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AN ELLIPTIC PROBLEM INVOLVING A POTENTIAL WITH EXPONENTIAL GROWTH

Abstract. We study the nonlinear weighted elliptic problem

$$-\nabla.(w_\beta(x)|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u = f(x,u), \quad u \in W_0^{1,N}(B, w_\beta),$$

where B is the unit ball of $\mathbb{R}^N$, $N > 2$, $w_\beta(x) = (1-\log|x|)^{\beta(N-1)}$, $\beta \in [0,1)$, is the singular logarithmic weight with the limiting exponent $N-1$ in the Trudinger–Moser embedding, and V is a continuous positive potential. The nonlinearities critical or subcritical growth in view of Trudinger–Moser inequalities. We prove the existence of nontrivial solutions via critical point theory. In the critical case, the associated energy functional does not satisfy the compactness condition. We give a new growth condition and we point out its importance for checking the Palais–Smale compactness condition.

1. Introduction and main results. In this paper, we extend the recent result of B. Dridi [13] and consider the following elliptic nonlinear problem:

$$(1.1)\quad \begin{cases} L_{N,w} = -\nabla.(w_\beta(x)|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u = f(x,u) & \text{in } B,\\ u > 0 & \text{in } B,\\ u = 0 & \text{on } \partial B,\end{cases}$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^N$, $N > 2$, $f(x,t)$ is a radial function with respect to x, and $V : \bar{B} \to \mathbb{R}$ is a positive continuous function

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satisfying some conditions. The weight $w_\beta(x)$ is given by

$$w_\beta(x) = (1 - \log|x|)^{\beta(N-1)}, \quad \beta \in [0,1). \tag{1.2}$$

In the case where $V(x) = 0$, Deng, Hu and Tang [11] studied the problem

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) = f(x,u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, $\rho(x) = \big(\log \frac{e}{|x|}\big)^{N-1}$, and the function $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{N}{N-1}}}\}$ as $t \to \infty$, for some $\alpha > 0$. The authors proved that there is a nontrivial solution to this problem using the Mountain Pass Theorem. They circumvented the loss of compactness of the associated energy function by an asymptotic condition on the nonlinearity and using appropriate Moser sequences. A similar result is proved in [22].

In this paper, we investigate the case $N > 2$ and use the Trudinger–Moser inequality to prove the existence of solutions to (1.1) without using the Ambrosetti–Rabinowitz condition.

Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function. The weighted Sobolev space is defined as

$$W_0^{1,N}(\Omega, w) = \text{closure}\Big\{u \in C_0^\infty(\Omega) : \int_\Omega |\nabla u|^N w(x)\,dx < \infty\Big\}.$$

We will restrict our attention to radial functions and consider the subspace

$$W_{0,\text{rad}}^{1,N}(\Omega, w) = \text{closure}\Big\{u \in C_{0,\text{rad}}^\infty(\Omega) : \int_\Omega |\nabla u|^N w(x)\,dx < \infty\Big\} \tag{1.3}$$

endowed with the norm

$$\|u\|_w = \Big(\int_\Omega |\nabla u|^N w(x)\,dx\Big)^{1/N}.$$

The choice of the weight and the space $W_{0,\text{rad}}^{1,N}(\Omega, w)$ are motivated by the following exponential inequalities.

Theorem 1.1 ([8]). *Let $\beta \in [0,1)$ and let w_β be given by* (1.2). *Then*

$$\int_B e^{|u|^\gamma}\,dx < \infty, \quad \forall u \in W_{0,\text{rad}}^{1,N}(B, w_\beta) \iff \gamma \leq \gamma_{N,\beta} \tag{1.4}$$

where

$$\gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and we have

(1.5)

$$\sup_{\substack{u\in W^{1,N}_{0,\mathrm{rad}}(B,w_\beta)\\ \|u\|_{w_\beta}\le 1}} \int_B e^{\alpha|u|^{\gamma_{N,\beta}}}\,dx<\infty \iff \alpha\le\alpha_{N,\beta}=N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and N' is the Hölder conjugate of N.

Let $\gamma:=\gamma_{N,\beta}$. In view of inequalities (1.4) and (1.5), we say that f has *subcritical growth* at $+\infty$ if

$$\lim_{s\to\infty}\frac{|f(x,s)|}{e^{\alpha s^\gamma}}=0,\qquad \forall\alpha \text{ with } \alpha_{N,\beta}\ge\alpha>0, \tag{1.6}$$

and f has *critical growth* at $+\infty$ if there exists some $0<\alpha_0\le\alpha_{N,\beta}$ such that

$$\begin{aligned}&\lim_{s\to\infty}\frac{|f(x,s)|}{e^{\alpha s^\gamma}}=0, &&\forall\alpha \text{ with } \alpha_0\le\alpha\le\alpha_{N,\beta},\\ &\lim_{s\to\infty}\frac{|f(x,s)|}{e^{\alpha s^\gamma}}=+\infty, &&\forall\alpha<\alpha_0.\end{aligned} \tag{1.7}$$

Our setting is defined as follows:

$$\mathbf{W}=\Big\{u\in W^{1,N}_{0,\mathrm{rad}}(B,w_\beta):\int_B V(x)|u|^N\,dx<\infty\Big\}.$$

It is a reflexive Banach space provided condition (V_1) below is satisfied; $\mathbf{W}$ is endowed with the norm

$$\|u\|=\Big(\int_B w_\beta(x)|\nabla u|^N\,dx+\int_B V(x)|u|^N\,dx\Big)^{1/N},$$

which is equivalent to the norm

$$\|u\|_{w_\beta}=\Big(\int_B w_\beta(x)|\nabla u|^N\,dx\Big)^{1/N}.$$

We denote by

$$\lambda_1=\inf_{\substack{u\in\mathbf{W}\\ u\neq 0}}\frac{\int_B(|\nabla u|^N w_\beta(x)+V(x)|u|^N)\,dx}{\int_B|u|^N\,dx}$$

the first eigenvalue of $(L_{N,w_\beta},\mathbf{W})$. It is well known that λ_1 is an isolated simple positive eigenvalue and has a positive bounded associated eigenfunction [12].

In this paper, we consider problem (1.1) with subcritical and critical growth nonlinearities $f(x,t)$. Furthermore, we suppose that $f(x,t)$ satisfies the following hypotheses:

$(f1)$ $f:\bar{B}\times\mathbb{R}\to\mathbb{R}$ is continuous, positive, radial in x, and $f(x,t)=0$ for $t\le 0$.

$(f2)$ There exist $t_0 > 0$ and $M > 0$ such that all $t > t_0$ and for all $x \in B$ we have

$$0 < F(x,t) \leq M f(x,t), \quad \text{where} \quad F(x,t) = \int_0^t f(x,s)\,ds.$$

$(f3)$ $0 < F(x,t) \leq \dfrac{1}{N} f(x,t)t, \forall t > 0, \forall x \in B.$

$(f4)$ $\displaystyle\limsup_{t\to 0} \frac{NF(x,t)}{t^N} < \lambda_1$ uniformly in x.

The potential V is continuous and satisfies

(V_1) $V(x) \geq V_0 > 0$ in B for some $V_0 > 0$.
(V_2) The function $1/V$ belongs to $L^{N-1}(B)$.

Let $\mathcal{J} : \mathbf{W} \to \mathbb{R}$ be the Euler–Lagrange functional associated to problem (1.1), that is,

$$\mathcal{J}(u) = \frac{1}{N}\int_B (|\nabla u|^N w_\beta(x) + V(x)|u|^N)\,dx - \int_B F(x,u)\,dx. \tag{1.8}$$

We say that u is a solution to problem (1.1) if u is a weak solution in the following sense:

Definition 1.2. We say that a function $u \in \mathbf{W}$ is a *solution* of problem (1.1) if

$$\int_B (w_\beta(x)|\nabla u|^{N-2}\nabla u \nabla\varphi + V|u|^{N-1}u\varphi)\,dx = \int_B f(x,u)\varphi\,dx, \qquad \forall \varphi \in \mathbf{W}.$$

In the subcritical exponential growth case, we will prove the following result:

Theorem 1.3. *Let $f(x,t)$ be a function that has subcritical growth at $+\infty$ and satisfies $(f1)$–$(f4)$. Then problem* (1.1) *has a nontrivial radial solution.*

In the case of critical exponential growth, the study of problem (1.1) becomes more difficult. Our Euler–Lagrange functional does not satisfy the Palais–Smale condition at all levels anymore. To overcome this lack of compactness, we choose testing functions which are extremal to the Trudinger–Moser inequality (1.5). Our result is as follows:

Theorem 1.4. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies conditions $(f1)$–$(f4)$. If in addition $f(x,t)$ satisfies the asymptotic condition*

$(f5)$ $\displaystyle\lim_{t\to\infty} \frac{f(x,t)t}{e^{\alpha_0 t^\gamma}} \geq \gamma_0$ *uniformly in x, with* $\gamma_0 > \dfrac{(1-\beta)^{N-1}N^{(N-1)(1-\beta)+1}}{\alpha_0^{(N-1)(1-\beta)}}$,

then problem (1.1) *has a nontrivial solution.*

We give an example of such a nonlinearity:

$$f(t) = F'(t) \quad \text{with} \quad F(t) = \frac{t^{N+1}}{N+1} + t^{\sigma} e^{\alpha_0 t^{\gamma}}, \ \sigma > N.$$

We point out that the special case $N = 2$ and $\beta = 1$, with the weight $\nu(x) = \log(\frac{e}{|x|})$, i.e. the problem

$$\begin{cases} L_{2,w} := -\operatorname{div}(\nu(x)\nabla u) = f(x,u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

was studied in [9].

Also, problem (1.1) without weight ($w = \text{const}$) and with $V = 0$ has been extensively studied by several authors; see e.g. [1, 10, 16, 18, 21] and references therein.

Finally, problem (1.1) is important and has several applications in non-Newtonian fluids, reaction-diffusion problems, turbulent flows in porous media and image treatment [3, 4, 19, 20].

This paper is organized as follows:

In Section 2, we present some necessary preliminary knowledge about our space. We also give some lemmas useful for compactness analysis. In Section 3, we prove that the energy has a mountain-pass geometry. In Section 4, we identify the first compactness level of the energy and prove the main results.

In this work, the constant C may change from line to line and sometimes we index the constants in order to show how they change.

2. Preliminaries and variational formulation

2.1. Weighted Lebesgue and Sobolev spaces setting. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. Following Drábek et al. and Kufner [15, 12], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} w(x)|u|^p \, dx < \infty \Big\}$$

for any real number $1 \leq p < \infty$. This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \Big(\int_{\Omega} w(x)|u|^p \, dx \Big)^{1/p},$$

and for $w(x) = 1$, we find the standard Lebesgue space $L^p(\Omega)$ and its norm

$$\|u\|_p = \Big(\int_{\Omega} |u|^p \, dx \Big)^{1/p}.$$

In [12], the corresponding weighted Sobolev space was defined as

$$W^{1,p}(\Omega, w) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega, w)\}$$

and equipped with the norm

$$\|u\|_{W^{1,p}(\Omega,w)} = (\|u\|_p^p + \|\nabla u\|_{p,w}^p)^{1/p}. \tag{2.1}$$

$L^p(\Omega, w)$ and $W^{1,p}(\Omega, w)$ are separable, reflexive Banach spaces provided that $w(x)^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$.

Furthermore, if $w \in L^1_{\mathrm{loc}}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, w)$, and we can introduce the space $W_0^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, w)$.

The space $W_0^{1,p}(\Omega, w)$ is equipped with the norm

$$\|u\|_{W_0^{1,p}(\Omega,w)} = \Big(\int_\Omega w(x)|\nabla u|^p\,dx\Big)^{1/p}, \tag{2.2}$$

which is equivalent to the one given by (2.1).

Also, we will use the space $W_0^{1,N}(\Omega, w)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,N}(\Omega, w)$, equipped with the norm

$$\|u\|_{W_0^{1,N}(\Omega,w)} = \Big(\int_\Omega w(x)|\nabla u|^N\,dx\Big)^{1/N}.$$

Let s be the real such that

$$s \in (1, \infty) \quad \text{and} \quad \omega^{-s} \in L^1(\Omega). \tag{2.3}$$

The last condition gives important embeddings of $W^{1,N}(\Omega, \omega)$ into usual Lebesgue spaces without weight. More precisely, following [12] we have

$$W^{1,N}(\Omega, \omega) \hookrightarrow L^N(\Omega) \text{ with compact injection} \tag{2.4}$$

and

$$W^{1,N}(\Omega, \omega) \hookrightarrow L^{N+\eta}(\Omega) \text{ with compact injection for } 0 \le \eta < N(s-1)$$

provided

$$\omega^{-s} \in L^1(\Omega) \text{ with } s \in (1, \infty).$$

2.2. Some useful lemmas for compactness analysis. In this section we will derive several technical lemmas for later use. We begin with a radial lemma.

LEMMA 2.1. *Assume that V is continuous and satisfies (V_1) and (V_2).*

(i) *Let u be a radially symmetric C_0^1 function on the unit ball B. Then*

$$|u(x)| \le \frac{\big|\log|x|\big|^{\frac{1-\beta}{N'}}}{\omega_{N-1}^{1/N'}(1-\beta)^{1/N'}}\|u\|,$$

where ω_{N-1} is the area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

(ii) *There exists a positive constant C such that for all $u \in \mathbf{W}$,*

$$\int_B V|u|^N\,dx \le C\|u\|^N$$

and so the norms $\|\cdot\|$ and $\|\cdot\|_{W^1_{0,\mathrm{rad}}(B,w_\beta)} = (\int_B w_\beta(x)|\nabla\cdot|^N\,dx)^{1/N}$ are equivalent.

(iii) *The following embedding is continuous:*

$$\mathbf{W} \hookrightarrow L^q(B) \quad \textit{for all } q \ge 1.$$

(iv) *The embedding in* (iii) *is compact.*

Proof. (i) See [7].

(ii) From (i) we have, for all $u \in \mathbf{W}$,

$$\begin{aligned}\int_B V|u|^N\,dx &\le \frac{m}{\omega_{N-1}^{N-1}(1-\beta)^{N-1}}\|u\|^N \int_B \big|\log|x|\big|^{(1-\beta)(N-1)}\,dx\\ &\le C\frac{m}{\omega_{N-1}^{N-1}(1-\beta)^{N-1}}\|u\|^N \le C\|u\|^N,\end{aligned}$$

where $m = \max_{x\in\bar B} V(x)$. Hence (ii) follows.

(iii) From (i) and (ii), we know that the following embeddings are continuous:

$$\mathbf{W} \hookrightarrow W^1_{0,\mathrm{rad}}(B) \hookrightarrow L^q(B) \quad \forall q \ge N.$$

By the Hölder inequality and (V_2) we have

$$\int_B |u|\,dx \le \bigg(\int_B \frac{1}{V^{\frac{1}{N-1}}}\,dx\bigg)^{\frac{N-1}{N}}\bigg(\int_B V|u|^N\,dx\bigg)^{\frac{1}{N}} \le \bigg(\int_B \frac{1}{V^{\frac{1}{N-1}}}\,dx\bigg)^{\frac{N-1}{N}}\|u\|.$$

For any $1 < \beta < N$,

$$\int_B |u|^\beta\,dx \le \int_B (|u| + |u|^N)\,dx \le \bigg(\int_B \frac{1}{V^{\frac{1}{N-1}}}\,dx\bigg)^{\frac{N-1}{N}}\|u\| + \frac{1}{V_0}\|u\|^N.$$

Thus we get the continuous embedding $\mathbf{W} \hookrightarrow L^q(B)$ for all $q \ge 1$.

(iv) Let $(u_k) \subset \mathbf{W}$ be a sequence such that $\|u_k\| \le C$ for all k. Then $\|u_k\|_{W^1_{0,\mathrm{rad}}} \le C$ for all k. On the other hand, we have the compact embedding [12]

$$W^1_{0,\mathrm{rad}} \hookrightarrow L^q \quad \text{for } 1 \le q < Ns \text{ with } s > 1.$$

Then up to a subsequence, there exists $u \in W^1_{0,\mathrm{rad}}$ such that u_k converges to u strongly in $L^q(B)$ for all q such that $1 \le q < Ns$. Without loss of generality, we may assume that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } \mathbf{W},\\ u_k \to u & \text{strongly in } L^1(B),\\ u_k(x) \to u(x) & \text{almost everywhere in } B.\end{cases} \tag{2.5}$$

For $q > 1$, it follows from (2.5) and the continuous embedding

$$\mathbf{W} \hookrightarrow L^s(B) \quad (s \geq 1)$$

that

$$\begin{aligned}\int_B |u_k - u|^q \, dx &= \int_B |u_k - u|^{1/2} |u_k - u|^{q-1/2} \, dx \\ &\leq \Big(\int_B |u_k - u| \, dx \Big)^{1/2} \Big(\int_B |u_k - u|^{2q-1} \, dx \Big)^{1/2} \\ &\leq C \Big(\int_B |u_k - u| \, dx \Big)^{1/2} \to 0.\end{aligned}$$

This concludes the proof. ■

LEMMA 2.2 ([14]). *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function. Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n) u_n| \, dx \leq C$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(\Omega).$$

We need a Lions type result [17] about an improved Trudinger–Moser inequality when we deal with weakly convergent sequences.

LEMMA 2.3. *Let $(u_k)_k$ be a sequence in $\mathbf{W}$. Suppose that $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in $\mathbf{W}$, $u_k(x) \to u(x)$ a.e. $x \in B$, $\nabla u_k(x) \to \nabla u(x)$ a.e. $x \in B$ and $u \not\equiv 0$. Then*

$$\sup_k \int_B e^{p\alpha_{N,\beta} |u_k|^\gamma} \, dx < \infty$$

for all $1 < p < U$ where U is given by

$$U = \begin{cases} \dfrac{1}{(1 - \|u\|^N)^{\gamma/N}} & \text{if } \|u\| < 1, \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. For $a, b \in \mathbb{R}$ and $q > 1$, if q' is the conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, by the Young inequality we have

$$e^{a+b} \leq \frac{1}{q} e^{qa} + \frac{1}{q'} e^{q'b}.$$

Also,

$$(1 + a)^q \leq (1 + \varepsilon) a^q + \Big(1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q-1}}} \Big)^{1-q}, \; \forall a \geq 0, \, \varepsilon > 0, \, q > 1. \tag{2.6}$$

So, we get

$$\begin{aligned} |u_k|^\gamma = |u_k - u + u|^\gamma &\le (|u_k - u| + |u|)^\gamma \\ &\le (1+\varepsilon)|u_k - u|^\gamma + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u|^\gamma, \end{aligned}$$

which implies that

$$\begin{aligned} \int_B e^{p\alpha_{N,\beta}|u_k|^\gamma}\,dx &\le \frac{1}{q}\int_B e^{pq\alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma}\,dx \\ &\quad + \frac{1}{q'}\int_B \exp\left(pq'\alpha_{N,\beta}\left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma}|u|^\gamma\right)dx \end{aligned}$$

for any $p > 1$. From (1.4), the last integral is finite. To complete the proof we have to prove that for every p such that $1 < p < U$,

$$\sup_k \int_B e^{pq\alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma}\,dx < \infty \tag{2.7}$$

for some $\varepsilon > 0$ and $q > 1$.

In the following, we suppose that $\|u\| < 1$; for $\|u\| = 1$, the proof is similar.

When

$$\|u\| < 1 \quad \text{and} \quad p < \frac{1}{(1 - \|u\|^N)^{\gamma/N}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^N)^{\gamma/N}(1+\nu) < 1.$$

On the other hand, by Brezis–Lieb's lemma [5] we have

$$\|u_k - u\|^N = \|u_k\|^N - \|u\|^N + o(1) \quad \text{as } k \to \infty. \tag{2.8}$$

Then

$$\|u_k - u\|^N = 1 - \|u\|^N + o(1),$$

hence,

$$\lim_{k\to\infty} \|u_k - u\|^\gamma = (1 - \|u\|^N)^{\gamma/N}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \ge 1$ such that

$$\|u_k - u\|^\gamma \le (1+\varepsilon)(1 - \|u\|^N)^{\gamma/N}, \qquad \forall k \ge k_\varepsilon.$$

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt[3]{1+\nu} - 1$, then for all $k \ge k_\varepsilon$, we have

$$pq(1+\varepsilon)\|u_k - u\|^\gamma \le 1.$$

Consequently,

$$\begin{aligned}\int_B e^{pq\alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma}\,dx &\le \int_B e^{(1+\varepsilon)pq\alpha_{N,\beta}(\frac{|u_k-u|}{\|u_k-u\|})^\gamma \|u_k-u\|^\gamma}\,dx\\ &\le \int_B e^{\alpha_{N,\beta}(\frac{|u_k-u|}{\|u_k-u\|})^\gamma}\,dx\\ &\le \sup_{\|u\|\le 1}\int_B e^{\alpha_{N,\beta}|u|^\gamma}\,dx<\infty. \ \blacksquare\end{aligned}$$

3. The mountain pass geometry of the energy $\mathcal{J}$. Since the nonlinearity $f(x,t)$ is critical or subcritical at $+\infty$, there exist $a, C > 0$ and $t_1 > 1$ such that

$$|f(x,t)| \le Ce^{at^\gamma}, \quad \forall |t| > t_1. \tag{3.1}$$

So, the functional $\mathcal{J}$ given by (1.8) is well defined and of class C^1 .

In order to prove the existence of a nontrivial solution to problem (1.1), we will prove the existence of a nonzero critical point of the functional $\mathcal{J}$ by using the Mountain Pass Theorem of Ambrosetti and Rabinowitz [2].

Definition 3.1. Let (u_n) be a sequence in a Banach space E, let $\mathcal{J} \in C^1(E,\mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence (u_n) is a *Palais–Smale sequence at level* c (or a $(\mathrm{PS})_c$ *sequence*) for the functional $\mathcal{J}$ if

$$\begin{aligned}\mathcal{J}(u_n)&\to c \quad \text{in } \mathbb{R} \text{ as } n\to\infty,\\ \mathcal{J}'(u_n)&\to 0 \quad \text{in } E' \text{ as } n\to\infty.\end{aligned}$$

We say that the functional $\mathcal{J}$ satisfies the *Palais–Smale condition* $(\mathrm{PS})_c$ at level c if every $(\mathrm{PS})_c$ sequence (u_n) is relatively compact in E.

Theorem 3.2 ([2]). *Let E be a Banach space and $\mathcal{J} : E \to \mathbb{R}$ a C^1 functional satisfying $\mathcal{J}(0) = 0$. Suppose that*

(i) *there exist $\rho, \beta > 0$ such that $\mathcal{J}(u) \ge \beta$ for all $u \in \partial B(0,\rho)$;*
(ii) *there exists $x_1 \in E$ such that $\|x_1\| > \rho$ and $\mathcal{J}(x_1) < 0$;*
(iii) *$\mathcal{J}$ satisfies the Palais–Smale condition* (PS), *that is, for all sequences (u_n) in E satisfying*

$$\mathcal{J}(u_n)\to d \quad \text{as } n\to\infty \tag{3.2}$$

for some $d \in \mathbb{R}$ and

$$\|\mathcal{J}'(u_n)\|_* \to 0 \quad \text{as } n\to\infty, \tag{3.3}$$

the sequence (u_n) is relatively compact.

Then $\mathcal{J}$ has a critical point u and the critical value $c = \mathcal{J}(u)$ satisfies

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],X) : \gamma(0) = 0$ and $\gamma(1) = x_1\}$ and $c \ge \beta$.

It follows from Lemma 2.1 that the embedding $W^{1,N}(B) \hookrightarrow L^q(B)$ is continuous for all $q \geq 1$, so there exists a constant $C > 0$ such that $\|u\|_{qN'} \leq c\|u\|$ for all $u \in \mathbf{W}$.

In the next lemma, we prove that the functional $\mathcal{J}$ satisfies condition (i) of Theorem 3.2.

LEMMA 3.3. *Suppose that $(f1)$ and $(f4)$ hold. Then there exist $\rho, \beta > 0$ such that $\mathcal{J}(u) \geq \beta$ for all $u \in \mathbf{W}$ with $\|u\| = \rho$.*

Proof. It follows from $(f4)$ that there exist $t_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$F(x,t) \leq \frac{1}{N}\lambda_1(1-\varepsilon_0)|t|^N \quad \text{for } |t| < t_0. \tag{3.4}$$

Indeed,

$$\limsup_{t\to 0} \frac{NF(x,t)}{t^N} < \lambda_1$$

or

$$\inf_{\beta>0} \sup\left\{\frac{NF(x,t)}{t^N} : 0 < t < \beta\right\} < \lambda_1.$$

Since this inequality is strict, there exists $\varepsilon_0 > 0$ such that

$$\inf_{\beta>0} \sup\left\{\frac{NF(x,t)}{t^N} : 0 < t < \beta\right\} < \lambda_1 - \varepsilon_0.$$

Hence, there exists $t_0 > 0$ such that

$$\sup\left\{\frac{NF(x,t)}{t^N} : 0 < t < t_0\right\} < \lambda_1 - \varepsilon_0.$$

Hence

$$\forall |t| < t_0 \quad F(x,t) \leq \frac{1}{N}\lambda_1(1-\varepsilon_0)|t|^N.$$

From $(f3)$ and (3.1), for all $q > N$ there exists a constant $C > 0$ such that

$$F(x,t) \leq C|t|^q e^{at^\gamma}, \quad \forall |t| > t_1. \tag{3.5}$$

So

$$F(x,t) \leq \frac{1}{N}\lambda_1(1-\varepsilon_0)|t|^N + C|t|^q e^{at^\gamma} \quad \text{for } t \in \mathbb{R}. \tag{3.6}$$

Since

$$\mathcal{J}(u) = \frac{1}{N}\|u\|^N - \int_B F(x,u)\,dx,$$

we get

$$\mathcal{J}(u) \geq \frac{1}{N}\|u\|^N - \frac{1}{N}\lambda_1(1-\varepsilon_0)|t|^N - C\int_B |u|^q e^{au^\gamma}\,dx.$$

But $\lambda_1\|u\|_N^N \le \|u\|^N$ and from the Hölder inequality, we obtain

$$\mathcal{J}(u) \ge \frac{\varepsilon_0}{N}\|u\|^N - C\Big(\int_B e^{aN|u|^\gamma}\,dx\Big)^{1/N}\|u\|_{qN'}^q. \tag{3.7}$$

From Theorem 1.1, if we choose $u \in \mathbf{W}$ such that

$$aN\|u\|^\gamma \le \alpha_{N,\beta}, \tag{3.8}$$

we get

$$\int_B e^{aN|u|^\gamma}\,dx = \int_B e^{aN\|u\|^\gamma(\frac{|u|}{\|u\|})^\gamma}\,dx < \infty.$$

On the other hand, $\|u\|_{N'q} \le C\|u\|$ (Lemma 2.1), so

$$\mathcal{J}(u) \ge \frac{\varepsilon_0}{N}\|u\|^N - C\|u\|^q$$

for all $u \in \mathbf{W}$ satisfying (3.8). Since $N < q$, we can choose $\rho = \|u\|$ small enough such that there exists $\beta > 0$ small such that $\mathcal{J}(u) \ge \beta > 0$. ∎

By the following lemma, we find that $\mathcal{J}$ satisfies condition (ii) of Theorem 3.2.

LEMMA 3.4. *Suppose that* $(f1)$ *and* $(f2)$ *hold. Let* φ_1 *be a normalized eigenfunction associated to* λ_1 *in* $\mathbf{W}$. *Then* $\mathcal{J}(t\varphi_1) \to -\infty$ *as* $t \to \infty$.

Proof. It follows from condition $(f2)$ that

$$f(x,t) = \frac{\partial}{\partial t}F(x,t) \ge \frac{1}{M}F(x,t)$$

for all $t \ge t_0$. So

$$F(x,t) \ge Ce^{t/M}, \qquad \forall t \ge t_0. \tag{3.9}$$

It follows that there exist $b > \lambda_1$ and $C > 0$ such that $F(x,t) \ge \frac{b}{N}t^N + C$ for all $t > 0$. Therefore

$$\mathcal{J}(t\varphi_1) \le \frac{t^N}{N}\|\varphi_1\|^N - \frac{b}{N}t^N\|\varphi_1\|_N^N - C|B|,$$

where $|B| = \text{meas}(B) = \text{Vol}(B)$. Then, from the definition of λ_1, we get

$$\mathcal{J}(t\varphi_1) \le t^N\frac{\lambda_1 - b}{N}\|\varphi_1\|_N^N < 0, \qquad \forall t > 0.$$

So, the lemma follows. ∎

4. Proof of Theorem 1.3. In the following proposition, we will show the lack of compactness of the energy in the critical case.

PROPOSITION 4.1. *Suppose that* $(f1)$*–*$(f3)$ *hold.*

(i) *If the function* $f(x,t)$ *satisfies condition* (1.7) *for some* $\alpha_0 > 0$*, then the functional* $\mathcal{J}$ *satisfies the Palais–Smale condition* $(\mathrm{PS})_c$ *for any*

$$c < \frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma},$$

where $\alpha_{N,\beta} = N[w_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}$ *and* ω_{N-1} *is the area of the unit sphere* S^{N-1} *in* $\mathbb{R}^N$*.*

(ii) *If* $f(x,t)$ *satisfies condition* (1.6)*, then* $\mathcal{J}$ *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c \in \mathbb{R}$*.*

Proof. (i) Consider a $(\mathrm{PS})_c$ sequence in $\mathbf{W}$, for some $c \in \mathbb{R}$, that is,

$$\mathcal{J}(u_n) = \frac{1}{N}\|u_n\|^N - \int_B F(x,u_n)\,dx \to c, \qquad n \to \infty, \tag{4.1}$$

and for all $\varphi \in \mathbf{W}$,

$$|\langle \mathcal{J}'(u_n), \varphi\rangle| = \left|\int_B w_\beta(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla\varphi\,dx + A\right| \le \varepsilon_n\|\varphi\|, \tag{4.2}$$

where $A = \int_B V|u_n|^{N-2}u_n v\,dx - \int_B f(x,u_n)\varphi\,dx$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Also, inspired by [9], it follows from $(f2)$ that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon t f(x,t) \quad \text{for all } |t| > t_\varepsilon \text{ and uniformly in } x \in B, \tag{4.3}$$

and so, by (4.1), for every $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\frac{1}{N}\|u_n\|^N \le C + \int_B F(x,u_n)\,dx,$$

hence

$$\frac{1}{N}\|u_n\|^N \le C + \int_{|u_n|\le t_\varepsilon} F(x,u_n)\,dx + \varepsilon\int_B f(x,u_n)u_n\,dx,$$

and so, from (4.2), we get

$$\frac{1}{N}\|u_n\|^N \le C_1 + \varepsilon\varepsilon_n\|u_n\| + \varepsilon\|u_n\|^N$$

for some constant $C_1 > 0$. Since

$$\left(\frac{1}{N} - \varepsilon\right)\|u_n\|^N \le C_1 + \varepsilon\varepsilon_n\|u_n\|, \tag{4.4}$$

we deduce that the sequence (u_n) is bounded in $\mathbf{W}$. As a consequence, there exists $u \in \mathbf{W}$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $\mathbf{W}$, $u_n \to u$ strongly in $L^q(B)$ for all $q \ge 1$ and $u_n(x) \to u(x)$ a.e. in B. For the proof of $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in B$ we refer the reader to [13].

Furthermore, from (4.1) and (4.2) we have

$$0 < \int_B f(x, u_n)u_n \le C, \tag{4.5}$$

$$0 < \int_B F(x, u_n) \le C. \tag{4.6}$$

Since by [14, Lemma 2.1], we have

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(B) \text{ as } n \to \infty, \tag{4.7}$$

it follows from $(f2)$ and the generalized Lebesgue dominated convergence theorem that

$$F(x, u_n) \to F(x, u) \quad \text{in } L^1(B) \text{ as } n \to \infty. \tag{4.8}$$

So,

$$\lim_{n\to\infty} \|u_n\|^N = N\Big(c + \int_B F(x, u)\, dx\Big). \tag{4.9}$$

So, using (4.1), we have

$$\lim_{n\to\infty} \int_B f(x, u_n)u_n\, dx = N\Big(c + \int_B F(x, u)\, dx\Big). \tag{4.10}$$

By condition $(f3)$,

$$\lim_{n\to\infty} N \int_B F(x, u_n)\, dx \le N\Big(c + \int_B F(x, u)\, dx\Big)$$

and so $c \ge 0$. Also, it follows from (4.1) and (4.2) that u is a weak solution of problem (1.1) and if we take $\varphi = u$ as a test function, we get

$$\int_B |\nabla u|^N w_\beta(x)\, dx + \int_B V(x)|u|^N\, dx = \int_B f(x, u)u\, dx.$$

So it follows from $(f3)$ that $\mathcal{J}(u) \ge 0$.

We will finish the proof by considering three cases for the level c.

Case 1: $c = 0$. In this case

$$0 \le \mathcal{J}(u) \le \liminf_{n\to\infty} \mathcal{J}(u_n) = 0.$$

So, $\mathcal{J}(u) = 0$ and hence

$$\lim_{n\to\infty} \frac{1}{N}\|u_n\|^N = \int_B F(x, u)\, dx = \frac{1}{N}\|u\|^N.$$

By Brezis–Lieb's lemma [5], we get $u_n \to u$ in $\mathbf{W}$.

Case 2: $c > 0$ and $u = 0$. We prove that this case cannot happen. From (4.1) and (4.2) with $v = u_n$ we have

$$\lim_{n\to\infty} \|u_n\|^N = Nc \quad \text{and} \quad \lim_{n\to\infty} \int_B f(x, u_n)u_n\, dx = Nc,$$

and by (4.2) we also have

$$\Big| \|u_n\|^N - \int_B f(x,u_n)u_n\,dx \Big| \le C\varepsilon_n.$$

We claim that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q\,dx \le C, \tag{4.11}$$

so

$$\|u_n\|^N \le C\varepsilon_n + \Big(\int_B |f(x,u_n)|^q\,dx\Big)^{1/q} \Big(\int_B |u_n|^{q'}\Big)^{1/q'},$$

where q' is the conjugate of q. Since (u_n) converges to $u = 0$ in $L^{q'}(B)$,

$$\lim_{n\to\infty} \|u_n\|^N = 0.$$

This is in contradiction with $c > 0$.

To prove the claim, note that since f has subcritical or critical growth, for every $\varepsilon > 0$ and $q > 1$ there exist $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \ge t_\varepsilon$, we have

$$|f(x,t)|^q \le Ce^{\alpha_0(\varepsilon+1)t^\gamma}. \tag{4.12}$$

Consequently,

$$\begin{aligned}\int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n|\le t_\varepsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q\,dx \\ &\le \omega_{N-1} \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(\varepsilon+1)|u_n|^\gamma}\,dx.\end{aligned}$$

Since $Nc < (\alpha_{N,\beta}/\alpha_0)^{N/\gamma}$, there exists $\eta \in (0,1/N)$ such that

$$Nc = (1-N\eta)\Big(\frac{\alpha_{N,\beta}}{\alpha_0}\Big)^{N/\gamma}.$$

On the other hand, $\|u_n\|^\gamma \to (Nc)^{\gamma/N}$, so there exists $n_\eta > 0$ such that for all $n \ge n_\eta$, we get $\|u_n\|^\gamma \le (1-\eta)\alpha_{N,\beta}/\alpha_0$. Therefore,

$$\alpha_0(1+\varepsilon)\Big(\frac{|u_n|}{\|u_n\|}\Big)^\gamma \|u_n\|^\gamma \le (1+\varepsilon)(1-\eta)\alpha_{N,\beta}.$$

We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le \alpha_{N,\beta},$$

therefore the second integral is uniformly bounded in view of (1.5), and the claim is proved.

Case 3: $c > 0$ and $u \neq 0$. In this case, we claim that $J(u) = c$ and therefore

$$\lim_{n\to\infty} \|u_n\|^N = N\Big(c + \int_B F(x,u)\,dx\Big) = N\Big(\mathcal{J}(u) + \int_B F(x,u)\,dx\Big) = \|u\|^N.$$

To prove the claim we remark that

$$\mathcal{J}(u) \le \frac{1}{N}\liminf_{n\to\infty} \|u_n\|^N - \int_B F(x,u)\,dx = c.$$

Suppose that $\mathcal{J}(u) < c$. We have

$$\|u\|^\gamma < \Big(N\Big(c + \int_B F(x,u)\,dx\Big)\Big)^{\gamma/N}. \tag{4.13}$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{(N(c + \int_B F(x,u)\,dx))^{1/N}}.$$

Then $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathbf{W}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 2.3,

$$\sup_n \int_B e^{p\alpha_{N,\beta}|v_n|^\gamma}\,dx < \infty \quad \text{for } 1 < p < (1 - \|v\|^N)^{-\gamma/N}.$$

As in Case 2, we are going to estimate $\int_B |f(x,u_n)|^q\,dx$. For $\varepsilon > 0$, one has

$$\begin{aligned}\int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n|\le t_\varepsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n|> t_\varepsilon\}} |f(x,u_n)|^q\,dx \\ &\le \omega_{N-1} \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(1+\varepsilon)|u_n|^\gamma}\,dx \\ &\le C_\varepsilon + C\int_B e^{\alpha_0(1+\varepsilon)\|u_n\|^\gamma|v_n|^\gamma}\,dx \le C\end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le p\alpha_{N,\beta}$ and $1 < p < (1 - \|v\|^N)^{-\gamma/N}$. Since

$$\begin{aligned}(1 - \|v\|^N)^{-\gamma/N} &= \bigg(\frac{N(c + \int_B F(x,u)\,dx)}{N(c + \int_B F(x,u)\,dx) - \|u\|^N)}\bigg)^{\gamma/N} \\ &= \bigg(\frac{c + \int_B F(x,u)\,dx}{c - \mathcal{J}(u)}\bigg)^{\gamma/N}\end{aligned}$$

and

$$\lim_{n\to\infty} \|u_n\|^\gamma = \Big(N\Big(c + \int_B F(x,u)\,dx\Big)\Big)^{\gamma/N},$$

we have

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le \alpha_0(1+2\varepsilon)\Big(N\Big(c + \int_B F(x,u)\,dx\Big)\Big)^{\gamma/N}.$$

But $\mathcal{J}(u) \geq 0$ and $c < \frac{1}{N}(\alpha_{N,\beta}/\alpha_0)^{N/\gamma}$, so if we choose $\varepsilon > 0$ small enough that

$$\frac{\alpha_0}{\alpha_{N,\beta}}(1+2\varepsilon) < \left(\frac{1}{N(c-\mathcal{J}(u))}\right)^{\gamma/N},$$

we have

$$(1+2\varepsilon)(c-\mathcal{J}(u))^{\gamma/N} < \frac{\alpha_{N,\beta}}{N^{\gamma/N}\alpha_0}$$

and so the sequence $(f(x,u_n))$ is bounded in L^q, $q>1$.

Since $\langle \mathcal{J}'(u_n),(u_n-u)\rangle = o_n(1)$, we get

$$\int_B w_\beta(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n-\nabla u)\,dx + \int_B V|u_n|^{N-2}u_n(u_n-u)\,dx - \int_B f(x,u_n)(u_n-u)\,dx = o_n(1). \tag{4.14}$$

On the other hand, since $u_n \rightharpoonup u$ weakly in $\mathbf{W}$, we get

$$\int_B w_\beta(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u)\,dx+\int_B V|u|^{N-2}u(u_n-u)\,dx = o_n(1). \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$\begin{aligned}\int_B w_\beta(x)(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u).(\nabla u_n-\nabla u)\,dx\\ + \int_B V(|u_n|^{N-2}u_n - |u|^{N-2}u)(u_n-u)\,dx\\ = \int_B f(x,u_n)(u_n-u)\,dx + o_n(1).\end{aligned} \tag{4.16}$$

Using the well known inequality

$$(|x|^{N-2}x-|y|^{N-2}y).(x-y) \geq 2^{2-N}|x-y|^N, \quad \forall x,y\in\mathbb{R}^N \text{ and } N\geq 2, \tag{4.17}$$

we obtain

$$\begin{aligned}0 &\leq 2^{2-N}\int_B w_\beta(x)|\nabla u_n-\nabla u|^N\,dx + \int_B V|u_n-u|^N\,dx\\ &\leq \int_B f(x,u_n)(u_n-u)\,dx + o_n(1).\end{aligned} \tag{4.18}$$

By the Hölder inequality, we obtain

$$2^{2-N}\Big(\int_B w_\beta(x)|\nabla u_n-\nabla u|^N\,dx+\int_B V|u_n-u|^N\,dx\Big) \tag{4.19}$$
$$\le \int_B f(x,u_n)(u_n-u)\,dx+o(1)$$
$$\le \Big(\int_B |f(x,u_n)|^q\,dx\Big)^{1/q}\Big(\int_B |u_n-u|^{q'}\Big)^{1/q'}dx+o(1).$$

So,

$$\|u_n-u\|\to 0 \quad \text{as } n\to\infty.$$

By Brezis–Lieb's lemma, up to a subsequence, we get

$$\lim_{n\to\infty}\|u_n\|^N=N\Big(c+\int_B F(x,u)\,dx\Big)=\|u\|^N,$$

and this contradicts (4.13).

(ii) From the proof of (i), up to a subsequence, there exists $M>0$ such that $\|u_n\|\le M$. By the subcritical case of f at ∞, for some $q>1$ there exist $\alpha\le \alpha_{N,\beta}/M^\gamma$ and positive constants $C_{1,q}$ and $C_{2,q}$ such that

$$\int_B |f(x,u_n)|^q\,dx = C_{1,q}+C_{2,q}\int_B e^{\alpha|u_n|^\gamma}\,dx \le C_{1,q}+C_{2,q}\int_B e^{\alpha\frac{|u_n|^\gamma}{\|u_n\|^\gamma}\|u_n\|^\gamma}\,dx$$
$$\le C_{1,q}+C_{2,q}\int_B e^{\alpha_{N,\beta}\frac{|u_n|^\gamma}{\|u_n\|^\gamma}}\,dx<\infty.$$

We conclude as in (i). ∎

Remark 4.2. We have $u>0$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\|\to\rho>0$. In addition, $\mathcal{J}'(u_n)\to 0$ leads to

$$\int_B (w_\beta(x)|\nabla u|^{N-2}\nabla u.\nabla\varphi+V(x)|u|^N)\,dx=\int_B f(x,u)\varphi\,dx, \qquad \forall\varphi\in\mathbf{W}.$$

By taking $\varphi=u^-$ with $u^\pm=\max(\pm u,0)$, we get $\|u^-\|^N=0$ and so $u=u^+\ge 0$. Since the nonlinearity has critical growth at $+\infty$ and from the Trudinger–Moser inequality (1.5), we have $f(\cdot,u)\in L^p(B)$ for all $p\ge 1$. So, by elliptic regularity $u\in W^{2,p}(B,\sigma)$ for all $p\ge 1$. Therefore, by Sobolev embedding, $u\in C^{0,\gamma}(\overline{B})$. Let $B_0=\{x\in B: u(x)=0\}$. Then $B_0=\emptyset$. Indeed, suppose $B_0\ne\emptyset$. Since $f(x,u)\ge 0$, from the Harnack inequality [12, Theorem 1.9] we can deduce that B_0 is an open and closed subset of B. In virtue of the connectedness of B, we have reached a contradiction.

Proof of Theorem 1.3. Since $f(x,t)$ satisfies (1.6) for all $\alpha_0>0$ such that $\alpha_{N,\beta}\ge\alpha_0>0$, Proposition 4.1 implies that $\mathcal{J}$ satisfies the (PS) condition (at each possible level d). So, by Lemmas 3.3 and 3.4, $\mathcal{J}$ has a nonzero critical point u in $\mathbf{W}$. From the maximum principle, the solution u of problem (1.1) is positive. ∎

5. Proof of Theorem 1.4. In Theorem 1.4, we suppose that the function $f(x,t)$ is critical, that is, it satisfies condition (1.7) for some α_0 with $\alpha_{N,\beta} \geq \alpha_0 > 0$.

We can still use Theorem 3.2 if we prove that the mountain pass level c satisfies

$$c < \frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

For this purpose, we will prove that there exists $v \in \mathbf{W}$ such

$$\max_{t\geq 0} \mathcal{J}(tv) < \frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}. \tag{5.1}$$

We consider the Moser-type sequence given by

$$v_n(x) = \frac{N^{1-\beta}}{\alpha_{N,\beta}^{1/\gamma}} \begin{cases} \dfrac{\left(\log \frac{1}{|x|}\right)^{1-\beta}}{n^{(1-\beta)/N}} & \text{if } e^{-n/N} \leq |x| \leq 1, \\ \dfrac{1}{N^{(1-\beta)}} n^{1/\gamma} & \text{if } 0 \leq |x| \leq e^{-n/N}. \end{cases} \tag{5.2}$$

Let $w_n = v_n/\|v_n\|$. Then $w_n \in \mathbf{W}$ and has norm 1. We also have

$$\int_B |\nabla v_n(x)|^N \left|\log\left(\frac{1}{|x|}\right)\right|^{\beta(N-1)} dx = \frac{N^{(1-\beta)}(1-\beta)}{n^{1-\beta}} \int_{e^{-n/N}}^{1} \frac{1}{r}\left(\log\frac{1}{r}\right)^{-\beta} dr = 1,$$

so

$$\|v_n\|^N = 1 + \int_B V(x)|v_n|^N\, dx.$$

Lemma 5.1. *Assume that V is continuous and (V_1) is satisfied. Then the following hold:*

(i) *We have*

$$\|v_n\|^\gamma \leq 1 + \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{N/\gamma}}(F+G) + o_n(1),$$

$$F = \omega_{N-1} n^{(1-\beta)(N-1)} e^{-n\frac{N}{N-1}}(1-e^{-n/N}),$$

$$G = \frac{1}{N^{(1-\beta)N}} \frac{\omega_{N-1} \quad n^{(N-1)(1-\beta)}e^{-n}}{N},$$

where $m = \max_{x\in\bar{B}} V(x)$. In addition,

$$\frac{1}{\|v_n\|^\gamma} \geq 1 - E(N;n;m),$$

$$E := E(N;n;m) = \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{N/\gamma}}(F+G) + o_n(1).$$

(ii) *For all x such that $|x| \leq e^{-n/N}$,*

$$\alpha_{N,\beta} w_n^{\gamma}(x) \geq n(1-E).$$

Proof. (i) We have

$$\|v_n\|^N = 1 + \int_B V(x)|v_n|^N \, dx \leq 1 + m \int_B |v_n|^N \, dx.$$

Hence

$$\|v_n\|^N \leq 1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{N/\gamma}} \times \left\{ \int_{e^{-n/N} \leq |x| \leq 1} \frac{\left(\log \frac{1}{|x|}\right)^{(1-\beta)N}}{n^{1-\beta}} \, dx + \int_{0 \leq |x| \leq e^{-n/N}} \frac{1}{N^{(1-\beta)N}} n^{(N-1)(1-\beta)} \, dx \right\}.$$

We have

$$\begin{aligned}
\int_{e^{-n/N} \leq |x| \leq 1} \frac{\left(\log \frac{1}{|x|}\right)^{(1-\beta)N}}{n^{1-\beta}} \, dx &= \frac{\omega_{N-1}}{n^{1-\beta}} \int_{e^{-n/N}}^{1} r^{N-1} \left(\log \frac{1}{r}\right)^{(1-\beta)N} dr \\
&\leq \frac{\omega_{N-1}}{n^{(1-\beta)}} e^{-n\frac{N}{N-1}} n^{(1-\beta)N} \int_{e^{-n/N}}^{1} dr \\
&= \omega_{N-1} n^{(1-\beta)(N-1)} e^{-n\frac{N}{N-1}} (1 - e^{-n/N}) = o_n(1).
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_{0 \leq |x| \leq e^{-n/N}} \frac{1}{N^{(1-\beta)N}} n^{(N-1)(1-\beta)} \, dx \\
&\qquad = \frac{1}{N^{(1-\beta)N}} \omega_{N-1} n^{(N-1)(1-\beta)} \int_0^{e^{-n/N}} r^{N-1} \, dr \\
&\qquad = \frac{1}{N^{(1-\beta)N}} \frac{\omega_{N-1} n^{(N-1)(1-\beta)} e^{-n}}{N} = o_n(1).
\end{aligned}$$

Hence,

$$\|v_n\|^N \leq 1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{N/\gamma}} (F+G).$$

Therefore,

$$\|v_n\|^{\gamma} \leq \left(1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{N/\gamma}} (F+G)\right)^{\frac{1}{(N-1)(1-\beta)}}.$$

So,

$$\|v_n\|^\gamma \le 1 + \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{N/\gamma}}(F+G) + o_n(1).$$

Consequently,

$$\frac{1}{\|v_n\|^\gamma} \ge 1 - E.$$

(ii) For all x such that $|x| \le e^{-n/N}$, we have

$$\alpha_{N,\beta} w_n^\gamma = \alpha_{N,\beta}\frac{|v_n|^\gamma}{\|v_n\|^\gamma} \ge n(1-E). \ \blacksquare$$

Now, we introduce the following elementary result.

Lemma 5.2. *For the sequence w_n induced by* (5.2), *we have*

$$\lim_{n\to\infty}\int_B e^{\alpha_{N,\beta}w_n^\gamma}\,dx \ge \frac{\omega_{N-1}}{N}(N+1). \tag{5.3}$$

Proof. Let

$$I_1 = \int_{e^{-n/N}\le|x|\le1} e^{\alpha_{N,\beta}w_n^\gamma}\,dx = \int_{e^{-n/N}\le|x|\le1} \exp\biggl(N^{N'}\frac{\bigl(\log\frac{1}{|x|}\bigr)^{N'}}{\|v_n\|^\gamma n^{1/(N-1)}}\biggr)\,dx,$$

$$I_2 = \int_{0\le|x|\le e^{-n/N}} e^{\alpha_{N,\beta}w_n^\gamma}\,dx.$$

Then

$$\lim_{n\to\infty}\int_B e^{\alpha_{N,\beta}|w_n|^\gamma}\,dx = \lim_{n\to\infty} I_1 + \lim_{n\to\infty} I_2.$$

On the one hand, from Lemma 5.1(ii) and using the fact that $nE \sim n^{N(1-\beta)+\beta}e^{-n}$,

$$I_2 = \int_{0\le|x|\le e^{-n/N}} e^{\alpha_{N,\beta}w_n^\gamma}\,dx \ge \int_{0\le|x|\le e^{-n/N}} e^{n(1-E)}\,dx$$
$$\ge \frac{\omega_{N-1}}{N}e^{-nE} \to \frac{\omega_{N-1}}{N} \quad \text{as } n\to\infty.$$

On the other hand,

$$I_1 = \int_{e^{-n/N}\le|x|\le1} \exp\biggl(N^{N'}\frac{\bigl(\log\frac{1}{|x|}\bigr)^{N'}}{n^{1/(N-1)}}(1-E)\biggr)\,dx,$$

so

$$I_1 = \omega_{N-1}\int_{e^{-n/N}}^{1} r^{N-1}\exp\biggl(N^{N'}\frac{\bigl(\log\frac{1}{r}\bigr)^{N'}}{n^{1/(N-1)}}(1-E)\biggr)\,dr.$$

We make the change of variable $|x| = r = e^{-t/N}$. Then we get

$$I_1 = \frac{\omega_{N-1}}{N} \int_0^n \exp\left(\frac{t^{N'}(1-E)}{n^{1/(N-1)}} - t \right) dt.$$

For any $n > 1$, let

$$\varphi_n(t) := \frac{t^{N'}(1-E)}{n^{1/(N-1)}} - t, \qquad t \geq 0.$$

The interval $[0, n]$ is then divided as follows:

$$[0, n] = [0, n^{1/N}] \cup [n^{1/N}, n - n^{1/N}] \cup [n - n^{1/N}, n].$$

First, we consider the interval $[0, n^{1/N}]$. Using the fact that $1 - E \leq 1$, we get

$$\begin{aligned} &\chi_{[0,n^{1/N}]} e^{\varphi_n(t)} \leq e^{1-t} \in L^1([0,\infty)), \\ &\chi_{[0,n^{1/N}]}(t) e^{\varphi_n(t)} \to e^{-t} \qquad \text{for a.e. } t \in [0,\infty) \text{ as } n \to \infty. \end{aligned}$$

Then, using the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^{n^{1/N}} e^{\varphi_n(t)}\, dt = \lim_{n\to\infty} \int_0^n \chi_{[0,n^{1/N}]} e^{\varphi_n(t)}\, dt = 1.$$

Now, we are going to study the limit of this integral on

$$[n^{1/N}, n - n^{1/N}] \quad \text{and} \quad \left[n - n^{1/N}, n\right].$$

So, we compute

$$\varphi_n(n^{1/N}) = 1 - E - n^{1/N} \leq 1 - n^{1/N} \qquad \text{for } n \text{ large enough}$$

and

$$\begin{aligned} \varphi_n(n^{1/N}) &\leq -\frac{1}{N-1} n^{1/N} \\ &= -(N'-1) n^{1/N} \qquad \text{for all } n \geq \left(\frac{N-1}{N-2}\right)^{1/N}. \end{aligned} \tag{5.4}$$

Also, for n large enough,

$$\begin{aligned} \varphi_n(n - n^{1/N}) &= \frac{(n - n^{1/N})^{N'}(1-E)}{n^{\frac{1}{N-1}}} - n + n^{1/N} \\ &\leq \frac{n^{\frac{N}{N-1}}(1 - n^{-1/N'})^{N'}}{n^{\frac{1}{N-1}}} - n + n^{1/N} \\ &= n\left(1 - \frac{N'}{n^{1/N'}} + o\left(\frac{1}{n^{1/N'}}\right)\right) - n + n^{1/N} \end{aligned}$$

$$= n\left(-\frac{N'}{n^{1/N'}} + \frac{1}{n^{1/N'}} + o\left(\frac{1}{n^{1/N'}}\right)\right)$$
$$= -N'n^{1/N} + o\left(\frac{1}{n^{1/N}}\right) \le -(N'-1)n^{1/N} + o\left(\frac{1}{n^{1/N}}\right).$$

Therefore, for every $\varepsilon \in (0,1)$ there exists $n_\varepsilon \ge 1$ such that

$$\varphi_n(n - n^{1/N}) \le -(N'-1)n^{1/N}(1-\varepsilon) \quad \text{for every } n \ge n_\varepsilon. \tag{5.5}$$

Fix n large enough. A qualitative study of φ_n in $[0,\infty)$ shows that there exists a unique $s_n \in (0,n)$ such that $\varphi_n'(s_n) = 0$ and consequently

$$\int_{n^{1/N}}^{n-n^{1/N}} e^{\varphi_n(t)}\,dt \le (n - 2n^{1/N})e^{\max[\varphi_n(n^{1/N}),\varphi_n(n-n^{1/N})]}.$$

In addition, from (5.4) and (5.5) with $\varepsilon < 1$, we obtain

$$\max\,[\varphi_n(n^{1/N}), \varphi_n(n-n^{1/N})] \le -\frac{1}{N-1}n^{1/N}$$

provided that n is large enough. Hence, there exists $\overline{n} \ge 1$ such that

$$\int_{n^{1/N}}^{n-n^{1/N}} e^{\varphi_n(t)}\,dt \le (n-2n^{1/N})e^{-\frac{1}{N-1}n^{1/N}} \quad \text{for all } n \ge \overline{n}.$$

Therefore,

$$\lim_{n\to\infty} \int_{n^{1/n}}^{n-n^{1/n}} \exp\left(\frac{t^{N'}}{n^{\frac{1}{N-1}}} - t\right) dt = 0.$$

Finally, we will study the limit on the interval $[n - n^{1/n}, n]$. We mention that for a fixed $n \ge 1$ large enough, φ_n is a convex function on $[n - n^{1/N}, \infty)$. Also, $\varphi_n(n) = n(1-E) - n \le 0$. Thus, we get the estimate

$$\varphi_n(t) \le \varphi_n(t) - \varphi_n(n) \le \frac{n-t}{n^{1/N}}\varphi_n(n-n^{1/N}), \quad t \in [n - n^{1/N}, n].$$

On the other hand, in view of (5.5), if $\varepsilon \in (0,1)$ and $n \ge n_\varepsilon$, we have

$$\varphi_n(t) \le (N'-1)(1-\varepsilon)(t-n), \quad t \in [n - n^{1/n}, n]. \tag{5.6}$$

Furthermore, using the fact that φ_n is convex on $[n - n^{1/N}, \infty)$ and $\varphi_n'(n) = N'(1-E) - 1$, we get

$$\begin{aligned} \varphi_n(t) &\ge \varphi_n(n) + \varphi_n'(n)(t-n) \\ &\ge (N'(1-E)-1)(t-n), \quad t \in [n-n^{1/N}, n]. \end{aligned} \tag{5.7}$$

Then by bringing together (5.6) and (5.7), we deduce

$$\lim_{n\to\infty}\frac{1-e^{-n^{1/N}}}{N'(1-E)-1}\le\lim_{n\to\infty}\int\limits_{n-n^{1/N}}^{n} e^{\varphi_n(t)}\,dt\le\lim_{n\to\infty}\frac{1-e^{-n^{1/N}}}{(N'-1)(1-\varepsilon)}.$$

Using the fact that $\lim_{n\to\infty}E=0$ and since ε is arbitrary , we get

$$\lim_{n\to\infty}\int\limits_{n-n^{1/N}}^{n} e^{\varphi_n(t)}dt=\frac{1}{N'-1}=N-1.$$

The lemma follows. ∎

LEMMA 5.3. *For the sequence* (w_n) *determined by* (5.2), *there exists* $n\ge1$ *such that*

$$\max_{t\ge0}\mathcal{J}(tw_n)<\frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}. \tag{5.8}$$

Proof. For contradiction, suppose that for all $n\ge1$,

$$\max_{t\ge0}\mathcal{J}(tw_n)\ge\frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

Then, for any $n\ge1$, there exists $t_n>0$ such that

$$\max_{t\ge0}\mathcal{J}(tw_n)=J(t_nw_n)\ge\frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}$$

and so

$$\frac{1}{N}t_n^N-\int\limits_B F(x,t_nw_n)\,dx\ge\frac{1}{N}\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

Then, by using $(H1)$,

$$t_n^N\ge\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}. \tag{5.9}$$

On the other hand,

$$\frac{d}{dt}\mathcal{J}(tw_n)\Big|_{t=t_n}=t_n^{N-1}-\int\limits_B f(x,t_nw_n)w_n\,dx=0,$$

that is,

$$t_n^N=\int\limits_B f(x,t_nw_n)t_nw_n\,dx. \tag{5.10}$$

Now, we claim that the sequence (t_n) is bounded in $(0,+\infty)$. Indeed, it follows from $(H5)$ that for all $\varepsilon>0$, there exists $t_\varepsilon>0$ such that

$$f(x,t)t\ge(\gamma_0-\varepsilon)e^{\alpha_0t^\gamma}\quad\forall|t|\ge t_\varepsilon,\text{ uniformly in }x\in B. \tag{5.11}$$

Using (5.10) and (5.11) we get

$$
\begin{aligned}
(5.12)\qquad t_n^N &\geq (\gamma_0 - \varepsilon) \int\limits_{0 \leq |x| \leq e^{-n/N}} e^{\alpha_0 t_n^\gamma w_n^\gamma}\, dx \\
&\geq (\gamma_0 - \varepsilon) \int\limits_{0 \leq |x| \leq e^{-n/N}} e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}} n(1-E)}\, dx \\
&= (\gamma_0 - \varepsilon) \frac{\omega_{N-1}}{N} e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}} n(1-E) - n}.
\end{aligned}
$$

Therefore (t_n) is bounded.

Also, from (5.9) we see that

$$\lim_{n\to\infty} t_n^N \geq \left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

Now, suppose that

$$\lim_{n\to\infty} t_n^N > \left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

Then for n large enough, there exists some $\delta > 0$ such that $t_n^\gamma \geq \frac{\alpha_{N,\beta}}{\alpha_0} + \delta$, and consequently the right hand side of (5.12) tends to infinity and this contradicts the boundedness of t_n. Since (t_n) is bounded, we thus get

$$(5.13)\qquad \lim_{n\to\infty} t_n^N = \left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma}.$$

Now, we want to use the expression of t_n^N given by (5.10) and the hypothesis (5.11). So let

$$B_{n,+} = \{x \in B : t_n v_n(x) \geq t_\varepsilon\}, \qquad B_{n,-} = \{x \in B : t_n v_n(x) < t_\varepsilon\}.$$

We have

$$\int\limits_B f(x, t_n w_n) t_n w_n\, dx = \int\limits_{B_{n,+}} f(x, t_n w_n) t_n w_n\, dx + \int\limits_{B_{n,-}} f(x, t_n w_n) t_n w_n\, dx.$$

Then

$$t_n^N \geq (\gamma_0 - \varepsilon) \int\limits_{B_{n,+}} e^{\alpha_0 t_n^\gamma w_n^\gamma}\, dx + \int\limits_{B_{n,-}} f(x, t_n w_n) t_n w_n\, dx,$$

and so

$$
\begin{aligned}
(5.14)\qquad t_n^N \geq{}& (\gamma_0 - \varepsilon) \int\limits_B e^{\alpha_0 t_n^\gamma w_n^\gamma}\, dx - (\gamma_0 - \varepsilon) \int\limits_{B_{n,-}} e^{\alpha_0 t_n^\gamma w_n^\gamma}\, dx \\
&+ \int\limits_{B_{n,-}} f(x, t_n w_n) t_n w_n\, dx.
\end{aligned}
$$

The sequence (w_n) converges to 0 in B and $\chi_{B_{n,-}}$ converges to 1 a.e. in B. By using the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{B_{n,-}} f(x, t_n w_n) t_n w_n \, dx = 0.$$

Also, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{B_{n,-}} e^{\alpha_0 t_n^\gamma w_n^\gamma} \, dx = \frac{\omega_{N-1}}{N}.$$

Using Lemma 5.2 and (5.7), we obtain

$$\lim_{n\to\infty} t_n^N \geq \lim_{n\to\infty} (\gamma_0 - \varepsilon) \int_B e^{\alpha_{N,\beta} t_n^\gamma w_n^\gamma} \, dx - (\gamma_0 - \varepsilon) \frac{\omega_{N-1}}{N} = (\gamma_0 - \varepsilon)\omega_{N-1}.$$

So, it follows that

$$\left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{N/\gamma} \geq (\gamma_0 - \varepsilon)\omega_{N-1} \tag{5.15}$$

for all $\varepsilon > 0$. Hence

$$\gamma_0 \leq \frac{(1-\beta)^{N-1} N^{(N-1)(1-\beta)+1}}{\alpha_0^{(N-1)(1-\beta)}},$$

which contradicts $(f5)$. Thus Theorem 1.4 is proved. ■

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