f$. Let $\widetilde{\mathcal{C}} \subseteq Z$ be any subset of Z homeomorphic to $\mathcal{C}$. Let $\phi : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a homeomorphism. For $c \in \widetilde{\mathcal{C}}$ we put $G_n(c) = H^*((\phi(c))_n, c)$. Then $\{G_n\}_{n\in\mathbb{N}}$ is a countable family of functions from $M((r(\mathcal{A}(\widetilde{\mathcal{C}}))_\sigma,\mathbb{R})$. Hence (see [2, Corollary 2.2]) there exists a function $h \in M((r(\mathcal{A}(\widetilde{\mathcal{C}}))_\sigma,\mathbb{R})$ such that

$$\bigcup_{n\in\mathbb{N}} G_n \cap h = \emptyset\,.$$

By [6, Corollary 2F] there exists $\widetilde{h} \in M((r(\mathcal{A}(Z)))_\sigma,\mathbb{R})$ which is an extension of h to the whole Z.

We claim that $\widetilde{h}$ is the required function. Suppose that $\{Z_n : n \in \mathbb{N}\}$ is any partition of Z such that

$$h|Z_n \in \mathrm{cl}(\underline{M}(\mathcal{A}(Z)|Z_n,\mathbb{R}) + \overline{M}(\mathcal{A}(Z)|Z_n,\mathbb{R}))$$

for each $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ let $c_n \in \mathcal{C}$ be such that the function $H_{c_n}^*$ extends $h|Z_n$. Let us choose $c \in \widetilde{\mathcal{C}}$ such that $(\forall n \in \mathbb{N})(c_n = (\phi(c))_n)$. Let $k \in \mathbb{N}$ be such that $c \in Z_k$. Then $h(c) = (h|Z_k)(c) = H_{c_k}^*(c) = G_k(c)$, hence $h(c) \in \{G_n(c) : n \in \mathbb{N}\}$, which gives a contradiction with the choice of the function h. ■

It is clear that we can directly apply Theorem 2 to any uncountable Polish space Z and the class of all $\Sigma_1^1(Z)$ sets or, more generally, to the classes $\Sigma_n^1(Z)$ for every natural number $n \geq 1$. Thus Theorem 2 strengthens [6, Corollary 3.23].

Let us recall that any Σ_2^1 set is a union of ω_1 Borel sets. Hence any function from the class $M((r(\Sigma_1^1(Z)))_\sigma,\mathbb{R})$ can be decomposed into ω_1 Borel functions. This observation and Theorem 2 imply that (for terminology see [2])

$$\mathrm{dec}(M((r(\Sigma_1^1(Z)))_\sigma,\mathbb{R}), R(\mathrm{cl}(\underline{M}(\Sigma_1^1(Z),\mathbb{R}) + \overline{M}(\Sigma_1^1(Z),\mathbb{R})))) = \omega_1\,.$$

Now we consider an abstract Borel hierarchy of sets starting from some σ-class $\mathcal{A} = \mathcal{A}_0$. For $0 < \alpha < \omega_1$, $\mathcal{A}_\alpha$ is defined inductively:

$$\mathcal{A}_\alpha = \Big(r\Big(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}\Big)\Big)_\sigma\,.$$

As one can see from Theorem 2 the conclusion of Sierpiński's Theorem 1 may not hold for $\mathcal{A}_0$. However, as it easily turns out, it is true for all higher classes $\mathcal{A}_\alpha$ of the hierarchy. Namely, the following holds:

FACT. $M(\mathcal{A}_{\alpha+1},\mathbb{R}) = \mathrm{cl}(\underline{M}(\mathcal{A}_\alpha,\mathbb{R}) + \overline{M}(\mathcal{A}_\alpha,\mathbb{R}))$ *if* $\alpha \geq 1$.

Proof. To see this it is enough to notice that $\mathcal{A}_{\alpha+1} = (r(\mathcal{A}_\alpha))_\sigma = (\mathcal{A}^c_\alpha)_\sigma$. Indeed, $\mathcal{A}^c_\alpha = ((r(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}))_\sigma)^c = (r(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}))_\delta$, whence $(\mathcal{A}^c_\alpha)_\sigma = (r(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}))_{\delta\sigma}$. Thus $\mathcal{A}_\alpha \subseteq (\mathcal{A}^c_\alpha)_\sigma$. ∎

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