APPLICATIONES MATHEMATICAE
51,1 (2024), pp. 75–93

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SPECTRAL DECOMPOSITION OF THE FRACTIONAL STURM–LIOUVILLE OPERATOR WITH ρ-GENERALIZED DERIVATIVE

Abstract. We study the ρ-generalized fractional version of the classical Sturm–Liouville operator. We establish the existence and uniqueness of a boundary value problem for this operator with homogeneous boundary conditions. Using this problem, we present a spectral decomposition of the ρ-generalized fractional Sturm–Liouville operator.

1. Introduction. The Sturm–Liouville operator is a fundamental concept in the field of differential equations and mathematical physics. It is named after the mathematicians Jacques Charles François Sturm and Joseph Liouville, who made significant contributions to the theory. The operator is defined in the context of the Sturm–Liouville problem, a second-order linear ordinary differential equation with certain boundary conditions. In its most general form, the Sturm–Liouville operator takes the following form:

$$Lu = -(p(x)u'(x))' + q(x)u(x), \quad x \in (a,b),$$

where p, q are known functions defined on a specified interval (a,b).

The Sturm–Liouville operator plays a crucial role in the theory of orthogonal functions and eigenvalue problems. It is closely related to self-adjoint differential operators and has many desirable properties, such as orthogonality of eigenfunctions and a real and discrete spectrum of eigenvalues.

2020 *Mathematics Subject Classification*: Primary 26A33; Secondary 34K08, 34A08, 34Bxx, 47B40.
Key words and phrases: generalized fractional derivative, boundary value problem, fractional Sturm–Liouville operator.
Received 9 June 2023; revised 2 November 2023.
Published online 22 July 2024.

DOI: 10.4064/am2482-5-2024

The classical Sturm–Liouville operator has a spectral decomposition, when it is expressed as the sum or integral of eigenfunctions, with associated eigenvalues. One may study and resolve Sturm–Liouville issues using the spectral decomposition (see for example [B10]).

The extension of the Sturm–Liouville operator to fractional calculus involves considering fractional order differential operators instead of traditional integer order operators. Fractional calculus deals with derivatives and integrals of non-integer order, allowing for a more nuanced analysis of systems with fractional dynamics.

A first version of the fractional Sturm–Liouville operator was defined by Rivero et al. [RTV13] as follows:

$$L_\alpha u(x) = D^\alpha_{b^-}(p(x) D^\alpha_{a^+} u(x)) + q(x)u(x), \qquad x \in (a, b),$$

where $D^\alpha_{a^+}, D^\alpha_{b^-}$ are the left and right Riemann–Liouville derivatives of order α and $p(x)$ and $q(x)$ are continuous functions on $[a, b]$.

Next, the authors of [KA13] gave another version:

$$L^c_\alpha u(x) = {}^c D^\alpha_{b^-}(p(x) D^\alpha_{a^+} u(x)) + q(x)u(x), \qquad x \in (a, b),$$

where $D^\alpha_{a^+}$ is the left Riemann–Liouville derivative and ${}^c D^\alpha_{b^-}$ is the right Caputo derivative of order α on (a, b). They studied some properties of the eigenvalues of L_α. Moreover, an application of this method to the Legendre equation was found.

In [MNT17], Muensawat et al. solved a system of fractional differential equations, using a version of the fractional Sturm–Liouville operator defined by

$$L_{\alpha,\beta} u(x) = {}^c D^\beta_{a^+}(p(x)\, {}^c D^\alpha_{a^+} + q(x))u(x), \qquad x \in [0, X].$$

This operator was employed in [TD$^+$20] to resolve a boundary value problem

$$\begin{cases} {}^c D^\beta_{a^+}(p(x)\, {}^c D^\alpha_{a^+} + q(x))u(x) = f(x, u(x)), & x \in [0, X], \\ u(0) + g(u) = u_0 \in \mathbb{R}, \quad {}^c D^\beta_{a^+} u(X) + h(u) = u_1 \in \mathbb{R}, \end{cases}$$

where $\alpha, \beta \in (0, 1], {}^c D^\beta_{a^+}, {}^c D^\alpha_{a^+}$ are the Caputo fractional derivatives, $p \in C([0, X], \mathbb{R})$ with $|p| \geq K > 0$, $q \in C([0, X], \mathbb{R})$, $g, h : C([0, X], \mathbb{R}) \to \mathbb{R}$ are continuous functions and $f \in C([0, X] \times \mathbb{R}, \mathbb{R})$ is continuous function.

For more examples, we refer for instance to [A09, H11, JR12, KN16, KOM14].

We are interested in the spectral decomposition of the following fractional Sturm–Liouville operator:

$$L^\rho_\alpha u(x) = {}^\rho D^\alpha_{b^-}(p(x)\, {}^\rho D^\alpha_{a^+} u(x)) + q(x)u(x), \qquad x \in (a, b),$$

where $0 < \alpha < 1$ and ${}^\rho D^\alpha_{a^+}$ and ${}^\rho D^\alpha_{b^-}$ are the left and right fractional derivatives, which were introduced by Katugampola [K14] on a bounded interval (a, b).

The spectral decomposition of the fractional Sturm–Liouville operator involves finding a set of eigenfunctions $\{w_n\}$ and eigenvalues $\{\lambda_n\}$ such that

$$L^\rho_\alpha w_n = \lambda_n w_n,$$

with appropriate boundary conditions.

The eigenfunctions are typically chosen to be orthogonal with respect to a suitable inner product.

The paper is organized as follows: In Section 2, we provide the necessary preliminaries, focusing on important spaces and concepts related to fractional calculus. In Section 3, we introduce and define the fractional Sobolev spaces ${}^\rho W^{\alpha,p}_{a^+}(a,b)$ $(0 < \alpha < 1,\ 1 \le p < +\infty)$ on the bounded interval (a,b), which is the functional framework of two-point ρ-fractional boundary value problems. In Section 4, we address the existence and uniqueness of solutions of equations associated with the operator L^ρ_α. Lastly, in the final section we introduce the spectral decomposition of the operator L^ρ_α, by unveiling its eigenfunctions and eigenvalues.

2. Preliminaries. In the rest of this work, we assume that $0 < \alpha < 1$, $\rho > 0$, $1 \le p < +\infty$ and $0 < a < b < +\infty$. So, we can write

$$m_\rho \le x^{\rho-1} \le M_\rho, \qquad x \in [a,b], \tag{2.1}$$

where

$$m_\rho = a^{\rho-1}, \quad M_\rho = b^{\rho-1} \quad \text{if } \rho \ge 1,$$
$$m_\rho = b^{\rho-1}, \quad M_\rho = a^{\rho-1} \quad \text{if } 0 < \rho < 1.$$

Let $AC^p(a,b)$ denote the space of all measurable functions f on (a,b) such that there exist $c \in \mathbb{R}$ and $\varphi \in L^p(a,b)$ satisfying

$$f(x) = c + \int_a^x \varphi(t)\,dt$$

for all $x \in (a,b)$.

Definition 2.1 ([K14]). The *left* and *right* *ρ-generalized fractional integrals of order* α of a function $f \in L^p(a,b)$ are defined, respectively, as follows:

$$({}^\rho I^\alpha_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t) t^{\rho-1}\,dt, \tag{2.2}$$

$$({}^\rho I^\alpha_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\alpha-1} f(t) t^{\rho-1}\,dt. \tag{2.3}$$

Remark 2.2. We have the following particular cases:

(i) If $\rho = 1$, we obtain the fractional integrals of Riemann–Liouville:

$$(I^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{2.4}$$

$$(I^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt. \tag{2.5}$$

(ii) If $\rho \to 0^+$, we obtain the fractional integrals of Hadamard:

$$({}^H I^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t)\,\frac{dt}{t}, \tag{2.6}$$

$$({}^H I^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t)\,\frac{dt}{t}. \tag{2.7}$$

Indeed, by employing the theorem of finite increments to the function t^ρ/ρ $(0 < \rho < 1)$, there exists $\tau \in [t, x]$ such that

$$\frac{x^\rho - t^\rho}{\rho} = \tau^{\rho-1}(x-t).$$

Then

$$\begin{aligned}
\left| \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} t^{\rho-1} f(t) \right| &= \tau^{(\alpha-1)(\rho-1)} t^{\rho-1} (x-t)^{\alpha-1} |f(t)| \\
&\le b^{(\alpha-1)(\rho-1)} a^{\rho-1} (x-t)^{\alpha-1} |f(t)| \\
&= (ab^{\alpha-1})^{\rho-1} (x-t)^{\alpha-1} |f(t)| \\
&\le \max\left\{1, \frac{1}{ab^{\alpha-1}}\right\} (x-t)^{\alpha-1} |f(t)|.
\end{aligned}$$

Since $\max\left\{1, \frac{1}{ab^{\alpha-1}}\right\} (x-t)^{\alpha-1}|f(t)| \in L^p(a,b)$, we apply Lebesgue's theorem and taking into account that

$$\lim_{\rho \to 0^+} \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} t^{\rho-1} f(t) = \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t},$$

we deduce that

$$\lim_{\rho \to 0^+} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} t^{\rho-1} f(t)\,dt = \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t)\,\frac{dt}{t}.$$

Relation (2.7) is proven in the same way.

PROPOSITION 2.3. *For all* $f \in L^p(a,b)$, *we have* ${}^\rho I^\alpha_{a+} f, {}^\rho I^\alpha_{b-} f \in L^p(a,b)$. *Moreover,*

$$\|{}^\rho I^\alpha_{a+} f\|_{L^p(a,b)} \le \frac{M_\rho (b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}, \tag{2.8}$$

$$\|{}^\rho I^\alpha_{b-} f\|_{L^p(a,b)} \le \frac{M_\rho (b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}, \tag{2.9}$$

where m_ρ, M_ρ *are given by* (2.1).

Proof. Let $f \in L^p(a,b)$. Then for all $x \in (a,b)$ we have

$$|({}^\rho I^\alpha_{a+} f)(x)| = \frac{1}{\Gamma(\alpha)}\left|\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t)t^{\rho-1}\,dt\right|.$$

As mentioned in Remark 2.2, there exists $\tau \in [t,x]$ such that

$$\begin{aligned}|({}^\rho I^\alpha_{a+} f)(x)| &\le \frac{1}{\Gamma(\alpha)}\int_a^x (\tau^{\rho-1})^{\alpha-1}(x-t)^{\alpha-1}|f(t)|t^{\rho-1}\,dt \\ &\le \frac{m_\rho^{(\alpha-1)} M_\rho}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t)|\,dt \\ &= \frac{M_\rho}{m_\rho^{1-\alpha}}(I^\alpha_{a+}|f|)(x).\end{aligned}$$

Hence,

$$\|{}^\rho I^\alpha_{a+} f\|_{L^p(a,b)} \le \frac{M_\rho}{m_\rho^{1-\alpha}}\big\|I^\alpha_{a+}|f|\big\|_{L^p(a,b)}.$$

Using [K06, (2.1.23)], we obtain

$$\|{}^\rho I^\alpha_{a+} f\|_{L^p(a,b)} \le \frac{M_\rho (b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\big\||f|\big\|_{L^p(a,b)} = \frac{M_\rho (b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}.$$

The second relation is proved in the same way. ■

The following proposition provides a similar formulation of integration by parts and can be found in [JA20].

PROPOSITION 2.4. *Let* $1 \le p, q \le +\infty$ *be such that* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$. *Then, for all* $f \in L^p(a,b)$ *and* $g \in L^q(a,b)$,

$$\int_a^b f(x)({}^\rho I^\alpha_{b-} g)(x)x^{\rho-1}\,dx = \int_a^b g(x)({}^\rho I^\alpha_{a+} f)(x)x^{\rho-1}\,dx. \tag{2.10}$$

DEFINITION 2.5. We denote $\delta_\rho = x^{1-\rho}\frac{d}{dx}$. The *left* and *right* ρ*-fractional derivatives of order* α of $f \in AC^p(a,b)$ are given by

$$({}^\rho D^\alpha_{a^+} f)(x) = \delta_\rho({}^\rho I^{1-\alpha}_{a^+} f)(x) = \frac{x^{1-\rho}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t)t^{\rho-1}\,dt, \tag{2.11}$$

$$({}^\rho D^\alpha_{b^-} f)(x) = (-\delta_\rho)({}^\rho I^{1-\alpha}_{b^-} f)(x) = -\frac{x^{1-\rho}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\alpha-1} f(t)t^{\rho-1}\,dt. \tag{2.12}$$

DEFINITION 2.6.

$$ {}^\rho AC^{\alpha,p}_{a^+}(a,b) = \left\{ \begin{matrix} u \mid \exists A \in \mathbb{R},\ \exists \xi \in L^p(a,b),\ \forall x \in (a,b): \\ u(x) = \frac{A}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a^+}\xi)(x) \end{matrix} \right\}, \tag{2.13}$$

$$ {}^\rho AC^{\alpha,p}_{a^+}(a,b) = \left\{ \begin{matrix} u \mid \exists B \in \mathbb{R},\ \exists \zeta \in L^p(a,b),\ \forall x \in (a,b): \\ u(x) = \frac{B}{\Gamma(\alpha)}\left(\frac{b^\rho - x^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{b^-}\zeta)(x) \end{matrix} \right\}. \tag{2.14}$$

The following theorem give a characterization of ${}^\rho AC^{\alpha,p}_{a^+}(a,b)$ and ${}^\rho AC^{\alpha,p}_{b^-}(a,b)$. The proof is the same as that of [DW13, Theorem 7].

THEOREM 2.7. *We have the following formulas:*

(i) *If* $f \in AC^{\alpha,p}_{a^+}(a,b)$, *then*

$$f(x) = \frac{({}^\rho I^{1-\alpha}_{a^+} f)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a^+}\, {}^\rho D^\alpha_{a^+} f)(x).$$

(ii) *If* $f \in AC^{\alpha,p}_{b^-}(a,b)$, *then*

$$f(x) = \frac{({}^\rho I^{1-\alpha}_{b^-} f)(b)}{\Gamma(\alpha)}\left(\frac{b^\rho - x^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{b^-}\, {}^\rho D^\alpha_{b^-} f)(x).$$

Now, we present a version of integration by parts which generalizes [DW13, (34)]:

THEOREM 2.8. *Let* $1 \le p, q < +\infty$ *be such that* $1/p < \alpha$ *and* $1/q < \alpha$. *Then, for all* $f \in {}^\rho AC^{\alpha,p}_{a^+}(a,b)$ *and* $g \in {}^\rho AC^{\alpha,q}_{b^-}(a,b)$,

$$\int_a^b f(x)({}^\rho D^\alpha_{b^-} g)(x)x^{\rho-1}\,dx = g(a)({}^\rho I^{1-\alpha}_{a^+} f)(a) - f(b)({}^\rho I^{1-\alpha}_{b^-} g)(b) + \int_a^b ({}^\rho D^\alpha_{a^+} f)(x)g(x)x^{\rho-1}\,dx. \tag{2.15}$$

Proof. We have

$$f(x) = \frac{A}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a+}\xi)(x),$$
$$g(x) = \frac{B}{\Gamma(\alpha)}\left(\frac{b^\rho - x^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{b-}\zeta)(x),$$

where $\xi, \zeta \in L^p(a,b)$ and

$$A = ({}^\rho I^{1-\alpha}_{a^+} f)(a), \qquad B = ({}^\rho I^{1-\alpha}_{b^-} g)(a).$$

Then

$$\begin{aligned}
&\int_a^b f(x)({}^\rho D^\alpha_{b-} g)(x) x^{\rho-1}\, dx \\
&\quad = \int_a^b \left[\frac{A}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a+}\xi)(x)\right]\zeta(x) x^{\rho-1}\, dx \\
&\quad = \frac{A}{\Gamma(\alpha)}\int_a^b \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1}\zeta(x) x^{\rho-1}\, dx + \int_a^b ({}^\rho I^\alpha_{a+}\xi)(x)\zeta(x) x^{\rho-1}\, dx \\
&\quad = A({}^\rho I^\alpha_{b-}\zeta)(a) + \int_a^b ({}^\rho I^\alpha_{a+}\xi)(x)\zeta(x) x^{\rho-1}\, dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_a^b ({}^\rho D^\alpha_{a+} f)(x) g(x) x^{\rho-1}\, dx \\
&\quad = \int_a^b \xi(x)\left[\frac{B}{\Gamma(\alpha)}\left(\frac{b^\rho - x^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{b-}\zeta)(x)\right] x^{\rho-1}\, dx \\
&\quad = \frac{B}{\Gamma(\alpha)}\int_a^b \left(\frac{b^\rho - x^\rho}{\rho}\right)^{\alpha-1}\xi(x) x^{\rho-1}\, dx + \int_a^b \xi(x)({}^\rho I^\alpha_{b-}\zeta)(x) x^{\rho-1}\, dx \\
&\quad = B({}^\rho I^\alpha_{a+}\xi)(b) + \int_a^b ({}^\rho I^\alpha_{a+}\xi)(x)\zeta(x) x^{\rho-1}\, dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_a^b f(x)({}^\rho D^\alpha_{b-} g)(x) x^{\rho-1}\, dx = &\int_a^b ({}^\rho D^\alpha_{a+} f)(x) g(x) x^{\rho-1}\, dx \\
&+ A({}^\rho I^\alpha_{b-}\zeta)(a) - B({}^\rho I^\alpha_{a+}\xi)(b).
\end{aligned}$$

Note that

$$A({}^\rho I^\alpha_{b-}\zeta)(a) = Ag(a) - \frac{AB}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1},$$
$$B({}^\sigma I^\alpha_{a+}\xi)(b) = Bf(b) - \frac{AB}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1}.$$

Therefore,

$$\begin{aligned} A({}^\rho I^\alpha_{b-}\zeta)(b) - B({}^\rho I^\alpha_{a+}\xi)(b) &= Ag(a) - Bf(b) \\ &= ({}^\rho I^{1-\alpha}_{a^+} f)(a)g(a) - ({}^\rho I^{1-\alpha}_{b^-} g)(b)f(b), \end{aligned}$$

and we arrive at (2.15). ■

3. Fractional Sobolev spaces with ρ-generalized operator

DEFINITION 3.1. A *fractional Sobolev space via the ρ-generalized operator* is defined as follows:

$$(3.1)\quad {}^\rho W^{\alpha,p}_{a^+}(a,b) = \begin{Bmatrix} u \in L^p(a,b) \mid \exists g \in L^p(a,b),\ \forall \varphi \in C^\infty_c(a,b): \\ \int_a^b u(x)({}^\rho D^\alpha_{b^-}\varphi)(x)x^{\rho-1}\,dx = \int_a^b g(x)\varphi(x)x^{\rho-1}\,dx \end{Bmatrix},$$

where $C^\infty_c(a,b)$ denotes the space of $C^\infty(a,b)$ functions with compact support.

PROPOSITION 3.2. *The function g in* (3.1) *coincides with ${}^\rho D^\alpha_{a^+}u$ a.e. in (a,b).*

Proof. Let $u \in {}^\rho W^{\alpha,p}_{a^+}(a,b)$ and $\varphi \in C^\infty_c(a,b)$. Then $({}^\rho D^\alpha_{b^-}\varphi)(x) \in L^q(a,b)$ for $1/p + 1/q = 1 \le 1+\alpha$, $\varphi(a) = \varphi(b) = 0$, and we have

$$({}^\rho D^\alpha_{b^-}\varphi)(x) = -\delta_\rho({}^\rho I^{1-\alpha}_{b^-}\varphi)(x) = -({}^\rho I^{1-\alpha}_{b^-}\delta_\rho\varphi)(x).$$

So,

$$\int_a^b u(x)({}^\rho D^\alpha_{b^-}\varphi)(x)x^{\rho-1}\,dx = -\int_a^b u(x)\,{}^\rho I^{1-\alpha}_{b^-}(t^{1-\rho}\varphi'(t))(x)x^{\rho-1}\,dx.$$

Applying (2.15), we obtain

$$\begin{aligned} \int_a^b u(x)({}^\rho D^\alpha_{b^-}\varphi)(x)x^{\rho-1}\,dx &= -\int_a^b ({}^\rho I^{1-\alpha}_{a^+}u)(x)(x^{1-\rho}\varphi'(x))x^{\rho-1}\,dx \\ &= -\int_a^b ({}^\rho I^{1-\alpha}_{a^+}u)(x)\varphi'(x)\,dx. \end{aligned}$$

Using the classical integration by parts, we get

$$\int_a^b u(x)({}^\rho D^\alpha_{b-}\varphi)(x)x^{\rho-1}\,dx$$
$$= [-({}^\rho I^{1-\alpha}_{a+}u)(x)\varphi(x)]_a^b + \int_a^b \left[\frac{d}{dx}({}^\rho I^{1-\alpha}_{a+}u)\right](x)\varphi(x)\,dx$$
$$= \int_a^b (\delta_\rho I^{1-\alpha}_{a+}u)(x)\varphi(x)x^{\rho-1}\,dx = \int_a^b ({}^\rho D^\alpha_{a+}u)(x)\varphi(x)x^{\rho-1}\,dx.$$

But we have

$$\int_a^b u(x)({}^\rho D^\alpha_{b-}\varphi)(x)x^{\rho-1}\,dx = \int_a^b g(x)\varphi(x)x^{\rho-1}\,dx.$$

Thus,

$$g(x) = ({}^\rho D^\alpha_{a+}u)(x) \quad \text{a.e. in } (a,b). \blacksquare$$

The following theorem characterizes the space ${}^\rho W^{\alpha,p}_{a+}(a,b)$.

THEOREM 3.3. *We have* ${}^\rho W^{\alpha,p}_{a+}(a,b) = {}^\rho AC^{\alpha,p}_{a+}(a,b) \cap L^p(a,b)$. *Moreover, for all* $u \in {}^\rho W^{\alpha,p}_{a+}(a,b)$,

$$u(x) = \frac{({}^\rho I^{1-\alpha}_{a+}u)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a+}\,{}^\rho D^\alpha_{a+}u)(x). \tag{3.2}$$

Proof. Let $u \in {}^\rho W^{\alpha,p}_{a+}(a,b)$. Then $u, {}^\rho D^\alpha_{a+}u \in L^p(a,b)$ and we have

$$({}^\rho I^\alpha_{a+}\,{}^\rho D^\alpha_{a+}u)(x) = u(x) - \frac{({}^\rho I^{1-\alpha}_{a+}u)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1}.$$

Hence,

$$u\,(x) = \frac{({}^\rho I^{1-\alpha}_{a+}u)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a+}\,{}^\rho D^\alpha_{a+}u)(x).$$

Putting $A = ({}^\rho I^{1-\alpha}_{a+}u)(a)$ and $\xi = {}^\rho D^\alpha_{a+}u$, we obtain the direct inclusion.

Conversely, let $u \in {}^\rho AC^{\alpha,p}_{a+}(a,b) \cap L^p(a,b)$. Then $u \in L^p(a,b)$ and there exist $A \in \mathbb{R}$ and $\xi \in L^p(a,b)$ such that

$$u(x) = \frac{A}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + ({}^\rho I^\alpha_{a+}\xi)(x).$$

Hence, ${}^\rho D^\alpha_{a+}u = \xi \in L^p(a,b)$, which gives the reverse inclusion. $\blacksquare$

Using (2.8) and the above result, we get

COROLLARY 3.4. *For all* $u \in {}^{\rho}W^{\alpha,p}_{a^+}(a,b)$,

$$\left\| u - \frac{({}^{\rho}I^{1-\alpha}_{a^+}u)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right\|_{L^p(a,b)} \leq \frac{M_\rho (b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\|{}^{\rho}D^{\alpha}_{a^+}u\|_{L^p(a,b)}, \tag{3.3}$$

where m_ρ, M_ρ *are given by* (2.1).

REMARK 3.5. If u takes the form (3.2), then $u \in L^p(a,b)$ if and only if

$$\frac{({}^{\rho}I^{1-\alpha}_{a^+}u)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} \in L^p(a,b).$$

Hence:

(i) If $(1-\alpha)p \geq 1$, then $u \in L^p(a,b)$ if and only if $({}^{\rho}I^{1-\alpha}_{a^+}u)(a) = 0$.
(ii) If $({}^{\rho}I^{1-\alpha}_{a^+}u)(a) \neq 0$, then $u \in L^p(a,b)$ if and only if $(1-\alpha)p < 1$.

DEFINITION 3.6. In ${}^{\rho}W^{\alpha,p}_{a^+}(a,b)$, we define two norms:

$$^1\|u\|^p_{{}^{\rho}W^{\alpha,p}_{a^+}(a,b)} = \|u\|^p_{L^p(a,b)} + \|{}^{\rho}D^{\alpha}_{a^+}u\|^p_{L^p(a,b)}, \tag{3.4}$$

$$^2\|u\|^p_{{}^{\rho}W^{\alpha,p}_{a^+}(a,b)} = |{}^{\rho}I^{1-\alpha}_{a^+}u(a)|^p + \|{}^{\rho}D^{\alpha}_{a^+}u\|^p_{L^p(a,b)}. \tag{3.5}$$

THEOREM 3.7. *The norm* $^1\|\cdot\|_{{}^{\rho}W^{\alpha,p}_{a^+}(a,b)}$ *is equivalent to* $^2\|\cdot\|_{{}^{\rho}W^{\alpha,p}_{a^+}(a,b)}$.

Proof. Let $u \in {}^{\rho}W^{\alpha,p}_{a^+}(a,b)$. We distinguish two cases.
(i) If $1-\alpha p < 1$, then according to (3.2), we get

$$u = \frac{{}^{\rho}I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + {}^{\rho}I^{\alpha}_{a^+}\,{}^{\rho}D^{\alpha}_{a^+}u(x).$$

Using the same arguments as for [DW13, Theorem 24], we obtain

$$\begin{aligned}\|u\|^p_{L^p(a,b)} &= \int_a^b \left| \frac{{}^{\rho}I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + {}^{\rho}I^{\alpha}_{a^+}\,{}^{\rho}D^{\alpha}_{a^+}u(x)\right|^p dx \\ &\leq 2^{p-1}\left|\frac{{}^{\rho}I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}\right|^p \int_a^b x^{1-\rho}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(\alpha-1)p} x^{\rho-1}\,dx \\ &\quad + 2^{p-1}\|{}^{\rho}I^{\alpha}_{a^+}\,{}^{\rho}D^{\alpha}_{a^+}u\|^p_{L^p(a,b)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|u\|^p_{L^p(a,b)} \leq{}& \frac{2^{p-1}M^p_\rho}{((\alpha-1)p+1)\Gamma^p(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{(\alpha-1)p+1} |{}^{\rho}I^{1-\alpha}_{a^+}u(a)|^p \\ &+ \frac{2^{p-1}M^p_\rho (b-a)^\alpha}{m_\rho^{(1-\alpha)p}\Gamma(2-\alpha)}\|{}^{\rho}D^{\alpha}_{a^+}u\|^p_{L^p(a,b)},\end{aligned}$$

where m_ρ and M_ρ are given by (2.1). Then

$$\begin{aligned}
{}^1\|u\|^p_{{}^\rho W^{\alpha,p}_{a^+}(a,b)} &= \|u\|^p_{L^p(a,b)} + \|{}^\rho D^\alpha_{a+}u\|^p_{L^p(a,b)} \\
&\le \frac{2^{p-1}M^p_\rho}{((\alpha-1)p+1)\Gamma^p(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{(\alpha-1)p+1} |{}^\rho I^{1-\alpha}_{a^+}u(a)|^p \\
&\quad + \left(1+\frac{2^{p-1}M^p_\rho(b-a)^\alpha}{m_\rho^{(1-\alpha)p}\Gamma(\alpha+1)}\right)\|{}^\rho D^\alpha_{a+}u\|^p_{L^p(a,b)} \\
&\le C_1(|{}^\rho I^{1-\alpha}_{a^+}u(b)|^p + \|{}^\rho D^\alpha_{a+}u\|^p_{L^p(a,b)}) \\
&= C_1\,{}^2\|u\|^p_{{}^\rho W^{\alpha,p}_{a^+}(a,b)},
\end{aligned}$$

where

$$C_1 = \max\left\{\frac{2^{p-1}M^p_\rho}{((\alpha-1)p+1)\Gamma^p(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{(\alpha-1)p+1}, 1+\frac{2^{p-1}m^p_\rho(b-a)^\alpha}{m_\rho^{1-\alpha}\Gamma(\alpha+1)}\right\}.$$

Conversely, according to the mean value theorem there exists $x_0 \in [a,b]$ such that

$${}^\rho I^{1-\alpha}_{a^+}u(x_0) = \frac{1}{b-a}\int_a^b {}^\rho I^{1-\alpha}_{a^+}u(x)\,dx.$$

Note that

$$\begin{aligned}
{}^\rho I^{1-\alpha}_{a^+}u(a) &= {}^\rho I^{1-\alpha}_{a^+}u(x_0) - \int_a^{x_o}\frac{d}{dx}\,{}^\rho I^{1-\alpha}_{a^+}u(x)\,dx \\
&= \frac{1}{b-a}\int_a^b {}^\rho I^{1-\alpha}_{a^+}u(x)\,dx - \int_a^{x_0}\delta_\rho\,{}^\rho I^{1-\alpha}_{a^+}u(x)x^{\rho-1}\,dx \\
&= \frac{1}{b-a}\int_a^b {}^\rho I^{1-\alpha}_{a^+}u(x)\,dx - \int_a^{x_0}{}^\rho D^\alpha_{a+}u(x)x^{\rho-1}\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
|{}^\rho I^{1-\alpha}_{a^+}u(a)| &\le \frac{1}{b-a}\int_a^b|{}^\rho I^{1-\alpha}_{a^+}u(x)|\,dx + \int_a^{x_0}|{}^\rho D^\alpha_{a+}u(x)|x^{\rho-1}\,dx \\
&\le \frac{1}{b-a}\|{}^\rho I^{1-\alpha}_{a^+}u\|_{L^1(a,b)} + M^{\rho-1}\|{}^\rho D^\alpha_{a+}u\|_{L^1(a,b)}.
\end{aligned}$$

Using the Hölder inequality, we obtain

$$|{}^\rho I^{1-\alpha}_{a^+}u(a)| \le \frac{1}{(b-a)^{1/p}}\|{}^\rho I^{1-\alpha}_{a^+}u\|_{L^p(a,b)} + M_\rho(b-a)^{1-1/p}\|{}^\rho D^\alpha_{a+}u\|_{L^p(a,b)}.$$

Applying (2.8) to the first term on the right, we obtain

$$|{}^\rho I^{1-\alpha}_{a^+}u(a)| \le \frac{M_\rho(b-a)^{1-\alpha-1/p}}{m^\alpha_\rho\Gamma(2-\alpha)}\|u\|_{L^p(a,b)} + M_\rho(b-a)^{1-1/p}\|{}^\rho D^\alpha_{a+}u\|_{L^p(a,b)}.$$

Then

$$|{}^\rho I_{a^+}^{1-\alpha}u(a)|^p \le 2^{p-1}\frac{M_\rho^p(b-a)^{(1-\alpha)p-1}}{m_\rho^{\alpha p}\Gamma^p(2-\alpha)}\|u\|_{L^p(a,b)}^p + 2^{p-1}M_\rho^p(b-a)^{p-1}\|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p.$$

We deduce that

$$\begin{aligned}
{}^2\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p &= |{}^\rho I_{a^+}^{1-\alpha}u(a)|^p + \|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p,\\
&\le \frac{2^{p-1}M_\rho^p(b-a)^{(1-\alpha)p-1}}{m_\rho^{\alpha p}\Gamma^p(2-\alpha)}\|u\|_{L^p(a,b)}^p\\
&\quad + (1+2^{p-1}M_\rho^p(b-a)^{p-1})\|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p\\
&\le C_2(\|u\|_{L^p(a,b)}^p + \|{}^\rho D_{a^+}^\alpha u\|_{L^1(a,b)}^p)\\
&= C_2^2\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p,
\end{aligned}$$

where

$$C_2 = \max\left\{\frac{2^{p-1}M_\rho^p(b-a)^{(1-\alpha)p-1}}{m_\rho^{\alpha p}\Gamma^p(2-\alpha)}, 1+2^{p-1}M_\rho^p(b-a)^{p-1}\right\}.$$

(ii) If $1-\alpha p \ge 1$, from Remark 3.5 we deduce that $({}^\rho I_{a^+}^{1-\alpha}u)(a) = 0$. Thus,

$${}^2\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p = \|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p \le {}^1\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p$$

and

$$\begin{aligned}
{}^1\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p &= \|u\|_{L^p(a,b)}^p + \|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p\\
&= \|{}^\rho I_{a^+}^\alpha\, {}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p + \|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p\\
&\le \left(1+\frac{M_\rho^p(b-a)^{\alpha p}}{m_\rho^{(1-\alpha)p}\Gamma^p(\alpha+1)}\right)\|{}^\rho D_{a^+}^\alpha u\|_{L^p(a,b)}^p\\
&= \left(1+\frac{M_\rho^p(b-a)^{\alpha p}}{m_\rho^{(1-\alpha)p}\Gamma^p(\alpha+1)}\right){}^2\|u\|_{{}^\rho W_{a^+}^{\alpha,p}(a,b)}^p.
\end{aligned}$$

Hence, we get the result. ■

THEOREM 3.8. *If $\alpha p > 1$, then the following set is closed:*

$$H_p^\rho = \{v \in {}^\rho W_{a^+}^{\alpha,p}(a,b) \mid {}^\rho I_{a^+}^{1-\alpha}v(a) = v(b) = 0\}. \tag{3.6}$$

Proof. Let (v_n) be a sequence in H_p^ρ that converges to v in ${}^\rho W_{a^+}^{\alpha,p}(a,b)$. Then $0 = {}^\rho I_{a^+}^{1-\alpha}v_n(a)$ converges to ${}^\rho I_{a^+}^{1-\alpha}v(a) = 0$.

On the other hand, ${}^2\|v_n\|_{{}^\rho W^{\alpha,p}_{a^+}(a,b)} = \|{}^\rho D^\alpha_{a^+} v_n\|^p_{L^p(a,b)}$ is bounded and we have $v_n(b) = {}^\rho I^\alpha_{a^+}({}^\rho D^\alpha_{a^+} v_n)(b)$. Thus,

$$\begin{aligned} |v_n(b)| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^b \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\alpha-1} {}^\rho D^\alpha_{a^+} v_n(t) t^{\rho-1}\,dt\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^b \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\alpha-1} |{}^\sigma D^\alpha_{a^+} v_n(t)| t^{\rho-1}\,dt. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} |v_n(b)| &\le \frac{1}{\Gamma(\alpha)}\left(\int_a^b \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\frac{(\alpha-1)p}{p-1}} t^{\frac{(\rho-1)p}{p-1}}\,dt\right)^{\frac{p-1}{p}} \left(\int_a^b |\,{}^\rho D^\alpha_{a^+} v_n(t)|^p\,dt\right)^{1/p} \\ &= \frac{1}{\Gamma(\alpha)}\left(\int_a^b \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\frac{(\alpha-1)p}{p-1}} t^{\rho-1} t^{\frac{\rho-1}{p-1}}\,dt\right)^{\frac{p-1}{p}} \|{}^\rho D^\alpha_{a^+} v_n\|_{L^p(a,b)} \\ &\le \frac{(p-1)M^{\frac{\rho-1}{p}}(b^\rho - a^\rho)^{\frac{\alpha p-1}{p}}}{(\alpha p-1)^{\frac{p-1}{p}}\Gamma(\alpha)} \|v_n\|_{{}^\rho W^{\alpha,p}_{a^+}(a,b)}. \end{aligned}$$

We deduce that $(v_n(b))$ is bounded, so we can extract a subsequence $v_{n_l}(b)$ that converges to $v(b) = 0$. Consequently, H^ρ_p is closed. ∎

THEOREM 3.9. *For all $1 \le q \le p$ such that $(1-\alpha)q < 1$ we have a compact embedding ${}^\rho W^{\alpha,p}_{a^+}(a,b) \hookrightarrow {}^\rho W^{\alpha,q}_{a^+}(a,b)$.*

Proof. Let (u_n) be a bounded sequence in ${}^\rho W^{\alpha,p}_{a^+}(a,b)$. Then from (3.2) we have

$$u_n(x) = \frac{{}^\sigma I^{1-\alpha}_{a^+} u_n(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + {}^\sigma I^\alpha_{a^+}({}^\rho D^\alpha_{a^+} u_n)(x),$$

and

$$\|u_n\|^p_{{}^\rho W^\alpha_{a^+}(a,b)} = |{}^\rho I^{1-\alpha}_{a^+} u_n(a)|^p + \|{}^\sigma D^\alpha_{a^+} u_n\|^p_{L^p(a,b)}.$$

From the continuity of ${}^\rho I^{1-\alpha}_{a^+}$, and the completeness of $\mathbb{R}$ and $L^p(a,b)$ we can extract a subsequence (u_{nk}) such that $({}^\rho I^{1-\alpha}_{a^+} u_n(a))$ converges to ${}^\rho I^{1-\alpha}_{a^+} u(a)$ in $\mathbb{R}$ and $({}^\sigma D^\alpha_{a^+} u_n)$ converges to φ in $L^p(a,b) \subset L^q(a,b)$.

Thus, $({}^\sigma D^\alpha_{a^+} u_n)$ converges to φ in $L^q(a,b)$ and we have

$$u(x) = \frac{{}^\sigma I^{1-\alpha}_{a^+} u(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + {}^\sigma I^\alpha_{a^+}({}^\rho D^\alpha_{a^+} \varphi)(x),$$

so $u \in {}^\rho W^{\alpha,q}_{a^+}(a,b)$ and $\varphi = {}^\rho D^\alpha_{a^+} u$. Hence, ${}^\rho W^{\alpha,p}_{a^+}(a,b) \hookrightarrow {}^\rho W^{\alpha,q}_{a^+}(a,b)$ compactly. ∎

The previous theorem yields

Corollary 3.10. *For any* $1 \le q \le p$ *such that* $(1-\alpha)q < 1$, *we have a compact embedding* ${}^\rho W^{\alpha,p}_{a^+}(a,b) \hookrightarrow L^q(a,b)$.

4. A fractional Sturm–Liouville problem with homogeneous boundary conditions. Consider the following two-point problem of fractional Sturm–Liouville type:

$$(P)\qquad \begin{cases} ({}^\rho D^\alpha_{b-}(\mu(x)\,{}^\rho D^\alpha_{a+}u))(x) + \lambda(x)u(x) = f(x) & \text{in } (a,b),\\ ({}^\rho I^{1-\alpha}_{a^+}u)(a) = u(b) = 0, \end{cases}$$

where $1/2 < \alpha < 1$, $\mu, \lambda \in L^\infty(a,b)$ and $f \in L^2(a,b)$.

Let H_ρ be the closed subspace of $W^{\alpha,2}_{a^+,\rho}(a,b)$ defined by

$$H_\rho = \{v \in W^{\alpha,2}_{a^+,\rho}(a,b) \mid ({}^\rho I^\alpha_{a+}v)(a) = 0,\ v(b) = 0\}. \tag{4.1}$$

Multiplying the first equation of (P) by a function $x^{\rho-1}v$ for $v \in H_\rho$, and integrating over (a,b), we get

$$\int_a^b ({}^\rho D^\alpha_{b-}(\mu(x)\,{}^\rho D^\alpha_{a+}u))(x)v(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u(x)v(x)x^{\rho-1}\,dx = \int_a^b f(x)v(x)x^{\rho-1}\,dx.$$

By using formula (2.8) of integration by parts in the first term, and taking into account $v \in H_\rho$ and the boundary conditions, we obtain the following variational problem:

$$(PV)\qquad \int_a^b \mu(x)({}^\rho D^\alpha_{a+}u)(x)({}^\rho D^\alpha_{a+}v)(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u(x)v(x)x^{\rho-1}\,dx = \int_a^b f(x)v(x)x^{\rho-1}\,dx.$$

Conversely, let $u \in H_\rho$ and suppose that for all $\varphi \in C^\infty_c(a,b)$ we have

$$\int_a^b \mu(x)({}^\rho D^\alpha_{a+}u)(x)({}^\rho D^\alpha_{a+}\varphi)(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u(x)\varphi(x)x^{\rho-1}\,dx = \int_a^b f(x)\varphi(x)x^{\rho-1}\,dx,$$

which can be written as

$$\int_a^b [({}^\rho D^\alpha_{b-}\mu(x)\,{}^\rho D^\alpha_{a+}u)(x) + \lambda(x)u(x) - f(x)]\varphi(x)x^{\rho-1}\,dx = 0,\ \forall\varphi \in C^\infty_c(a,b).$$

We deduce that

$$({}^\rho D^\alpha_{b-}\, {}^\rho D^\alpha_{a+} u)(x) + \lambda(x)u(x) = f(x) \quad \text{a.e. in } (a,b).$$

The following theorem ensures the existence and uniqueness of solutions of problem (PV).

THEOREM 4.1. *Assume that there exist $\mu_0, \lambda_0 > 0$ such that*

$$\mu(x) \geq \mu_0, \qquad \lambda(x) \geq \lambda_0 \qquad \textit{a.e. in } (a,b). \tag{4.2}$$

Then problem (PV) has a unique solution $u \in H_\rho$.

Proof. Set

$$\begin{aligned} A_\rho(u,v) = &\int_a^b ({}^\rho D^\alpha_{a+} u)(x)({}^\rho D^\alpha_{a+} v)(x) x^{\rho-1}\,dx \\ &+ \int_a^b \lambda(x)u(x)v(x)x^{\rho-1}\,dx, \qquad u,v \in H_\rho, \end{aligned}$$

and

$$L_\rho(v) = \int_a^b f(x)v(x)x^{\rho-1}\,dx, \qquad v \in H_\rho.$$

Then A_ρ is a bilinear form and L_ρ is a linear form, and we have

$$\begin{aligned} |A_\rho(u,v)| &\leq \int_a^b \mu(x)|({}^\rho D^\alpha_{a+} u)(x)({}^\rho D^\alpha_{a+} v)(x)| x^{\rho-1}\,dx \\ &\quad + \int_a^b \lambda(x)|u(x)v(x)|x^{\rho-1}\,dx \\ &\leq M_\rho\big(\|\mu\|_{L^\infty(a,b)} \|{}^\rho D^\alpha_{a+} u\|_{L^2(a,b)} \|{}^\rho D^\alpha_{a+} v\|_{L^2(a,b)} \\ &\quad + \|\lambda\|_{L^\infty(a,b)} \|u\|_{L^2(a,b)} \|v\|_{L^2(a,b)}\big) \\ &\leq M_\rho \max\{\|\mu\|_{L^\infty(a,b)}, \|\lambda\|_{L^\infty(a,b)}\} \big(\|u\|_{L^2(a,b)}\|v\|_{L^2(a,b)} \\ &\quad + \|{}^\rho D^\alpha_{a+} u\|_{L^2(a,b)} \|{}^\rho D^\alpha_{a+} v\|_{L^2(a,b)}\big) \\ &\leq M_\rho \max\{\|\mu\|_{L^\infty(a,b)}, \|\lambda\|_{L^\infty(a,b)}\} \|u\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)} \|v\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)}, \end{aligned}$$

and

$$|L_\rho(v)| \leq M_\rho \|f\|_{L^2(a,b)} \|v\|_{L^2(a,b)} \leq M_\rho \|f\|_{L^2(a,b)} \|v\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)},$$

where m_ρ and M_ρ are given by (2.1).

Thus, A_ρ and L_ρ are continuous.

Now, let $u \in H_\rho$. Then

$$\begin{aligned} A(u,u) &= \int_a^b \mu(x)({}^\rho D^\alpha_{a+}u)^2(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u^2(x)x^{\rho-1}\,dx, \\ &\geq m_\rho(\mu_0\|{}^\rho D^\alpha_{a+}u\|^2_{L^2(a,b)} + \lambda_0\|u\|^2_{L^2(a,b)}) \\ &\geq m_\rho \min\{\mu_0,\lambda_0\}\,\|u\|^2_{{}^\rho W^{\alpha,2}_{a+}(a,b)}. \end{aligned}$$

Hence, A_ρ is coercive. According to the Lax–Milgram theorem, problem (PV) admits a unique solution $u \in H_\rho$. ■

5. Spectral decomposition of a Sturm–Liouville operator of fractional type. In this section, we introduce a decomposition of a ρ-generalized Sturm–Liouville operator, defined by

$$L^\rho_\alpha u(x) = ({}^\rho D^\alpha_{b-}(\mu(x)\,{}^\rho D^\alpha_{a+}u))(x) + \lambda(x)u(x), \qquad x \in (a,b), \tag{5.1}$$

where ρ, α, μ and λ are defined in the preceeding section.

The main result of this section is the following theorem:

THEOREM 5.1. *Let $1/2 < \alpha < 1$ and let $\mu, \lambda \in L^\infty(a,b)$ satisfy (4.2). Then there exists a real sequence $\{\eta_n\}_{n=1}^\infty$ and a Hilbert basis $\{w_n\}_{n=1}^\infty$ of $L^2(a,b)$ such that $\lim_{n\to\infty}\eta_n = +\infty$ and*

$$\begin{cases} ({}^\rho D^\alpha_{b-}(\mu(x)\,{}^\rho D^\alpha_{a+}w_n))(x) + \lambda(x)w_n(x) = \eta_n w_n(x) \quad \text{in } (a,b), \\ ({}^\rho I^{1-\alpha}_{a^+}w_n)(a) = 0, \quad w_n(b) = 0. \end{cases}$$

$\{\eta_n\}_{n=1}^\infty$ is a sequence of eigenvalues of L^ρ_α defined by (5.1), the fractional Sturm–Liouville operator of ρ-generalized type, and $\{w_n\}_{n=1}^\infty$ is a sequence of associated eigenvectors.

Proof. Consider the boundary value problem (P), introduced in the preceding section, where μ, λ satisfy (4.2) and $f \in L^2(a,b)$. We know from Theorem 4.1 that problem (P) admits a unique solution u which belongs to the closed subspace H_ρ of ${}^\rho W^{\alpha,2}_{a^+}(a,b)$, given by (4.1). So, H_ρ is closed in $L^2(a,b)$.

Let T be the operator from $L^2(a,b)$ to $L^2(a,b)$ that associates to $f \in L^2(a,b)$ the function u solving problem (P).

We will prove that it has the properties guaranteeing the existence of a spectral decomposition.

(i) *T is continuous from $L^2(a,b)$ to ${}^\rho W^{\alpha,2}_{a^+}(a,b)$*: Since $(I^{1-\alpha}_{a^+}(\cdot))(a) = 0$ in H_ρ, we can consider $\|{}^\rho D^\alpha_{a+} u\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)}$ to be a norm on H_ρ. Then

$$\begin{aligned}\min\{\mu_0,\lambda_0\}\,\|u\|^2_{{}^\rho W^{\alpha,2}_{a+}(a,b)} &\le \int_a^b \mu(x)({}^\rho D^\alpha_{a+}u)^2(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u^2(x)x^{\rho-1}\,dx\\ &= \int_a^b f(x)u(x)x^{\rho-1}\,dx \le M_\rho\|f\|_{L^2(a,b)}\|u\|_{L^2(a,b)}\\ &\le M_\rho\|f\|_{L^2(a,b)}\|u\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)}.\end{aligned}$$

Thus, $\|u\|_{{}^\rho W^{\alpha,2}_{a+}(a,b)} \le \frac{M_\rho}{\min\{\mu_0,\lambda_0\}}\|f\|_{L^2(a,b)}$, so T is continuous.

(ii) *T is compact*: As T is continuous from $L^2(a,b)$ in ${}^\rho W^{\alpha,2}_{a^+}(a,b)$ and from Corollary 3.10 we have ${}^\rho W^{\alpha,2}_{a^+}(a,b) \hookrightarrow L^2(a,b)$ compactly, the operator T is compact from $L^2(a,b)$ into $L^2(a,b)$.

(iii) *T is self-adjoint*: Let $f,g \in L^2(a,b)$ and let $u = Tf, v = Tg$ be solutions of problem (P) associated with f,g respectively. Since $u,v \in H_\rho$, we can write

$$\int_a^b \mu(x)({}^\rho D^\alpha_{a+}u)(x)({}^\rho D^\alpha_{a+}v)(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)u(x)v(x)x^{\rho-1}\,dx = \int_a^b f(x)v(x)x^{\rho-1}\,dx$$

and

$$\int_a^b \mu(x)({}^\rho D^{\alpha,\rho}_{a^+}v)(x)({}^\rho D^\alpha_{a+}u)(x)x^{\rho-1}\,dx + \int_a^b \lambda(x)v(x)u(x)x^{\rho-1}\,dx = \int_a^b g(x)v(x)x^{\rho-1}\,dx.$$

Thus,

$$\int_a^b f(x)v(x)x^{\rho-1}\,dx = \int_a^b g(x)u(x)x^{\rho-1}\,dx,$$

i.e. $(Tf,g)_{L^2(a,b)} = (f,Tg)_{L^2(a,b)}$, where

$$(f,g) = \int_a^b f(x)g(x)x^{\rho-1}\,dx, \qquad f,g \in L^2(a,b).$$

Hence, T is self-adjoint.

(iv) $f \equiv 0$ if and only if $u \equiv 0$. Thus, $\ker(T) = \{0\}$.

According to [B10, Theorem 6.11], there exists a sequence of distinct, positive $\{\kappa_n\}_{n=1}^{\infty}$, and a Hilbertian basis $\{w_n\}_{n=1}^{\infty} \subset L^2(a,b)$ such that

$$\begin{cases} ({}^\rho D_{b^-}^{\alpha}(\mu(x)\, {}^\rho D_{a^+}^{\alpha} \kappa_n w_n))(x) + \lambda(x)\kappa_n w_n(x) = w_n(x) \quad \text{in } (a,b), \\ ({}^\rho I_{a^+}^{1-\alpha} \kappa_n w_n)(a) = \kappa_n w_n(b) = 0, \end{cases}$$

where $\lim_{n\to\infty} \kappa_n = 0^+$. Setting $\eta_n = 1/\kappa_n$, we get

$$\begin{cases} ({}^\rho D_{b^-}^{\alpha}(\mu(x)\, {}^\rho D_{a^+}^{\alpha} w_n))(x) + \lambda(x) w_n(x) = \eta_n w_n(x) \quad \text{in } (a,b), \\ ({}^\rho I_{a^+}^{1-\alpha} w_n)(a) = 0,\ w_n(b) = 0, \end{cases}$$

where $\lim_{n\to\infty} \eta_n = +\infty$. ∎

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