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# NON-COMPLETENESS OF SOME CONVERGENCE ON $l^1$

BY

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The aim of this paper is to solve the following problem posed by J. Mikusiński [1]:

In the space $l^1$ of absolutely summable sequences of real numbers we consider $p$-convergence: $x^n \twoheadrightarrow x$. This means that

(a) $\lim_n x_i^n = x_i$ for every coordinate $i$,

(b) in $l^1$ there is some $y$ such that $|x_i^n| \leqslant y_i$.

A sequence $(x_n)$ is called *p-Cauchy* iff $x^{n_{k+1}} - x^{n_k} \twoheadrightarrow 0$ whenever $(n_k)$ is an increasing sequence of integers. Is every $p$-Cauchy sequence $p$-convergent to some vector in $l^1$?

The answer is negative. We are going to show it by constructing a $p$-Cauchy sequence which is not $p$-convergent.

Denote by $\|\cdot\|_1$ the usual norm on $l^1$, thus

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| \quad \text{for } x = (x_i).$$

We first give a finite reformulation of the problem. It is convenient to introduce the following notation: If $x^1, x^2, \ldots, x^N$ is a finite set of vectors in $l^1$, define

$$a(x^1, x^2, \ldots, x^N) = \sum_i \max_{1 \leqslant n \leqslant N} |x_i^n|$$

and

$$\beta(x^1, x^2, \ldots, x^N) = \max_{1 \leqslant n \leqslant N} \|x^n\|_1 + \max_{1 \leqslant n_1 < n_2 < \ldots \leqslant N} a(x^{n_1} - x^{n_2}, x^{n_2} - x^{n_3}, \ldots).$$

Clearly, $\beta(x^1, x^2, \ldots, x^N) \leqslant 3a(x^1, x^2, \ldots, x^N)$.

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(1) P 1190 in Colloquium Mathematicum 43 (1980), p. 388.

Assume the following holds:

(*) For each $\varepsilon > 0$ there are vectors $x^1, x^2, \ldots, x^N$ in $l^1$ such that

$$\alpha(x^1, x^2, \ldots, x^N) = 1 \quad \text{and} \quad \beta(x^1, x^2, \ldots, x^N) < \varepsilon.$$

Then the required sequence can be constructed.

To see this, using (*) construct finite sequences $S_r = (x^{r,1}, \ldots, x^{r,N_r})$ in $l^1$ satisfying

1° $\alpha(S_r) = 1$ for each $r$,

2° $\beta(S_r) < 2^{-r}$ for each $r$,

3° the $S_r$ are disjointly supported

and take the sequence $S = (S_1, S_2, \ldots, S_r, \ldots)$ in $l^1$ obtained by piecing the $S_r$ together.

Then $S$ is $\|\cdot\|_1$-convergent to 0 but is not $p$-convergent since condition (b) is not fulfilled. Now, for any increasing sequence $(n_k)$ of integers, it is easily verified that for $S = (x^n)$

$$\sum_i \sup_k |x_i^{n_{k+1}} - x_i^{n_k}| \leqslant \sum_r 2\beta(S_r) < \infty,$$

and hence $x^{n_{k+1}} - x^{n_k} \twoheadrightarrow 0$.

Condition (*) has the following equivalent statement in terms of functions $f_1, f_2, \ldots, f_N$ in some Lebesgue space $L^1(\mu)$:

(**) For each $\varepsilon > 0$ there exist functions $f_1, f_2, \ldots, f_N$ in $L^1(\mu)$ such that

(a) $\int \max\limits_{1 \leqslant n \leqslant N} |f_n| \, d\mu = 1$,

(b) $\max\limits_{1 \leqslant n \leqslant N} \|f_n\|_1 < \varepsilon$,

(c) if $1 \leqslant n_1 < n_2 < \ldots \leqslant N$, then

$$\int \max_k |f_{n_{k+1}} - f_{n_k}| \, d\mu < \varepsilon.$$

Our measure space will be the circle $T$ equipped with Haar measure $m$. The functions $f_1, f_2, \ldots, f_N$ will be realized as the translations $(f_\varphi)$, $\varphi \in [0, 2\pi]$, of a function $f$ on $T$. Conditions (a), (b), and (c) of (**) become then

(a′) $\int \sup |f_\varphi| \, dm = 1$,

(b′) $\|f\|_1 < \varepsilon$,

(c′) if $0 \leqslant \varphi_1 < \varphi_2 < \ldots \leqslant 2\pi$, then

$$\int \max_k |f_{\varphi_{k+1}} - f_{\varphi_k}| \, dm < \varepsilon.$$

So it remains to construct the function $f$ and to show that (a′), (b′), (c′) hold.

Fix $\varepsilon > 0$ and let $d$ be an integer satisfying $d > 18/\varepsilon$. Proceeding by induction, we introduce $d$ positive continuous functions $f^1, f^2, \ldots, f^d$ on $T$ and numbers $1 > \delta_1 > \delta_2 > \ldots > \delta_d > 0$ such that

(i) $\|f^i\|_\infty = 1$ $(1 \leqslant i \leqslant d)$,

(ii) $f^i(1) = 1$ $(1 \leqslant i \leqslant d)$,

(iii) $\|f^{i+1}\|_1 < \delta_i/d^2$ $(1 \leqslant i < d)$,

(iv) $\|f^i - f^i_\psi\|_\infty < 1/d^2$ if $0 \leqslant \varphi < \psi \leqslant 2\pi$ and $|\varphi - \psi| \leqslant \delta_i$.

The fact that this can be done is straightforward and we let the verification to the reader.

Define now

$$f = \frac{1}{d}\sum_{i=1}^{d} f^i.$$

Since $f(1) = 1$, $\sup\limits_{\varphi} f_\varphi$ is the constant function 1 and (a′) holds. Also

$$\|f\|_1 \leqslant \frac{1}{d} + \frac{1}{d}\sum_{i=2}^{d}\frac{1}{d^2} < \frac{2}{d} < \varepsilon,$$

thus (b′) is satisfied.

The verification of (c′) is more complicated. Fix $0 \leqslant \varphi_1 < \varphi_2 < \ldots$ $\ldots < \varphi_K \leqslant 2\pi$. We introduce the following partition of the set $\{1, 2, \ldots, K\}$:

$$N_1 = \{k = 1, \ldots, K;\ \varphi_{k+1} - \varphi_k > \delta_1\},$$

$$N_i = \{k = 1, \ldots, K;\ \delta_{i-1} \geqslant \varphi_{k+1} - \varphi_k > \delta_i\} \quad (1 < i \leqslant d),$$

$$N_{d+1} = \{k = 1, \ldots, K;\ \delta_d \geqslant \varphi_{k+1} - \varphi_k\}.$$

It follows directly from this definition that $\operatorname{card}(N_i) < 2\pi/\delta_i$ for $i = 1, 2, \ldots, d$. Let us now estimate

$$\int \max_k |f_{\varphi_{k+1}} - f_{\varphi_k}| = \int \max_{1 \leqslant i \leqslant d+1} \max_{k \in N_i} |f_{\varphi_{k+1}} - f_{\varphi_k}|.$$

For each $i = 1, 2, \ldots, d$ we have

$$\begin{aligned}\max_{k\in N_i} |f_{\varphi_{k+1}} - f_{\varphi_k}| &\leqslant \frac{1}{d}\sum_{j=1}^{d}\max_{k\in N_i}|f^j_{\varphi_{k+1}} - f^j_{\varphi_k}|\\ &\leqslant \frac{1}{d}\sum_{j<i}\max_{k\in N_i}\|f^j_{\varphi_{k+1}} - f^j_{\varphi_k}\|_\infty + \frac{1}{d}\max_{k\in N_i}|f^i_{\varphi_{k+1}} - f^i_{\varphi_k}| +\\ &\quad + \frac{1}{d}\sum_{j>i}\max_{k\in N_i}(|f^j_{\varphi_k}| + |f^j_{\varphi_{k+1}}|).\end{aligned}$$

By construction, $\|f^j_{\varphi_{k+1}} - f^j_{\varphi_k}\|_\infty < 1/d^2$ for all $k \in N_i$ and $j < i$. Hence

$$\max_{k \in N_i} |f_{\varphi_{k+1}} - f_{\varphi_k}| < \frac{1}{d^2} + \frac{2}{d} + \frac{1}{d} \sum_{j > 1} \sum_{k \in N_i} (|f^j_{\varphi_k}| + |f^j_{\varphi_{k+1}}|),$$

and therefore

$$\max_{1 \leqslant i \leqslant d} \max_{k \in N_i} |f_{\varphi_{k+1}} - f_{\varphi_k}| \leqslant \frac{1}{d^2} + \frac{2}{d} + \frac{1}{d} \sum_{i=1}^{d} \sum_{j > i} \sum_{k \in N_i} (|f^j_{\varphi_k}| + |f^j_{\varphi_{k+1}}|).$$

Now, again by (iv) we see that

$$\|f_\varphi - f_\psi\|_\infty < \frac{1}{d^2} \quad \text{if } 0 \leqslant \varphi < \psi \leqslant 2\pi \text{ and } |\varphi - \psi| \leqslant \delta_d.$$

Thus

$$\max_{k \in N_{d+1}} |f_{\varphi_{k+1}} - f_{\varphi_k}| \leqslant \frac{1}{d^2}.$$

Combining these facts, we obtain

$$\int \max_k |f_{\varphi_{k+1}} - f_{\varphi_k}| \leqslant \frac{1}{d^2} + \frac{2}{d} + \frac{1}{d} \sum_{i=1}^{d} \sum_{j > i} \sum_{k \in N_i} \Big( \int |f^j_{\varphi_k}| + \int |f^j_{\varphi_{k+1}}| \Big)$$

$$\leqslant \frac{1}{d^2} + \frac{2}{d} + \frac{1}{d} \sum_{i=1}^{d} \sum_{j > i} 2 \operatorname{card}(N_i) \|f^j\|_1$$

$$\leqslant \frac{1}{d^2} + \frac{2}{d} + \frac{2}{d} \sum_{i=1}^{d} \sum_{j > i} \frac{2\pi}{\delta_i} \frac{\delta_{j-1}}{d^2} \leqslant \frac{1}{d^2} + \frac{2}{d} + \frac{4\pi}{d} < \varepsilon,$$

establishing (c′).

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