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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE p-LAMÉ DIRICHLET PROBLEM BY TOPOLOGICAL DEGREE

Abstract. We consider a mathematical model named the generalized Lamé system (p-Lamé), which describes the displacement u from the natural state of a nonhomogeneous elastic solid subjected to a volume density of forces f that depends on the displacement u in a domain Ω of $\mathbb{R}^N$. Using the topological degree theory for a class of demicontinuous operators of generalized (S_+) type, we prove the existence and uniqueness of the weak solution.

1. Introduction. Let Ω be a connected open bounded domain of $\mathbb{R}^N$ with Lipschitz boundary Γ which is composed of two disjoint parts Γ_1 and Γ_2 such that $\mathrm{mes}(\Gamma_1) > 0$. We first consider the general mathematical model of the elasticity system

$$-L_p u + F(u) = 0,$$

with

$$L_p u = \mu \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + (\lambda + \mu)\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i}\right), \qquad 1 \le j \le N,$$

and $F(u)$ is the perturbation. Here p, q are real numbers with $1/p + 1/q = 1$ and $p \in \,]1, +\infty[$, and λ and μ are the Lamé coefficients such that $\lambda > 0$ and $\lambda + \mu \ge 0$ (μ is the shear modulus, also called the second Lamé coefficient). In particular, the Lamé coefficients are expressed as functions of the Young

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modulus E and the Poisson ratio ν:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

Given a function $f = (f_1, \dots, f_N)$ and a square matrix $\varphi = (\varphi_{i,j})_{1\le i,j\le N}$ such that

(i) $\varphi_{i,j} = \varphi_{j,i} \in C^{0,1}(\overline{\Omega})$ (the space of Hölder functions),
(ii) $\varphi_{i,j} > 0$ for all $x \in \Gamma_2$,

the problem is to find a function $u = (u_1, \dots, u_N)$ solving the nonlinear elliptic equation of the form

$$\begin{cases} -L_p u + F(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \sigma(u)\cdot\eta + \varphi(x)(u) = 0 & \text{on } \Gamma_2. \end{cases}$$

Here σ is the stress tensor defined by Hook's law

$$\sigma_{ij} = \lambda(\operatorname{div} u)\delta_{ij} + 2\mu\varepsilon(u), \quad 1 \le i,j \le N,$$

where $\varepsilon(u)$ is the deformation tensor, whose components are

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i,j \le N,$$

and $\eta = (\eta_1, \dots, \eta_N)$ is the outward unit vector normal to the boundary Γ of Ω.

From this model, we can consider several boundary value problems depending on the perturbation $F(u)$ and the nature of the boundary conditions imposed on Γ. More precisely, $F(u)$ can be, for example,

- $F(u) = 0$,
- $F(u) = |u|^{p-2}u, p > 2$,
- $F(u) = u^3$ and $F(u) = a(x,t)u^2$, where a is a function that depends on x and t.

As for the boundary conditions, we distinguish several cases:

(i) When $\Gamma_2 = \emptyset$, the problem becomes a Dirichlet problem.
(ii) When $\Gamma_1 = \emptyset$ and $\varphi(x) = 0$ on Γ_2, the problem becomes a Neumann problem. Of course in this case, one supposes that the necessary condition is satisfied, namely orthogonality of the data to the rigid displacements:

$$\int_\Omega f\cdot v\,dx = \int_\Gamma 0\cdot v\,ds = 0 \qquad \text{for any function } v \text{ of the form}$$

$$v(x,y) = \begin{pmatrix} a+cy \\ b-cx \end{pmatrix},$$

where a, b, c are any real numbers and $\Omega \subset \mathbb{R}^2$. From the mechanical point of view, this condition makes sense even if the volumetric and surface forces are equivalent to zero.

(iii) When $\Gamma_1 \neq \emptyset$ and $\varphi(x) \neq 0$ on Γ_2, the problem becomes a mixed problem.

For $p = 2$, we find the classical Lamé system (elasticity) which has already been studied by several authors. In [6, 7, 8], Merouani has studied the singularity of the solutions to some boundary values problems for the Lamé system in a polygonal domain in the case where $F(u) = 0$ and $\varphi(x) = 0$. Further, for a class of Sobolev spaces, Benseridi [2] studied some transmission problems related to this problem in a polyhedron. In [10], Said has shown an existence result for some problems governed by the Lamé operator multiplied by a weight function $\lambda(r) \in C^{0,\infty}(\overline{\Omega}_\psi)$, where Ω_ψ is the infinite plane sector with angle ψ defined by

$$\Omega_\psi = \{(r, \theta) : r > 0 \text{ and } 0 < \theta < \psi < 2\pi\}.$$

He found that the solution of these problems depends on the angles of the polygon; on the other hand, if we perturb the Lamé operator by a suitable weight function, the solution does not depend on the angles. In [9], using the Galerkin method, the authors have studied, in a bounded domain of $\mathbb{R}^N$, the existence, uniqueness and regularity of solution of the nonlinear problem

$$\begin{cases} -L_p u + |u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \sigma(u) \cdot \eta + \varphi(x)u = 0 & \text{on } \Gamma_2, \end{cases}$$

where the right hand side f is independent of u. Recently, in [12], Zoubai and Merouani have studied the mixed problem for a nonlinear elasticity system by the compactness method using Schauder's topological degree with f depending on u.

The main purpose of this paper is to prove the existence and uniqueness of the solution for the following nonlinear problem, by using the topological degree theory for a class of demicontinuous operators of generalized (S_+) type used in [1, 4]:

$$(1.1) \qquad \begin{cases} -\mu \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) & \\ \qquad - (\lambda + \mu)\dfrac{\partial}{\partial x_j}\left(\displaystyle\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i}\right) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases}$$

for all $1 \leq j \leq N$, where Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary Γ and f is a Carathéodory function satisfying some non-standard growth condition. The paper is organized as follows:

In Section 2, we use Green's formula to obtain a variational formulation of problem (1.1) in a Sobolev space. In Section 3, we introduce some classes of operators of generalized (S_+) type and the extended topological degree theory of Leray–Schauder. In the fourth section, we present the essential properties of the p-Lamé operator. In the fifth section, under certain hypotheses, we prove the existence of a weak solution via the technique of topological degree. Finally, Section 6 is devoted to the uniqueness of solution of problem (1.1).

2. Weak formulation. Firstly, we assume that a solution u of problem (1.1) exists and belongs to $(W^{2,p}(\Omega))^N$.

We take the space $V = [W_0^{1,p}(\Omega)]^N$, where $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for } 1 \le i \le N \right\},$$

equipped with the norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^p,$$

where

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

Note that the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space, and the norm $\|u\|_{1,p}$ on $W_0^{1,p}(\Omega)$ is equivalent to the norm $\|\cdot\|$ given by

$$\|u\| = \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{1/p} \qquad \text{for } u \in W_0^{1,p}(\Omega),$$

and the embedding $I : W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Multiplying the first equation in (1.1) by a test function $v \in V$, then integrating on Ω, we obtain

$$-\mu \sum_{i=1}^{N} \int_\Omega \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \cdot v \, dx - (\lambda + \mu) \int_\Omega \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i} \right) \cdot v \, dx = \int_\Omega f(x,u) \cdot v \, dx,$$

which gives, thanks to Green's formula and the fact that $v \in V$,

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx + (\lambda + \mu) \int_{\Omega} \operatorname{div} u \operatorname{div} v \, dx = \int_{\Omega} f(x,u) v \, dx, \qquad \forall v \in V.$$

3. Some classes of operators and topological degree. In this section, we need to recall some definitions and lemmas which are needed for our result.

Let X and Y be real Banach spaces and Σ a nonempty subset of X. The symbol $\to$ $[\rightharpoonup]$ stands for strong [weak] convergence.

An operator $F : X \supset \Sigma \to Y$ is said to be

- *bounded* if it takes any bounded set into a bounded set;
- *demicontinuous* if for any sequence $(u_n) \subset \Sigma$, $u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$;
- *compact* if it is continuous and the image of any bounded set is relatively compact.

Let X be a real reflexive Banach space with dual X^*. An operator $F : X \supset \Sigma \to X^*$ is said to be

- *monotone* if $\langle F(u) - F(v), u - v \rangle \geq 0$ for all $u, v \in D_F$, where D_F is the domain of F;
- *strongly monotone* if there is a constant $C > 0$ such that
$$\langle F(u) - F(v), u - v \rangle \geq C \|u - v\|^2, \qquad \forall u, v \in D_F;$$
- *of class* (S_+) if for any sequence $(u_n) \subset \Sigma$ with $u_n \rightharpoonup u$ and
$$\limsup \langle F(u_n), u_n - u \rangle \leq 0,$$
we have $u_n \to u$ in X;
- *quasimonotone* if for any sequence $(u_n) \subset \Sigma$ with $u_n \rightharpoonup u$, we have
$$\limsup \langle F(u_n), u_n - u \rangle \geq 0.$$

For any operator $F : X \supset \Sigma \to X$ and any operator $T : X \supset \Sigma_1 \to X^*$ such that $\Sigma \subset \Sigma_1$, we say that F

- *satisfies condition* $(S_+)_T$ if for any sequence $(u_n) \subset \Sigma$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y \in X^*$ and $\limsup \langle F(u_n), u_n - u \rangle \leq 0$, we have $u_n \to u$ in X;
- F *has property* $(QM)_T$ if for any sequence $(u_n) \subset \Sigma$ with $u_n \rightharpoonup u$ and $y_n := Tu_n \rightharpoonup y \in X^*$, we have $\limsup \langle F(u_n), u_n - u \rangle \geq 0$.

Definition 3.1. We say that T is *coercive* if

$$\lim_{\|u\|_X \to +\infty} \frac{\langle Tu, u \rangle_{X^*,X}}{\|u\|_X} = +\infty.$$

Now, we consider the following classes of operators. Let D_F be the domain of F. For any $\Sigma \subset D_F$ and $T \in \mathcal{F}_1(\Sigma)$, set

$$\mathcal{F}_1(\Sigma) := \{F : X \supset \Sigma \to X^* \mid F \text{ is bounded, demicontinuous and satisfies } (S_+)\},$$

$$\mathcal{F}_T(\Sigma) := \{F : \Sigma \to X \mid F \text{ is demicontinuous and satisfies } (S_+)_T\}.$$

Let $\mathcal{O}$ be the collection of all bounded open sets in X. Define

$$\mathcal{F}(X) := \{F \in \mathcal{F}_T(\overline{G}) \mid G \in \mathcal{O},\, T \in \mathcal{F}_1(\overline{G})\}.$$

Here T is called an *essential inner map* of F.

The following lemma shows that the Hammerstein operator of the form $I + S \circ T$ belongs to the class $\mathcal{F}(X)$.

LEMMA 3.2 ([4]). *Let G be a bounded open set in a real reflexive Banach space X. Suppose that $T \in \mathcal{F}_1(\overline{G})$ is continuous and $S : X^* \supset D_S \to X$ is demicontinuous with $T(\overline{G}) \subset D_S$.*

(i) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*
(ii) *If S satisfies condition (S_+), then $S \circ T \in \mathcal{F}_T(\overline{G})$.*

DEFINITION 3.3. Let G be a bounded open set in a real reflexive Banach space X, let $T \in \mathcal{F}_1(\overline{G})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{G})$. The affine homotopy $H : [0,1] \times \overline{G} \to X$ defined by

$$H(t,u) = (1-t)Fu + tSu \quad \text{for } (t,u) \in [0,1] \times \overline{G}$$

is called an *admissible affine homotopy* and satisfies the $(S_+)_T$ condition with the common continuous essential inner map T.

In what follows, as in [4] we introduce a suitable topological degree for the class $\mathcal{F}(X)$ that extends the degree theory of Berkovits [3] to all demicontinuous operators satisfying condition $(S_+)_T$.

THEOREM 3.4. *Let*

$$M = \{(F,G,h) \mid G \in \mathcal{O},\, T \in \mathcal{F}_1(\overline{G}),\, F \in \mathcal{F}_T(\overline{G}),\, h \notin F(\partial G)\}.$$

There exists a unique degree function $d : M \to \mathbb{Z}$ that satisfies the following classical properties:

(i) (Existence) *If $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
(ii) (Additivity) *Let $F \in \mathcal{F}_T(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then*

$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$

(iii) (Homotopy invariance) *Suppose that* $H : [0,1]\times\overline{G} \to X$ *is a continuous path in* X *such that* $h(t) \notin H(t, \partial G)$ *for all* $t \in [0,1]$, *the value of* $d(H(t,\cdot), G, h(t))$ *is constant for all* $t \in [0,1]$.

(iv) (Normalization) *For any* $h \in G$, *we have*

$$d(I, G, h) = 1.$$

(v) (Boundary dependence) *If* $F, S \in \mathcal{F}_T(\overline{G})$ *coincide on* $\partial\Omega$ *and* $h \in F(\partial G)$ *then*

$$d(F, G, h) = d(S, G, h).$$

4. Properties of the p-Lamé operator. In this section, we discuss the nonlinear operator

$$L_p u = \mu \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + (\lambda+\mu)\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i}\right), \qquad 1 \le j \le N.$$

The mapping defined on $(L^p(\Omega))^N$ by

$$u \mapsto |u|^{p-2}u$$

has values in $(L^q(\Omega))^N$, is continuous and measurable.

In fact, for $1/p + 1/q = 1$,

$$\int_\Omega \big||u|^{p-2}u\big|^q \, dx = \int_\Omega |u|^p \, dx < \infty.$$

So, for all $u \in (W^{1,p}(\Omega))^N$, $\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i} \in (L^q(\Omega))^N$ for $1 \le i \le N$.

Hence we can define the following numerical map on $(W_0^{1,p}(\Omega))^2$:

$$(u,v) \mapsto a(u,v) = \mu \sum_{i,j=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} \, dx + (\lambda+\mu)\int_\Omega \operatorname{div} u \operatorname{div} v \, dx.$$

Since for any $u \in (W_0^{1,p}(\Omega))^N$, the mapping $v \mapsto a(u,v)$ is continuous linear from $(W_0^{1,p}(\Omega))^N$ into $\mathbb{R}$, there exists a unique $J(u) \in (W^{-1,q}(\Omega))^N$ such that

$$a(u,v) = \langle J(u), v\rangle, \qquad \forall v \in (W_0^{1,p}(\Omega)^N.$$

The mapping $(W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N, u \mapsto J(u)$, denoted by

$$-L_p u = -\mu \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - (\lambda+\mu)\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i}\right), \qquad 1 \le j \le N,$$

is called the *p-Lamé operator*.

THEOREM 4.1. *The operator* $J : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ *is*

(i) *bounded, continuous and strongly monotone,*

(ii) *coercive,*

(iii) *of class* (S_+),
(iv) *a homeomorphism.*

Proof. (i) We first show that J is bounded. We have

$$\|J(u)\|_{V'} = \sup_{\|v\|_{1,p}=1} |\langle J(u), v\rangle|, \qquad \forall u \in V = (W_0^{1,p}(\Omega))^N.$$

According to Hölder's inequality,

$$\begin{aligned}
|\langle J(u), v\rangle| &= \Big|\mu \sum_{i,j=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u_j}{\partial x_i}\, \frac{\partial v_j}{\partial x_i}\, dx + (\lambda+\mu)\int_\Omega \operatorname{div} u \, \operatorname{div} v \, dx\Big| \\
&\le \mu \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \Big(\sum_{i,j=1}^{N} \Big|\frac{\partial u_j}{\partial x_i}\Big|^2\Big)^{1/2} \Big(\sum_{i,j=1}^{N} \Big|\frac{\partial v_j}{\partial x_i}\Big|^2\Big)^{1/2} dx \\
&\quad + (\lambda+\mu)\int_\Omega |\operatorname{div} u|\, |\operatorname{div} v|\, dx \\
&\le \mu \int_\Omega |\nabla u|^{p-1} |\nabla v|\, dx + (\lambda+\mu)\int_\Omega |\operatorname{div} u|\, |\operatorname{div} v|\, dx. \\
&\le \mu \Big(\int_\Omega |\nabla u|^{q(p-1)}\, dx\Big)^{1/q} \Big(\int_\Omega |\nabla v|^p\, dx\Big)^{1/p} \\
&\quad + (\lambda+\mu)\Big(\int_\Omega |\nabla u|^p\, dx\Big)^{1/p} \Big(\int_\Omega (|\nabla v|^q\, dx\Big)^{1/q}.
\end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $\frac{1}{q} = \frac{p-1}{p}$, so that

$$\begin{aligned}
|\langle J(u), v\rangle| &\le \mu \Big(\int_\Omega |\nabla u|^p\, dx\Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\nabla v|^p\, dx\Big)^{1/p} \\
&\quad + (\lambda+\mu)\|\nabla u\|_{L^p(\Omega)} \|\nabla v\|_{L^q(\Omega)} \\
&\le \mu \|\nabla u\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(\Omega)} + C(\lambda+\mu)\|\nabla u\|_{L^p(\Omega)} \|\nabla v\|_{L^p(\Omega)},
\end{aligned}$$

where we have used the fact that

$$\|\nabla v\|_{L^q(\Omega)} \le C\|\nabla v\|_{L^p(\Omega)} \qquad \text{for } p > q.$$

So

$$|\langle J(u), v\rangle| \le \mu \|u\|_V^{p-1} \|v\|_V + C(\lambda+\mu)\|u\|_V \|v\|_V.$$

Let U be a bounded subset of V. Then

$$\exists M > 0, \forall u \in U, \qquad \|u\| \le M.$$

Hence

$$\|J(u)\|_{V'} \le \text{const} \cdot M.$$

Consequently, J is bounded.

We check now that the operator J is continuous. To do this, we consider a sequence $(u_n)_n \subset V$ such that $u_n \to u$ in V. Then

$$
\begin{aligned}
\|J(u_n) - J(u)\|_{V'} &= \sup_{\|v\|_{1,p}=1} |\langle J(u_n) - J(u), v\rangle| \\
&= \sup_{\|v\|_{1,p}=1} \bigg| \mu \int_\Omega \sum_{i,j=1}^N \bigg(\left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} \frac{\partial u_{n,j}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \bigg) \frac{\partial v_j}{\partial x_i}\, dx \\
&\qquad + (\lambda+\mu) \int_\Omega (\operatorname{div} u_n - \operatorname{div} u) \operatorname{div} v\, dx \bigg| \\
&\le \sup_{\|v\|_{1,p}=1} \mu \bigg(\sum_{i,j=1}^N \int_\Omega \bigg| \left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} \frac{\partial u_{n,j}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \bigg|^q dx \bigg)^{1/q} \\
&\qquad \times \bigg(\sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^p dx \bigg)^{1/p} \\
&\quad + (\lambda+\mu) \sup_{\|v\|_{1,p}=1} \bigg(\int_\Omega |\operatorname{div} u_n - \operatorname{div} u|^q\, dx \bigg)^{1/q} \bigg(\int_\Omega |\operatorname{div} v|^p\, dx \bigg)^{1/p} \\
&\le \sup_{\|v\|_{1,p}=1} \mu \bigg(\sum_{i,j=1}^N \int_\Omega \bigg| \left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} \frac{\partial u_{n,j}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \bigg|^q dx \bigg)^{1/q} \left\| \frac{\partial v}{\partial x_i} \right\|_p \\
&\quad + (\lambda+\mu) \sup_{\|v\|_{1,p}=1} \bigg(\int_\Omega |\operatorname{div} u_n - \operatorname{div} u|^q\, dx \bigg)^{1/q} \left\| \frac{\partial v}{\partial x_i} \right\|_p \\
&\le \mu C_{\mathrm{emb}} \bigg(\sum_{i,j=1}^N \int_\Omega \bigg| \left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} \frac{\partial u_{n,j}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \bigg|^q dx \bigg)^{1/q} \\
&\quad + (\lambda+\mu) C_{\mathrm{emb}} \bigg(\int_\Omega |\operatorname{div} u_n - \operatorname{div} u|^q\, dx \bigg)^{1/q},
\end{aligned}
$$

which tends to zero as $n \to \infty$, where C_{emb} is the constant of the embedding of $(W_0^{1,p}(\Omega))^N$ into $(L^p(\Omega))^N$, because the map $u \mapsto |u|^{p-2}u$ is continuous in $(L^q(\Omega))^N$. Thus $J(u_n) \to J(u)$ in V', and so J is continuous.

To show that J is strongly monotone, let $u, v \in V$. We have

$$
\begin{aligned}
\langle J(u) - J(v), u - v\rangle_{V'\times V} &= \mu \sum_{i,j=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \bigg(\frac{\partial u_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i} \bigg) dx \\
&\quad - \mu \sum_{i,j=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v_j}{\partial x_i} \bigg(\frac{\partial u_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i} \bigg) dx \\
&\quad + (\lambda+\mu) \int_\Omega \operatorname{div}(u-v) \operatorname{div}(u-v)\, dx.
\end{aligned}
$$

Using the following elementary inequality of Lindqvist [5]:

$$|x-y|^p \le 2^p(|x|^{p-2}x - |y|^{p-2}y)(x-y) \qquad \text{if } p \ge 2, \text{for all } x, y \in \mathbb{R}^N,$$

we obtain

$$\begin{aligned}\langle J(u) - J(v), u - v\rangle_{V'\times V} &\\ &\ge \frac{\mu}{2^p}\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right|^p dx + (\lambda+\mu)\int_{\Omega}(\operatorname{div}(u-v))^2\,dx\\ &\ge \frac{\mu}{2^p}\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right|^p dx.\end{aligned}$$

Then

$$\langle J(u) - J(v), u - v\rangle_{V'\times V} \ge \frac{\mu}{2^p}\|\nabla u - \nabla v\|_p^p = \frac{\mu}{2^p}\|u-v\|_V^p,$$

so J is strongly monotone.

(ii) Now, let us show that J is coercive:

$$\begin{aligned}\frac{\langle J(u), u\rangle_{V',V}}{\|u\|_V} &= \frac{1}{\|u\|_V}\left[\mu\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\left|\frac{\partial u}{\partial x_i}\right|^2 dx + (\lambda+\mu)\int_{\Omega}(\operatorname{div} u)^2\,dx\right]\\ &\ge \frac{1}{\|u\|_V}\mu\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^p dx = \frac{\mu}{\|u\|_V}\|u\|_V^p.\end{aligned}$$

This implies that

$$\lim_{\|u\|_V\to\infty}\frac{\langle J(u), u\rangle_{V',V}}{\|u\|_V} \ge \infty.$$

Then J is coercive. Thus J is a surjection.

(iii) Let us verify that condition (S_+) holds. If $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty}\langle J(u_n) - J(u), u_n - u\rangle \le 0,$$

then

$$\lim_{n\to\infty}\langle J(u_n) - J(u), u_n - u\rangle = 0,$$

and as J is strongly monotone according to (i), we have

$$\langle J(u_n) - J(u), u_n - u\rangle \ge C\|u_n - u\|_V^p,$$

which implies that

$$\lim_{n\to\infty}\|u_n - u\|_V = 0. \tag{4.1}$$

From (4.1), $u_n \to u$, i.e. J is of type (S_+).

(iv) By strong monotonicity, J is an injection. From (ii),

$$\lim_{\|u\|_V\to\infty}\frac{\langle J(u), u\rangle_{V',V}}{\|u\|_V} = \infty,$$

Since J is coercive, it is a surjection in view of the Minty–Browder Theorem (see [11]). Hence J has an inverse mapping

$$J^{-1} : (W^{-1,q}(\Omega))^N \to (W_0^{1,p}(\Omega))^N.$$

Therefore, the continuity of J^{-1} is sufficient to ensure that J is a homeomorphism.

If $f_n, f \in (W^{-1,q}(\Omega))^N$ with $f_n \to f$, let

$$u_n = J^{-1}(f_n), \qquad u = J^{-1}(f).$$

Then

$$f_n = J(u_n), \qquad f = J(u).$$

So $(u_n)_n$ is bounded in $(W_0^{1,p}(\Omega))^N$; without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $f_n \to f$, we have

$$\lim_{n\to\infty} \langle J(u_n) - J(u_0), u_n - u_0\rangle = \lim_{n\to\infty} \langle f_n, u_n - u_0\rangle = 0. \tag{4.2}$$

Since J is of type (S_+) and $u_n \rightharpoonup u_0$, we deduce that $u_n \to u_0$, so J^{-1} is continuous. Consequently, J is a homeomorphism. ■

5. Existence of solutions. In this section, we present a result on existence of weak solutions of problem (1.1) based on the topological degree theory of Section 3. But first we recall a definition.

DEFINITION 5.1 (Carathéodory function). Let $N, p, q \in \mathbb{N}^*$, and let Ω be an open subset of $\mathbb{R}^N$. We say that $f : \Omega \times \mathbb{R}^p \to \mathbb{R}^q$ is a *Carathéodory function* if $f(x, \cdot)$ is continuous on $\mathbb{R}^p$ for almost all $x \in \Omega$ and $f(\cdot, s)$ is measurable on Ω for all $s \in \mathbb{R}^p$.

Assume that Ω is a bounded domain with Lipschitz boundary Γ and $f : \Omega \times \mathbb{R}^p \to \mathbb{R}^N$ is a function such that

(H_1) f satisfies the Carathéodory condition,

(H_2) f satisfies the growth condition

$$|f(x, s)| \le c(d(x) + |s|^{p'-1})$$

for almost all $x \in \Omega$ and $s \in \mathbb{R}^p$, where c is a positive constant, $1 < p' < p$ and $d \in (L^q(\Omega))^N$.

Note that the embedding $I : (W_0^{1,p}(\Omega))^N \to (L^{p^*}(\Omega))^N$ is continuous, where

$$p^* = \frac{Np}{N-p},$$

and the embedding $I : (W_0^{1,p}(\Omega))^N \to (L^p(\Omega))^N$ is compact.

Definition 5.2. $u \in (W_0^{1,p}(\Omega))^N$ is called a *weak solution* of equation (1.1) if

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx + (\lambda+\mu)\int_{\Omega} \operatorname{div} u \operatorname{div} v\, dx = \int_{\Omega} f(x,u)v\, dx, \quad \forall v \in (W_0^{1,p}(\Omega))^N.$$

Lemma 5.3. *Under assumptions* $(H_1), (H_2)$, *the operator* $S : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ *defined by*

$$\langle S(u), v\rangle = -\int_{\Omega} f(x,u)v\, dx \qquad \textit{for } u, v \in (W_0^{1,p}(\Omega))^N$$

is compact.

Proof. Let $\phi : (W_0^{1,p}(\Omega))^N \to (L^q(\Omega))^N$ be the operator defined by

$$\phi(u)(x) = -f(x,u(x)) \qquad \text{for } u \in (W_0^{1,p}(\Omega))^N \text{ and } x \in \Omega.$$

First, we show that ϕ is bounded and continuous. For each $u \in (W_0^{1,p}(\Omega))^N$, using the growth condition (H_2), we get

$$\|\phi(u)\|_q = \Big(\int_{\Omega} |f(x,u(x))|^q\, dx\Big)^{1/q} \le c\Big(\int_{\Omega} |(d(x) + |u(x)|^{p'-1})|^q\, dx\Big)^{1/q} \le c(\|d\|_q + \|u\|_{q(p'-1)}^{p'-1}).$$

As the embedding $(L^p(\Omega))^N \hookrightarrow (L^r(\Omega))^N$, where $r = q(p'-1) < p$, is continuous, i.e.

$$\|u\|_r \le C_{\mathrm{emb}}\|u\|_p \quad \text{ for all } u \in (L^p(\Omega)^N,$$

we deduce that

$$\|\phi(u)\|_q \le C_{\mathrm{emb}}(\|d\|_q + \|u\|_p^{p'-1}).$$

This implies that ϕ is bounded on $(W_0^{1,p}(\Omega))^N$.

Now, we will show that ϕ is continuous. For this, let $(u_n) \subset (W_0^{1,p}(\Omega))^N$ be a sequence such that $u_n \to u$ in $(W_0^{1,p}(\Omega))^N$. Then

$$u_n \to u \text{ in } (L^p(\Omega))^N \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ in } (L^p(\Omega))^N \text{ for } 1 \le i \le N.$$

So, there exist a subsequence (u_j) and measurable functions L, K_i in $(L^p(\Omega))^N$ for $1 \le i \le N$ such that

$$u_j(x) \to u(x) \quad \text{and} \quad \frac{\partial u_j}{\partial x_i}(x) \to \frac{\partial u}{\partial x_i}(x),$$
$$|u_j(x)| \le L(x) \quad \text{and} \quad \left|\frac{\partial u_j}{\partial x_i}(x)\right| \le K_i(x),$$

for almost every $x \in \Omega$ and all $j \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain

$$f(x, u_j(x)) \to f(x, u(x)) \quad \text{for almost all } x \in \Omega,$$

and according to condition (H_2) and $v \in (L^p(\Omega))^N \subset (L^{q(p'-1)})^N$,

$$|f(x, u_j(x))| \le c(d(x) + |u_j(x)|^{p'-1}) \le c(d(x) + |L(x)|^{p'-1})$$

for almost every $x \in \Omega$ and all $j \in \mathbb{N}$, since

$$(d(x) + |u_j(x)|^{p'-1}) \in (L^q(\Omega))^N.$$

Thus, applying Lebesgue's dominated convergence theorem, we find

$$\|\phi(u_j) - \phi(u)\|_q^q = \int_\Omega |f(x, u_j(x)) - f(x, u(x))|^q \, dx \to 0$$

as $n \to \infty$. So, $\phi(u_j) \to \phi(u)$ in $(L^q(\Omega))^N$, and we deduce that ϕ is continuous.

Since the embedding $I : (W_0^{1,p}(\Omega))^N \to (L^p(\Omega))^N$ is compact, the adjoint operator $I^* : (L^q(\Omega))^N \to (W^{-1,q}(\Omega))^N$ is also compact. Therefore, the composition $I^* \circ \phi : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ is compact, which gives the result. ∎

The following main existence theorem establishes our main result in this paper by using topological degree theory [1, 4].

Theorem 5.4. *Under assumptions* $(H_1), (H_2)$, *problem* (1.1) *has a weak solution* $u \in (W_0^{1,p}(\Omega))^N$.

Proof. Let $S : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ be as in Lemma 5.3 and $J : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ as in Theorem 4.1, defined by

$$\langle J(u), v\rangle = \mu \sum_{i,j=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx + (\lambda + \mu) \int_\Omega \operatorname{div} u \operatorname{div} v \, dx$$

for all $u, v \in (W_0^{1,p}(\Omega))^N$. Then $u \in (W_0^{1,p}(\Omega))^N$ is a weak solution of (1.1) if and only if

$$Ju = -Su. \tag{5.1}$$

The operator $J : (W_0^{1,p}(\Omega))^N \to (W^{-1,q}(\Omega))^N$ is bounded, continuous, strongly monotone and coercive. In particular, it satisfies condition (S_+). By the main theorem on monotone operators due to Minty and Browder [11, Theorem 26A], the inverse $T = J^{-1} : (W^{-1,q}(\Omega))^N \to (W_0^{1,p}(\Omega))^N$ is bounded, continuous and satisfies condition (S_+). Moreover, by Lemma 5.3, the operator S is bounded, continuous and quasimonotone. Consequently, equation (5.1) is equivalent to

$$u = Tv \quad \text{and} \quad v + S \circ Tv = 0. \tag{5.2}$$

To solve (5.2), we will apply the topological degree theory introduced in Section 3. To do this, we first claim that the set

$$B := \{v \in (W^{-1,q}(\Omega))^N \mid v + tS \circ Tv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u = Tv$. Then $\|Tv\|_{1,p} = \|\nabla u\|_p$ and by the growth condition (H_2), we get

$$\begin{aligned}
\|Tv\|_{1,p}^p &= \frac{1}{\mu}\Big(\langle Ju, u\rangle - (\lambda+\mu)\int_\Omega (\operatorname{div} u)^2\,dx\Big)\\
&= \frac{1}{\mu}\Big(\langle v, Tv\rangle - (\lambda+\mu)\int_\Omega (\operatorname{div} u)^2\,dx\Big)\\
&= \frac{1}{\mu}\Big(-t\langle S\circ Tv, Tv\rangle - (\lambda+\mu)\int_\Omega (\operatorname{div} u)^2\,dx\Big)\\
&\le \frac{t}{\mu}\int_\Omega f(x,u(x))u(x)\,dx\\
&\le \frac{c}{\mu}\int_\Omega (d(x) + |u(x)|^{p'-1})u(x)\,dx\\
&\le \frac{c}{\mu}(\|d\|_q\|u\|_p + \|u\|_{p'}^{p'}).
\end{aligned}$$

From $p > 2$ and $p > p'$ it follows that the set $\{Tv : v \in B\}$ is bounded. Since the operator S is bounded, and from the definition of B, the set B is bounded in $(W^{-1,q}(\Omega))^N$. So, there exists a positive R such that

$$\|v\|_{-1,q} < R \quad \text{for all } v \in B.$$

This means that

$$v + tS\circ Tv \neq 0 \quad \text{for all } v \in \partial B_R(0) \text{ and } t \in [0,1].$$

According to Lemma 3.2,

$$I + S\circ T \in \mathcal{F}_T(\overline{B_R(0)}) \quad\text{and}\quad I = J\circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

To continue, consider the homotopy $H : [0,1]\times\overline{B_R(0)} \to (W^{-1,q}(\Omega))^N$ given by

$$H(t,v) = v + tS\circ Tv \quad \text{for } (t,v) \in [0,1]\times\overline{B_R(0)},$$

and applying the homotopy invariance and normalization properties of the topological degree d quoted in Theorem 3.4, we get

$$d(I + S\circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

which shows that there exists an element $v \in B_R(0)$ such that

$$v + S\circ Tv = 0.$$

Consequently, $u = Tv$ is a weak solution of problem (1.1). This completes the proof. ■

6. Uniqueness. In general, there is no reason for uniqueness to hold. The uniqueness of solution of problem (1.1) is given by the following theorem.

THEOREM 6.1. *Assume that, in addition of* $(H_1), (H_2)$, *the function* f *is decreasing with respect to the second variable, i.e.*

$$f(x, s_1) < f(x, s_2) \quad \text{for all } x \in \Omega, s_1, s_2 \in \mathbb{R}, s_1 > s_2.$$

Then there exists a unique weak solution $u \in (W_0^{1,p}(\Omega))^N$ *of the problem*

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx + (\lambda + \mu) \int_{\Omega} \operatorname{div} u \operatorname{div} v\, dx = \int_{\Omega} f(x, u) v\, dx \tag{6.1}$$

for all $u, v \in (W_0^{1,p}(\Omega))^N$.

Proof. Let u_1, u_2 be two solutions of (6.1). We have

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left|\frac{\partial u_1}{\partial x_i}\right|^{p-2} \frac{\partial u_{1,j}}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx + (\lambda + \mu) \int_{\Omega} \operatorname{div} u_1 \operatorname{div} v\, dx = \int_{\Omega} f(x, u_1) v\, dx,$$

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left|\frac{\partial u_2}{\partial x_i}\right|^{p-2} \frac{\partial u_{2,j}}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx + (\lambda + \mu) \int_{\Omega} \operatorname{div} u_2 \operatorname{div} v\, dx = \int_{\Omega} f(x, u_2) v\, dx.$$

Taking the difference of these two equalities, we get

$$\mu \sum_{i,j=1}^{N} \int_{\Omega} \left(\left|\frac{\partial u_1}{\partial x_i}\right|^{p-2} \frac{\partial u_{1,j}}{\partial x_i} - \left|\frac{\partial u_2}{\partial x_i}\right|^{p-2} \frac{\partial u_{2,j}}{\partial x_i} \right) \frac{\partial v_j}{\partial x_i}\, dx + (\lambda + \mu) \int_{\Omega} (\operatorname{div} u_1 - \operatorname{div} u_2) \operatorname{div} v\, dx = \int_{\Omega} (f(x, u_1) - f(x, u_2)) v\, dx.$$

For $v = u_1 - u_2 \in (W_0^{1,p}(\Omega))^N$, using the monotonicity of the operator J, the decrease of the function f and Poincaré's inequality, we obtain

$$\frac{\mu}{C^p} \left\| \frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} \right\|_p^p \le \mu \left\| \frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} \right\|_{(W_0^{1,p}(\Omega))^N}^p \le 0,$$

so that

$$\|u_1 - u_2\|_{(W_0^{1,p}(\Omega))^N}^p = 0, \quad \text{so} \quad u_1 = u_2 \text{ in } (W_0^{1,p}(\Omega))^N.$$

This completes the proof. ∎

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