APPLICATIONES MATHEMATICAE
51,1 (2024), pp. 13–46

Mykola Krasnoshchok (Sloviansk)

ON A ONE-DIMENSIONAL TIME-FRACTIONAL STEFAN PROBLEM

Abstract. We consider a one-dimensional one-phase Stefan problem with the Riemann–Liouville fractional derivatives. By reducing to a nonlinear integral equation, the problem is shown to admit a local in time classical solution.

1. Introduction. By the Stefan problem one usually means the class of mathematical models describing diffusion (heat) processes accompanied by phase changes of the media and absorption of latent heat. The boundary between two phases is unknown beforehand and must be determined together with the solution of the diffusion (heat) equation [8, 5]. Although a number of theoretical results related to the Stefan problem have been reported in the case of classical Fick's (Fourier's) law [21, 18, 3, 26], the same cannot be said of anomalous diffusion. Derivations of physical models are presented in [15, 24]. In [15, 25, 28, 17, 2] self-similar solutions were found for some specific initial and boundary conditions. In [14, 22] the authors studied fractional heat equation in prescribed non-cylindrical domains. To the author's best knowledge there are no results published on the solvability of the time-fractional Stefan problem even in the one-dimensional case.

Before formulating the problem, we recall the definitions of the left-sided fractional Riemann–Liouville integrals and derivatives of order $\alpha \in (0, 1)$:

2020 *Mathematics Subject Classification*: Primary 35R35; Secondary 35R11, 45G15.
Key words and phrases: free boundary, Stefan problem, fractional calculus, layer potential, integral equation.
Received 6 December 2023; revised 17 April 2024.
Published online 22 July 2024.

DOI: 10.4064/am2508-5-2024

$$
\begin{aligned}
I^{\alpha}_{t_0+}f(t) &= \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau,\\
D^{\alpha}_{t_0+}f(t) &= \frac{d}{dt}I^{1-\alpha}_{t_0+}f(t),
\end{aligned} \tag{1.1}
$$

where $\Gamma(\alpha)$ is Euler's gamma function.

In its simplest form, the Stefan problem consists in determining two functions $\varrho(t)$ and $u(x,t)$ such that the pair (ϱ,u) satisfies the equation

$$
u_t(x,t) - u_{xx}(x,t) = 0 \quad \text{in } 0 < x < \varrho(t),\ 0 < t \le \sigma, \tag{1.2}
$$

the boundary and initial conditions

$$
\begin{aligned}
&u(0,t) = f(t) > 0, \quad u(\varrho(t),t) = 0, \quad 0 \le t \le \sigma,\\
&u(x,0) = \varphi(x) \quad \text{for } 0 \le x \le b, \quad \varrho(0) = b,
\end{aligned} \tag{1.3}
$$

and the free boundary (Stefan) condition

$$
\dot{\varrho}(t) = -u_x(\varrho(t),t) \quad \text{for } 0 < t \le \sigma, \tag{1.4}
$$

where σ is a positive number.

Remark 1.1. The strong maximum principle and the Hopf lemma imply that

$$
\dot{\varrho}(t) = -u_x(\varrho(t),t) > 0 \quad \text{for } 0 < t \le \sigma,
$$

which means that ϱ must be a strictly increasing function of t. Therefore there exists an inverse function $l(x)$ of $\varrho(t)$ on the interval $(\varrho(0),\varrho(\sigma)]$. We put $l(x) = 0$ when $x \in [0,\varrho(0)]$.

The heat equation (1.2) can be derived by application of the energy conservation law and Fourier's law

$$
q(x,t) = -u_x(x,t), \tag{1.5}
$$

where q is the heat flux and u is the temperature of a slab. In this paper we use the following nonlocal generalization of (1.5) [19, 20, 15]:

$$
q(x,t) = -D^{1-\alpha}_{0+}u_x(x,t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_0^t \frac{u_x(x,\tau)}{(t-\tau)^{\alpha}}\,d\tau,\ \alpha \in (0,1) \tag{1.6}
$$

(see [28] for applications). In order to reformulate problem (1.2)–(1.4) in the moving domain we note that $q(x,t)$ in (1.5) makes sense only if $x \le \varrho(\tau)$ for all $\tau \in (0,t)$. From the previous study (see Remark 1.1) and explicit forms of self-similar solutions in [25] one can expect that the free boundary $\varrho(t)$ is strictly increasing. So the expression (1.6) should be replaced in a natural way by

$$
q(x,t) = -D^{1-\alpha}_{l(x)+}u_x(x,t) \tag{1.7}
$$

(cf. [15, 24, 28]).

Finally, the fractional Stefan problem is formulated as follows. We wish to find a pair (ϱ, u) that satisfies

$$
(1.8)\qquad
\begin{aligned}
&u_t(x,t) = (D^{1-\alpha}_{l(x)+} u_x)_x(x,t) \quad \text{in } 0 < x < \varrho(t),\ 0 < t \le \sigma,\\
&u(0,t) = f(t) > 0, \quad u(\varrho(t),t) = 0, \quad 0 \le t \le \sigma,\\
&u(x,0) = \varphi(x) \quad \text{for } 0 \le x \le b, \quad \varrho(0) = b,\\
&\dot{\varrho}(t) = -\lim_{x\to\varrho(t)-} D^{1-\alpha}_{l(x)+} u_x(x,t) \quad \text{for } 0 < t \le \sigma.
\end{aligned}
$$

The aim of this paper is to prove a local existence theorem for problem (1.8). The actual proof will be carried out by means of a layer potential technique apparently similar to that of A. Friedman [6]. This approach involves three steps. In the first step one obtains an integral representation $u(x,t) = \mathcal{U}[\varphi, f, \varrho, u_x(\varrho(\cdot),\cdot)](x,t)$ for the solution of (1.2)–(1.4). The second step is devoted to the integral equation

$$
v(t) = \mathcal{U}_x[\varphi, f, \varrho, v](\varrho(t),t), \quad \text{where} \quad \varrho(t) = b - \int_0^t v(s)\,ds.
$$

The third step is the verification that $v(t)$, $\varrho(t)$, $\mathcal{U}[\varphi, f, \varrho, v](x,t)$ satisfy (1.2)–(1.4). In our case, we also start with the integral representation of the form

$$
(1.9)\qquad u(x,t) = \mathcal{U}\Big[\varphi, f, \varrho, \lim_{x\to\varrho(t)-} D^{1-\alpha}_{l(x)+} u_x(x,t)\Big](x,t).
$$

However, it is impossible to calculate $D^{1-\alpha}_{l(x)+}\mathcal{U}_x[\dots](x,t)$, since both ϱ and l are unknown. Therefore, in the second step we consider an integral equation that gives

$$
(1.10)\quad \mathrm{v}(t) = \lim_{x\to\rho(t)-} D^{1-\alpha}_{0+}\mathcal{U}_x[\varphi, f, \rho, \mathrm{v}](\rho(t),t) \quad \text{and} \quad \rho(t) = b - \int_0^t \mathrm{v}(s)\,ds.
$$

In the final part we introduce the function

$$
\mathrm{u}(x,t) = \mathcal{U}[\varphi, f, \rho, \mathrm{v}](x,t)
$$

and prove that

$$
(1.11)\qquad \mathrm{u}(x,t) = 0 \quad \text{in } \hat{Q}^{\rho}_{\sigma} = \{(x,t) : x > \rho(t),\ t \in (0,\sigma)\}.
$$

This yields

$$
D^{1-\alpha}_{l(x)+}\mathrm{u}_x(x,t) = D^{1-\alpha}_{0+}\mathrm{u}_x(x,t)(x,t) \qquad \text{in } Q^{\rho}_{\sigma} = \{(x,t) : x \in (0,\rho(t)),\ t \in (0,\sigma)\},
$$

and allows us to verify that v, ρ, u in fact satisfy (1.8).

The paper is organized as follows. In Section 2 we state the main result. In Section 3 we collect some basic facts about the Wright functions and the fundamental solutions of time-fractional diffusion equations. Sections 4 and 5 are devoted to the integral representation (1.9) of u and its derivatives. Section 6 gives a well-posedness result for (1.10). Finally, in Section 7 we establish (1.11) and finish the proof of the main result. This paper is an extension of the study initiated in [11].

2. Main result. Denote

$$|f|_\sigma := \|f\|_{C([0,\sigma])} = \sup_{t\in(0,\sigma)} |f(t)|.$$

We consider functions defined on $(0,\sigma]$ which have (or whose derivatives have) a prescribed singularity at $t=0$. The space $C_{1-\alpha}([0,\sigma])$ is defined by

$$C_{1-\alpha}([0,\sigma]) = \{f(t) : t^{1-\alpha} f(t) \in C([0,\sigma])\}$$

with the norm

$$|f|_{\sigma,1-\alpha} := \|f\|_{C_{1-\alpha}([0,\sigma])} = \sup_{t\in(0,\sigma)} |t^{1-\alpha} f(t)|,$$

while the space $C^1_{1-\alpha}([0,\sigma])$ is defined by

$$C^1_{1-\alpha}([0,\sigma]) = \{f(t) : f(t) \in C([0,\sigma]),\ t^{1-\alpha}\dot{f}(t) \in C([0,\sigma])\}$$

with the norm

$$|f|^{(1)}_{\sigma,1-\alpha} := \|f\|_{C^1_{1-\alpha}([0,\sigma])} = |f|_\sigma + |\dot{f}|_{\sigma,1-\alpha}.$$

Lemma 2.1. *If $f \in C^1_{1-\alpha}([0,\sigma])$ then f is absolutely continuous on $[0,\sigma]$.*

The proof is straightforward.

We will assume that the data in (1.8) are subject to the following conditions:

(H1) $b>0$, $\varphi \in C^2([0,b])$, $\varphi(b)=0$, $\dot{\varphi}(b)<0$;
(H2) $f \in C^1_{1-\alpha}([0,\sigma])$;
(H3) $f(0)=\varphi(0)$.

Denote

$$M_f = |f|^{(1)}_{\sigma,1-\alpha},\ M_0 = \sup_{x\in(0,b)} |\varphi(x)|,\ M_1 = \sup_{x\in(0,b)} |\dot{\varphi}(x)|,\ M_2 = \sup_{x\in(0,b)} |\ddot{\varphi}(x)|.$$

Definition 2.2. We say that the pair (ϱ, u) is a solution of problem (1.8) for all $t\in(0,\sigma)$ if

(A1) $\varrho \in C^1_{1-\alpha}([0,\sigma])$, $\dot{\varrho}>0$ for all $t\in(0,\varrho)$, so there exist its inverse $l(x)$: $\varrho(l(x)) = x$ if $x>b$, and $l(x)=0$ if $x\in[0,b]$;
(A2) $u(x,t)$ is continuous for $0\le x\le \varrho(t)$, $0\le t\le\sigma$;
(A3) $u_x(x,t)$ is continuous for $0\le x\le\varrho(t)$, $0<t\le\sigma$, and for $0\le x<b$, $0\le t\le\sigma$; and $u_x(x,t)$ is bounded over $0\le x\le\varrho(t)$, $0\le t\le\sigma$;

(A4) for any $x \in [0, \varrho(\sigma)]$, $I^{\alpha}_{l(x)+} u_x(x,t)$ is continuous in t over $[l(x), \sigma]$, and

$$v(t) = \lim_{x \to \varrho(t)-} D^{1-\alpha}_{l(x)+} u_x(x,t) \in C_{1-\alpha}([0,\sigma]);$$

(A5) for any $x \in [0, b]$, $I^{\alpha}_{0+} u_x(x,t) \in C^1_{1-\alpha}([0,\sigma])$;

(A6) $D^{1-\alpha}_{l(x)+} u_x(x,t)$ is continuous for $0 \le x \le \varrho(t)$, $0 < t \le \sigma$;

(A7) $u_t(x,t)$, $(D^{1-\alpha}_{l(x)+} u_x)_x(x,t)$ are continuous for $0 < x < \varrho(t)$, $0 < t \le \sigma$;

(A8) the equations (1.8) are satisfied.

The main result is the following.

THEOREM 2.3. *Let assumptions* (H1)–(H3) *be satisfied. Then for sufficiently small σ there exists a solution* (ϱ, u) *of problem* (1.8).

3. Preliminaries. First of all define the right-sided Riemann–Liouville integrals

$$I^{\alpha}_{T-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(\tau)}{(\tau - t)^{1-\alpha}}\, d\tau,$$

$$D^{\alpha}_{T-} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{f(\tau)}{(\tau - t)^{\alpha}}\, d\tau.$$

Denote by $\partial^{\alpha}_{0+} f(t)$ the left-sided Caputo derivative of order $\alpha \in (0,1)$:

$$\partial^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot f(\tau)}{(t-\tau)^{\alpha}}\, d\tau.$$

Define the Wright functions (see [23, formula (7)])

$$\phi(-\beta, \delta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\delta - \beta n)}, \qquad \beta \in (0,1),\ \delta \in \mathbb{R}.$$

Throughout this work, the symbols C and κ will denote generic positive constants, depending only on the structural quantities of the problem.

LEMMA 3.1 ([23, §2, §3]). (1) *For all $\nu, c, z > 0$ and $\delta \in \mathbb{R}$ the following formulas hold:*

$$\frac{d}{dz} \phi(-\beta, \delta; -z) = -\phi(-\beta, \delta - \beta; -z), \tag{3.1}$$

$$\int_0^{\infty} z^{\nu-1} \phi(-\beta, \delta; -z)\, dz = \frac{\Gamma(\nu)}{\Gamma(\beta\nu + \delta)}, \tag{3.2}$$

$$I^{\nu}_{0+} t^{\delta-1} \phi(-\beta, \delta; -ct^{-\beta}) = t^{\delta+\nu-1} \phi(-\beta, \delta + \nu; -ct^{-\beta}), \tag{3.3}$$

$$D^{\nu}_{0+} t^{\delta-1} \phi(-\beta, \delta; -ct^{-\beta}) = t^{\delta-\nu-1} \phi(-\beta, \delta - \nu; -ct^{-\beta}). \tag{3.4}$$

(2) *For all* $z, \delta > 0$,

$$0 < \phi(-\beta, \delta; -z) \le c \exp(-\kappa z^{\frac{1}{1-\beta}}), \tag{3.5}$$

and for all $z > 0$,

$$0 < \phi(-\beta, 0; -z) \le c\, z \exp(-\kappa z^{\frac{1}{1-\beta}}). \tag{3.6}$$

Consider the Cauchy problems

$$\begin{aligned} &\partial^{\alpha}_{0+} \mathrm{w}(x,t) = \mathrm{w}_{xx}(x,t) + g(x,t), \quad x \in \mathbb{R},\ t > 0, \\ &\mathrm{w}(x,0) = \mathrm{w}_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &w_t(x,t) = (D^{1-\alpha}_{0+} w_x)_x(x,t) + g(x,t), \quad x \in \mathbb{R},\ t > 0, \\ &w(x,0) = w_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.8}$$

Define the kernels

$$K(x,t) = \tfrac{1}{2} t^{\alpha/2-1} \phi(-\alpha/2, \alpha/2; -|x| t^{-\alpha/2}), \tag{3.9}$$

$$\mathcal{K}(x,t) = \tfrac{1}{2} t^{-\alpha/2} \phi(-\alpha/2, 1-\alpha/2; -|x| t^{-\alpha/2}). \tag{3.10}$$

They are the fundamental solutions to problems (3.7) and (3.8) [13] in the sense that

$$\mathrm{w}(x,t) = \int_0^t d\tau \int_{\mathbb{R}} K(x-y, t-\tau) g(y,\tau)\, dy + \int_{\mathbb{R}} \mathcal{K}(x-y,t) \mathrm{w}_0(y)\, dy,$$

$$w(x,t) = \int_0^t d\tau \int_{\mathbb{R}} \mathcal{K}(x-y, t-\tau) g(y,\tau)\, dy + \int_{\mathbb{R}} \mathcal{K}(x-y,t) w_0(y)\, dy.$$

The representation of w can be found for example in [13, formulas (3.11)–(3.13)]. More general results are obtained in [23, Theorem 1].

Concerning problem (3.8) we can closely follow [16, Ch. IV, §1] and verify that $w(x,t)$ is a solution to this problem in the sense that

(B1) $w(x,t)$ is continuous over $\mathbb{R} \times [0,\sigma]$;

(B2) $w_t(x,t)$, $w_x(x,t)$, $w_{xx}(x,t)$, $D^{1-\alpha}_{0+} w_{xx}(x,t)$ are continuous for $x \in \mathbb{R}$, $t \in (0,\sigma)$;

(B3) $w(x,t)$ satisfies (3.8),

under the assumptions that $w_0 \in C^{\beta_1}(\mathbb{R})$, $g \in C((0,\sigma]; C^{\beta_2}(\mathbb{R}))$, where

$$\sup_{x\in\mathbb{R}} |w_0(x)| + \sup_{x\in\mathbb{R}} \frac{|w_0(x) - w_0(y)|}{|x-y|^{\beta_1}} < \infty,$$

$$\sup_{(x,t)\in\mathbb{R}\times(0,\sigma)} |t^{\gamma} g(x,t)| + \sup_{x,t\in\mathbb{R}} t^{\gamma} \frac{|g(x,t) - g(y,t)|}{|x-y|^{\beta_2}} < \infty,$$

$$w_0(x) = 0 \text{ for all } |x| \ge r \text{ and } g(x,t) = 0 \text{ for all } |x| \ge r,\ t \in (0,\sigma],$$

with some $\beta_1 \in (0,1]$, $\beta_2, \gamma \in (0,1)$ and $r > 0$.

By the subordination principle [4] the functions K and $\mathcal{K}$ can be written in the form (see [12])

$$K(x,t)=\int_0^\infty \Gamma(x,\lambda)\,t^{-1}\phi(-\alpha,0;-\lambda t^{-\alpha})\,d\lambda,$$

$$\mathcal{K}(x,t)=\int_0^\infty \Gamma(x,\lambda)\,t^{-\alpha}\phi(-\alpha,1-\alpha;-\lambda t^{-\alpha})\,d\lambda,$$

where $\Gamma(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{|x|^2}{4t}\right)$ is the fundamental solution for the classical heat equation. These representations are more suitable for differentiation with respect to x.

The following identities follow from (3.3), (3.4) ([23], see also [13, 12]):

$$\mathcal{K}(x,t)=I_{0+}^{1-\alpha}K(x,t),\qquad D_{0+}^{1-\alpha}\mathcal{K}(x,t)=K(x,t), \tag{3.11}$$

$$\mathcal{K}_t(x,t)=K_{xx}(x,t)=D_{0+}^{\alpha}K(x,t)=D_{0+}^{1-\alpha}\mathcal{K}_{xx}(x,t), \tag{3.12}$$

Lemma 15 in [23] gives the estimates

$$|D_{0+}^m D^n K(x,t)|\le Ct^{\alpha/2-1-m-n\alpha/2}\exp\left(-\kappa\left[\frac{|x|}{t^{\alpha/2}}\right]^{2/(2-\alpha)}\right), \tag{3.13}$$

$$|D_{0+}^m D^n \mathcal{K}(x,t)|\le Ct^{-\alpha/2-m-n\alpha/2}\exp\left(-\kappa\left[\frac{|x|}{t^{\alpha/2}}\right]^{2/(2-\alpha)}\right), \tag{3.14}$$

where $m=0,1$, $n=0,1,2$, $2m+n\le 2$. We have (see [11, formulas (19), (22)–(24)])

$$K_x(x,t)=-\frac{x}{t^\alpha}L(x,t), \tag{3.15}$$

$$L(x,t)=\frac{1}{4\sqrt{\pi}}\int_0^\infty \lambda^{-3/2}\exp\left(-\frac{|x|^2}{4\lambda}\right)t^{\alpha-1}\phi(-\alpha,0;-\lambda t^{-\alpha})\,d\lambda, \tag{3.16}$$

$$|D_x^n L(x,t)|\le Ct^{\alpha/2-1-n\alpha/2}\exp\left(-\kappa\left[\frac{|x|}{t^{\alpha/2}}\right]^{2/(2-\alpha)}\right). \tag{3.17}$$

By definition of $\mathcal{K}$ and (3.3) we have

$$I_{0+}^{\alpha}\mathcal{K}_x(x,t)=-\frac{1}{4\sqrt{\pi}}\int_0^\infty \frac{x}{\lambda^{3/2}}\exp\left(-\frac{|x|^2}{4\lambda}\right)t^{\alpha-1}\phi(-\alpha,\alpha;-\lambda t^{-\alpha})\,d\lambda, \tag{3.18}$$

and [23, Lemma 15] implies

$$|I_{0+}^{\alpha}\mathcal{K}_x(x,t)|\le C\exp\left(-\kappa\left[\frac{|x|}{t^{\alpha/2}}\right]^{2/(2-\alpha)}\right). \tag{3.19}$$

We introduce Green's functions on the half-plane $x>0$:

$$G(x,t,\xi,\tau) = K(x-\xi,t-\tau) - K(x+\xi,t-\tau),$$
$$\mathcal{G}(x,t,\xi,\tau) = \mathcal{K}(x-\xi,t-\tau) - \mathcal{K}(x+\xi,t-\tau),$$
$$N(x,t,\xi,\tau) = K(x-\xi,t-\tau) + K(x+\xi,t-\tau),$$
$$\mathcal{N}(x,t,\xi,\tau) = \mathcal{K}(x-\xi,t-\tau) + \mathcal{K}(x+\xi,t-\tau).$$

In order to avoid cumbersome notations for the functions depending on two time variables (t and τ) we just set

$$(3.20)\qquad I_{0+}^{1-\alpha}G(x,t,\xi,\tau) = \frac{1}{\Gamma(1-\alpha)}\int_\tau^t (t-\theta)^{-\alpha}G(x,\theta,\xi,\tau)\,d\theta,$$
$$I_{t-}^{1-\alpha}\mathcal{G}(x,t,\xi,\tau) = \frac{1}{\Gamma(1-\alpha)}\int_\tau^t (\theta-\tau)^{-\alpha}\mathcal{G}(x,t,\xi,\theta)\,d\theta.$$

Then we have for example

$$\mathcal{G}(x,t,\xi,\tau) = I_{0+}^{1-\alpha}G(x,t,\xi,\tau).$$

Indeed, by the first identity in (3.11),

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)}&\int_\tau^t (t-\theta)^{-\alpha}K(x\pm\xi,\theta-\tau)\,d\theta\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\tau}(t-\tau-\vartheta)^{-\alpha}K(x\pm\xi,\vartheta)\,d\vartheta\\ &= I_{0+}^{1-\alpha}K(x\pm\xi,t-\tau) = \mathcal{K}(x\pm\xi,t-\tau),\end{aligned}$$

and hence

$$\begin{aligned}I_{0+}^{1-\alpha}G(x,t,\xi,\tau) &= \frac{1}{\Gamma(1-\alpha)}\int_\tau^t (t-\theta)^{-\alpha}K(x-\xi,\theta-\tau)\,d\theta\\ &\quad - \frac{1}{\Gamma(1-\alpha)}\int_\tau^t (t-\theta)^{-\alpha}K(x+\xi,\theta-\tau)\,d\theta\\ &= \mathcal{K}(x-\xi,t-\tau) - \mathcal{K}(x+\xi,t-\tau) = \mathcal{G}(x,t,\xi,\tau).\end{aligned}$$

The following lemma states the jump conditions on the moving boundary $x=\varrho(t)$.

Lemma 3.2. *Let $h \in C_{1-\alpha}([0,\sigma])$, and $|\varrho(t)-\varrho(\tau)| \le \kappa|t-\tau|^\alpha$ for any $t,\tau\in[0,\sigma]$. Then for all $t\in(0,\sigma]$,*

$$(3.21)\qquad \begin{aligned}\lim_{x\to\varrho(t)\mp}\frac{\partial}{\partial x}&\int_0^t h(\tau)K(x-\varrho(\tau),t-\tau)\,d\tau\\ &= \pm\frac{1}{2}h(t) + \int_0^t h(\tau)K_x(\varrho(t)-\varrho(\tau),t-\tau)\,d\tau.\end{aligned}$$

REMARK 3.3. The limiting behaviour of this double layer potential as $x \to \varrho(t)-$ is studied in [11] under the stronger assumption $h \in C([0,\sigma])$. That proof is applicable in the present situation with minor changes (cf. [9]).

On the other hand, the double layer potential with kernel $\mathcal{K}_x$ is continuous.

LEMMA 3.4. *Let $h \in C_{1-\alpha}([0,\sigma])$ and $\varrho \in C[0,\sigma]$. Then for all $t \in (0,\sigma]$,*

$$\lim_{x\to\varrho(t)} \frac{\partial}{\partial x}\int_0^t h(\tau)\mathcal{K}(x-\varrho(\tau),t-\tau)\,d\tau = \int_0^t h(\tau)\mathcal{K}_x(\varrho(t)-\varrho(\tau),t-\tau)\,d\tau. \tag{3.22}$$

Proof. Fix any $t \in (0,\sigma]$ and $\epsilon > 0$. Let $\delta \in (0,t)$ be chosen below. We have

$$\begin{aligned}
&\int_0^t h(\tau)\mathcal{K}_x(x-\varrho(\tau),t-\tau)\,d\tau - \int_0^t h(\tau)\mathcal{K}_x(\varrho(t)-\varrho(\tau),t-\tau)\,d\tau \\
&\quad = \int_{t-\delta}^t h(\tau)\mathcal{K}_x(x-\varrho(\tau),t-\tau)\,d\tau - \int_{t-\delta}^t h(\tau)\mathcal{K}_x(\varrho(t)-\varrho(\tau),t-\tau)\,d\tau \\
&\qquad + \int_0^{t-\delta} h(\tau)[\mathcal{K}_x(x-\varrho(\tau),t-\tau) - \mathcal{K}_x(\varrho(t)-\varrho(\tau),t-\tau)]\,d\tau \\
&\quad = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3.
\end{aligned}$$

Applying estimates (3.14) we obtain

$$\begin{aligned}
|\mathcal{K}_1| + |\mathcal{K}_2| &\le 2C|h|_{\sigma,1-\alpha}\int_{t-\delta}^t \frac{\tau^{\alpha-1}}{(t-\tau)^\alpha}\,d\tau = C|h|_{\sigma,1-\alpha}\int_0^\delta \frac{(t-\tau)^{\alpha-1}}{\tau^\alpha}\,d\tau \\
&= 2C|h|_{\sigma,1-\alpha}\bigg(\int_0^\delta \frac{(t-\tau)^\alpha}{t\tau^\alpha}\,d\tau + \int_0^\delta \frac{\tau^{1-\alpha}}{t(t-\tau)^{1-\alpha}}\,d\tau\bigg) \\
&\le C|h|_{\sigma,1-\alpha}\bigg(\frac{\delta^{1-\alpha}}{t^{1-\alpha}} + \frac{\delta^{1-\alpha}}{t}\int_0^\delta (t-\tau)^{\alpha-1}\,d\tau\bigg) \\
&\le C|h|_{\sigma,1-\alpha}\bigg(\frac{\delta^{1-\alpha}}{t^{1-\alpha}} + \frac{\delta}{t}\bigg) \le \frac{\epsilon}{2}
\end{aligned} \tag{3.23}$$

for sufficiently small δ. Henceforth δ is fixed. We get

$$\mathcal{K}_3 = \int_0^{t-\delta} h(\tau)\,d\tau\int_0^1 \mathcal{K}_{xx}(\lambda(x-\varrho(\tau)) + (1-\lambda)(\varrho(t)-\varrho(\tau),t-\tau)\,d\lambda\big(x-\varrho(t)\big)$$

and once more by estimate (3.14)

$$
\begin{aligned}
(3.24)\quad |\mathcal{K}_3| &\le C|h|_{\sigma,1-\alpha} \int\limits_0^{t-\delta} \frac{\tau^{\alpha-1}}{(t-\tau)^{3\alpha/2}}\, d\tau\, |x-\varrho(t)| \\
&= C|h|_{\sigma,1-\alpha} \int\limits_\delta^{t} \frac{(t-s)^{\alpha-1}}{s^{3\alpha/2}}\, ds\, |x-\varrho(t)| \\
&\le C|h|_{\sigma,1-\alpha} \int\limits_0^{t} \frac{(t-s)^{\alpha-1}}{\delta^{3\alpha/2}}\, ds\, |x-\varrho(t)| \le C\frac{t^\alpha}{\delta^{3\alpha/2}}\, |x-\varrho(t)| \le \frac{\epsilon}{2}.
\end{aligned}
$$

for all x such that the difference $|x-\varrho(t)|$ is sufficiently small. The assertion of the lemma follows from (3.23) and (3.24). ■

4. Integral representation of u. In this section the solution of (1.8) will be represented by the sum of layer potentials. The function $\mathcal{G}(x,t,\xi,\tau)$ is the Green's function of the Dirichlet problem on the half-plane $x>0$ (see, for example, [13]). Multiply both sides of (1.8) by $\mathcal{G}(x,t,\xi,s)$ and integrate on $Q_t^\varrho = \{(\xi,\tau) : \xi \in (0,\varrho(\tau)),\ \tau\in(0,t)\}$. Let $\delta\in(0,t)$ and $t_\delta = t-\delta$. We have

$$
(4.1)\qquad \int\limits_0^{\varrho(t)} d\xi \int\limits_{l(\xi)}^{t} u_\tau(\xi,\tau)\mathcal{G}(x,t,\xi,\tau)\, d\tau = \lim_{\delta\to 0} I_\delta(x,t),
$$

where

$$
(4.2)\qquad I_\delta(x,t) = \int\limits_0^{\varrho(t_\delta)} d\xi \int\limits_{l(\xi)}^{t_\delta} u_\tau(\xi,\tau)\mathcal{G}(x,t,\xi,\tau)\, d\tau.
$$

Integrating by parts we obtain

$$
\begin{aligned}
I_\delta(x,t) = &\int\limits_0^{\varrho(t_\delta)} \big(u(\xi,t_\delta)\mathcal{G}(x,t,\xi,t_\delta) - u(\xi,l(\xi))\mathcal{G}(x,t,\xi,l(\xi))\big)\, d\xi \\
&- \int\limits_0^{\varrho(t_\delta)} d\xi \int\limits_{l(\xi)}^{t_\delta} u(\xi,\tau)\mathcal{G}_\tau(x,t,\xi,\tau)\, d\tau \\
= &\int\limits_0^{\varrho(t_\delta)} u(\xi,t_\delta)\mathcal{G}(x,t,\xi,t_\delta)\, d\xi - \int\limits_0^{b} \varphi(\xi)\mathcal{G}(x,t,\xi,0)\, d\xi \\
&- \int\limits_0^{\varrho(t_\delta)} d\xi \int\limits_{l(\xi)}^{t_\delta} u(\xi,\tau)\mathcal{G}_\tau(x,t,\xi,\tau)\, d\tau \\
= &\sum_{i=1}^{3} I_\delta^{(i)}(x,t).
\end{aligned}
$$

We are interested only in the first integral

$$I_\delta^{(1)}(x,t) = \int_0^{\varrho(t_\delta)} u(\xi,t_\delta)\mathcal{K}(x-\xi,\delta)\,d\xi - \int_0^{\varrho(t_\delta)} u(\xi,t_\delta)\mathcal{K}(x+\xi,\delta)\,d\xi$$
$$= I_\delta^{(1,1)}(x,t) - I_\delta^{(1,2)}(x,t).$$

Denote $\overline{M} = \sup_{(x,t)\in Q_\sigma^\varrho} |u(x,t)|$. We change variable $\xi \to \eta = \frac{x+\xi}{\delta^{\alpha/2}}$ and apply (3.1) and (3.5):

$$|I_\delta^{(1,2)}(x,t)| \le \frac{\overline{M}}{2}\int_{x\delta^{-\alpha/2}}^{\infty} \phi\Big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; -\eta\Big)\,d\eta = \frac{\overline{M}}{2}\phi\Big(-\frac{\alpha}{2}, 1, -\frac{x}{\delta^{\alpha/2}}\Big).$$

We get

$$\lim_{\delta\to 0} I_\delta^{(1,2)}(x,t) = 0 \quad \text{for all } x,t>0. \tag{4.3}$$

Define

$$\overline{u}(x,t) = \begin{cases} u(x,t) & \text{for } x \in [0,\varrho(t)),\\ 0 & \text{for } x\in[\varrho(t),\infty). \end{cases}$$

Then

$$I_\delta^{(1,1)}(x,t) = \int_0^\infty \overline{u}(\xi,t_\delta)\mathcal{K}(x-\xi,\delta)\,d\xi.$$

We claim that

$$\lim_{\delta\to 0} I_\delta^{(1,1)}(x,t) = \overline{u}(x,t) \quad \text{for all } x,t>0. \tag{4.4}$$

First of all we use (3.1):

$$\int_0^\infty \mathcal{K}(x-\xi,\delta)\,d\xi = \frac{1}{2\delta^{\alpha/2}}\int_0^\infty \phi\Big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}, -\frac{|x-\xi|}{\delta^{\alpha/2}}\Big)\,d\xi \tag{4.5}$$
$$= \frac{1}{2}\int_{-\infty}^{x\delta^{-\alpha/2}} \phi\Big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}, -|\eta|\Big)\,d\eta = 1 - \frac{1}{2}\phi\Big(-\frac{\alpha}{2}, 1, -x\delta^{-\alpha/2}\Big).$$

By (4.5) we deduce that

$$I_\delta^{(1,1)}(x,t) - \overline{u}(x,t) \tag{4.6}$$
$$= \int_0^\infty (\overline{u}(\xi,t_\delta) - \overline{u}(x,t))\mathcal{K}(x-\xi,\delta)\,d\xi - \frac{1}{2}\overline{u}(x,t)\phi\Big(-\frac{\alpha}{2}, 1, -\frac{x}{\delta^{\alpha/2}}\Big)$$
$$= \hat{I}_\delta^{(1,1)}(x,t) - \frac{1}{2}\overline{u}(x,t)\phi\Big(-\frac{\alpha}{2}, 1, -\frac{x}{\delta^{\alpha/2}}\Big)$$

and

$$\lim_{\delta\to 0} I_\delta^{(1,1)}(x,t) - \overline{u}(x,t) = \lim_{\delta\to 0} \hat{I}_\delta^{(1,1)}(x,t) \quad \text{for all } x, t > 0. \tag{4.7}$$

Let $\zeta \in (0,x)$, to be chosen below. We split $\hat{I}_\delta^{(1,1)}(x,t)$ into the sum

$$\hat{I}_\delta^{(1,1)}(x,t) = \int\limits_0^{x-\zeta} \dots d\xi + \int\limits_{x-\zeta}^{x+\zeta} \dots d\xi + \int\limits_{x+\zeta}^{\infty} \dots d\xi = \hat{\imath}_{1,\delta} + \hat{\imath}_{2,\delta} + \hat{\imath}_{3,\delta}. \tag{4.8}$$

Since $\overline{u}(x,t)$ is continuous (see (A2)) there exist ζ and δ_* such that for all ξ, δ_*, $|x-\xi|<\zeta$, $0<\delta<\delta_*$ we have $|\overline{u}(\xi,t_\delta)-u(x,t)|<\epsilon/3$ and

$$|\hat{\imath}_{2,\delta}| \le \frac{\epsilon}{3} \int\limits_{x-\zeta}^{x+\zeta} \mathcal{K}(x-\xi,\delta)\, d\xi \le \frac{\epsilon}{3}. \tag{4.9}$$

We change the variable $\xi \to \eta = \frac{x-\xi}{\delta^{\alpha/2}}$ and get

$$\begin{aligned} |\hat{\imath}_{1,\delta}| + |\hat{\imath}_{3,\delta}| &\le \overline{M}\Bigg(\int\limits_{\zeta\delta^{-\alpha/2}}^{\infty} \phi\Big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}, -|\eta|\Big)\, d\eta \\ &\quad + \int\limits_{-\infty}^{-\zeta\delta^{-\alpha/2}} \phi\Big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}, -|\eta|\Big)\, d\eta\Bigg) \\ &\le 2\overline{M}\,\phi\Big(-\frac{\alpha}{2}, 1, -\zeta\delta^{-\alpha/2}\Big). \end{aligned} \tag{4.10}$$

Now take ζ from (4.9) and δ_* depending on α, $\overline{M}$, ζ, in order to pick $\delta<\delta_*$ such that

$$|\hat{\imath}_{1,\delta}| + |\hat{\imath}_{3,\delta}| \le \frac{2\epsilon}{3}. \tag{4.11}$$

Combining (4.7), (4.11), (4.9) we infer that (4.4) holds true. Substitution of (4.3) and (4.4) into (4.1) yields

$$\begin{aligned} &\int\limits_0^{\varrho(t)} d\xi \int\limits_{l(\xi)}^{t} u_\tau(\xi,\tau)\mathcal{G}(x,t,\xi,\tau)\, d\tau \\ &= u(x,t) - \int\limits_0^b \varphi(\xi)\mathcal{G}(x,t,\xi,0)\, d\xi - \int\limits_0^{\varrho(t)} d\xi \int\limits_{l(\xi)}^{t} u(\xi,\tau)\mathcal{G}_\tau(x,t,\xi,\tau)\, d\tau. \end{aligned} \tag{4.12}$$

Denote

$$q(\xi,t) = D_{l(\xi)+}^{1-\alpha} u_\xi(\xi,t), \qquad v(\tau) = \lim_{\xi\to\varrho(\tau)-} q(\xi,\tau).$$

Since $\mathcal{G}(x,t,0,\tau)=0$ we have

$$\int_0^t d\tau \int_0^{\varrho(\tau)} q_\xi(\xi,\tau)\mathcal{G}(x,t,\xi,\tau)\,d\xi \tag{4.13}$$
$$= \int_0^t v(\tau)\mathcal{G}(x,t,\varrho(\tau),\tau)\,d\tau - \int_0^{\varrho(t)} d\xi \int_{l(\xi)}^t q(\xi,\tau)\mathcal{G}_\xi(x,t,\xi,\tau)\,d\tau$$
$$= J_1 - J_2.$$

Applying (3.11) gives

$$\mathcal{K}(x,t-\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\tau} \frac{K(x,s)}{(t-\tau-s)^\alpha}\,ds = \frac{1}{\Gamma(1-\alpha)}\int_\tau^t \frac{K(x,t-s)}{(s-\tau)^\alpha}\,ds.$$

and $\mathcal{G}(x,t,\xi,\tau) = I_{t-}^{1-\alpha}G(x,t,\xi,\tau)$. Our purpose is to prove that the integrals

$$j_1 = \int_0^{\varrho(t)} d\xi \int_{l(\xi)}^t |q(\xi,t)\mathcal{G}_\xi(x,t,\xi,\tau)|\,d\tau, \tag{4.14}$$

$$j_2 = \int_0^{\varrho(t)} d\xi \int_{l(\xi)}^t |I_{l(\xi)+}^{1-\alpha} q(\xi,t)G_\xi(x,t,\xi,\tau)|\,d\tau \tag{4.15}$$

are bounded. Then one can conclude by Fubini's theorem that (cf. [10, Lemma 2.7])

$$J_2 = \int_0^{\varrho(t)} d\xi \int_{l(\xi)}^t I_{l(\xi)+}^{1-\alpha} q(\xi,t)G_\xi(x,t,\xi,\tau)\,d\tau. \tag{4.16}$$

Combining (A5), (A6) and (3.14) we obtain

$$|j_1| \le C \sup_{(\xi,\tau)\in Q_\sigma^\varrho} |\tau^{1-\alpha}q(\xi,\tau)| \int_0^{\varrho(t)} d\xi \int_0^t \tau^{\alpha-1}(t-\tau)^{-\alpha}\,d\tau \le C. \tag{4.17}$$

Since

$$|I_{l(\xi)+}^{1-\alpha} q(\xi,\tau)| \le C\int_{l(\xi)}^\tau \frac{|q(\xi,s)|}{(\tau-s)^\alpha}\,ds \le C \sup_{(\xi,s)\in Q_\tau^\varrho} |s^{1-\alpha}q(\xi,s)| \int_0^\tau \frac{s^{\alpha-1}}{(\tau-s)^\alpha}\,ds \le C,$$

and (by (3.15) and (3.17))

$$\int_0^t |K_x(x,\tau)|\,d\tau \le C\int_0^t \frac{|x|}{\tau^{\alpha/2+1}} \exp\biggl(-\kappa\biggl[\frac{|x|}{\tau^{\alpha/2}}\biggr]^{\frac{2}{2-\alpha}}\biggr)\,d\tau \tag{4.18}$$
$$\le C\int_0^\infty \exp(-\kappa\eta^{\frac{2}{2-\alpha}})\,d\eta \le C$$

we infer that the integral J_2 converges. Thus (4.16) holds. By [10, Lemma 2.5] and the last statement in (A3) we get

$$
\begin{aligned}
(4.19)\qquad I^{1-\alpha}_{l(\xi)+}q(\xi,\tau) &= I^{1-\alpha}_{l(\xi)+}D^{1-\alpha}_{l(\xi)+}u_\xi(\xi,\tau) \\
&= u_\xi(\xi,\tau) - \frac{(\tau-l(\xi))^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)}\lim_{\tau\to l(\xi)}\int_{l(\xi)}^{\tau}(\tau-s)^{\alpha-1}u_\xi(\xi,s)\,ds = u_\xi(\xi,\tau).
\end{aligned}
$$

We go back to J_2 (see (4.16), (4.19)) and integrate by parts:

$$
\begin{aligned}
(4.20)\qquad J_2 &= \int_0^t \big(u(\varrho(\tau),\tau)G_\xi(x,t,\varrho(\tau),\tau) - u(0,\tau)G_\xi(x,t,0,\tau)\big)\,d\tau \\
&\quad - \int_0^t d\tau \int_0^\varrho u(\xi,\tau)G_{\xi\xi}(x,t,\xi,\tau)\,d\xi \\
&= -2\int_0^t f(\tau)K_x(x,t-\tau)\,d\tau - \int_0^t d\tau\int_0^\varrho u(\xi,\tau)G_{\xi\xi}(x,t,\xi,\tau)\,d\xi.
\end{aligned}
$$

Taking into account (1.8), (3.12), (4.12), (4.13), (4.20) we finally obtain

$$
\begin{aligned}
(4.21)\qquad u(x,t) &= \int_0^b \varphi(\xi)\mathcal{G}(x,t,\xi,0)\,d\xi - 2\int_0^t f(\tau)K_x(x,t-\tau)\,d\tau \\
&\quad + \int_0^t v(\tau)\mathcal{G}(x,t,\varrho(\tau),\tau)\,d\tau \\
&= u_1(x,t) + u_2(x,t) + u_3(x,t).
\end{aligned}
$$

Remark 4.1. It is clear that all integrals in (4.21) and the Riemann–Liouville integral $I^\alpha_{0+}u(x,t)$ are defined for all $x>0$ and $t\in(0,\sigma)$. Thus one can calculate the derivatives $u_t(x,t)$, $u_x(x,t)$, $D^{1-\alpha}_{0+}u_x(x,t)$, $(D^{1-\alpha}_{0+}u_x)_x(x,t)$ at least for $x\neq\varrho(t)$. Here we look at the fractional derivatives $D^{1-\alpha}_{0+}$ as "local" derivatives D_t of the fractional integrals I^α_{0+}.

5. Integral representation of derivatives. It is of particular interest to find the derivatives of u_3. Keeping in mind Lemma 3.4 we obtain (cf. (3.20))

$$
u_{3,x}(x,t) = \int_0^t v(\tau)\mathcal{G}_x(x,t,\varrho(\tau),\tau)\,d\tau,
$$

$$
\begin{aligned}
(5.1)\qquad I^\alpha_{0+}u_{3,x}(x,s) &= \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\,dt\int_0^t v(\tau)\mathcal{G}_x(x,t,\varrho(\tau),\tau)\,d\tau \\
&= \int_0^s v(\tau)I^\alpha_{0+}\mathcal{G}_x(x,s,\varrho(\tau),\tau)\,d\tau,
\end{aligned}
$$

and

$$D_{0+}^{1-\alpha}u_{3,x}(x,s) = \lim_{\delta\to 0}\left[\frac{\partial}{\partial s}\int_0^{s-\delta} v(\tau)I_{0+}^{\alpha}\mathcal{G}_x(x,s,\varrho(\tau),\tau)\,d\tau\right]$$
$$= \lim_{\delta\to 0} v(s-\delta)I_{0+}^{\alpha}\mathcal{G}_x(x,s,\varrho(s-\delta),s-\delta)$$
$$+\int_0^s v(\tau)D_{0+}^{1-\alpha}\mathcal{G}_x(x,s,\varrho(\tau),\tau)\,d\tau.$$

Assume that $x \neq \varrho(s)$ ($|x-\varrho(s)| = \omega > 0$). Then

$$\left(\frac{|x-\varrho(s)|}{\delta^{\alpha/2}}\right)^{\frac{2}{2-\alpha}} \le \left(\frac{|x-\varrho(s-\delta)| + |\varrho(s-\delta)-\varrho(s)|}{\delta^{\alpha/2}}\right)^{\frac{2}{2-\alpha}}$$
$$\le C_\alpha\left[\left(\frac{|x-\varrho(s-\delta)|}{\delta^{\alpha/2}}\right)^{\frac{2}{2-\alpha}} + (|\varrho|_{\sigma,1-\alpha}^{(1)}\delta^{\alpha/2})^{\frac{2}{2-\alpha}}\right]$$

and

$$-\left(\frac{|x-\varrho(s-\delta)|}{\delta^{\alpha/2}}\right)^{\frac{2}{2-\alpha}} \le -\frac{1}{C_\alpha}\left(\frac{\omega}{\delta^{\alpha/2}}\right)^{\frac{2}{2-\alpha}} + C(|\varrho|_{\sigma,1-\alpha}^{(1)})\delta^{\frac{\alpha}{2-\alpha}}. \tag{5.2}$$

By virtue of (3.19) and (5.2) we get

$$\begin{aligned}
&|I_{0+}^{\alpha}\mathcal{G}_x(x,s,\varrho(s-\delta),s-\delta)| \\
&\le C\left[\exp\left(-\kappa\left[\frac{|x-\varrho(s-\delta)|}{\delta^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) + \exp\left(-\kappa\left[\frac{|x+\varrho(s-\delta)|}{\delta^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right] \\
&\le C\left[\exp\left(-\kappa\left[\frac{\omega}{\delta^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) + \exp\left(-\kappa\left[\frac{b}{\delta^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right].
\end{aligned} \tag{5.3}$$

Estimate (5.3) and identity (3.11) imply

$$D_{0+}^{1-\alpha}u_{3,x}(x,s) = \int_0^s v(\tau)G_x(x,s,\varrho(\tau),\tau)\,d\tau \quad \text{for } x \neq \varrho(s). \tag{5.4}$$

In a similar way we deduce

$$u_{3,t}(x,t) = \lim_{\delta\to 0}\left[\frac{\partial}{\partial t}\int_0^{t-\delta} v(\tau)\mathcal{G}(x,t,\varrho(\tau),\tau)\,d\tau\right]$$
$$= \int_0^t v(\tau)\mathcal{G}_t(x,t,\varrho(\tau),\tau)\,d\tau \quad \text{for } x \neq \varrho(t), \tag{5.5}$$
$$(D_{0+}^{1-\alpha}u_{3,x})_x(x,t) = \int_0^t v(\tau)G_{xx}(x,t,\varrho(\tau),\tau)\,d\tau \quad \text{for } x \neq \varrho(t).$$

Denote $\hat{Q}^{\varrho}_{\sigma} = \{(x,t) : x > \varrho(t),\ 0 < t < \sigma\}$. The function u_3 can be regarded as a solution of the equation

$$u_{3,t}(x,t) = (D^{1-\alpha}_{0+} u_{3,x})_x(x,t) \quad \text{in } Q^{\varrho}_{\sigma} \cup \hat{Q}^{\varrho}_{\sigma}. \tag{5.6}$$

Since $\mathcal{G}(0,t,\varrho(\tau),\tau) = 0$ one can obtain

$$u_3(0,t) = 0, \qquad t > 0, \tag{5.7}$$

Estimate (3.14) and (A5) yield

$$u_3(x,0) = 0, \qquad x > 0. \tag{5.8}$$

Denote $\tilde{u}(x,t) = u_1(x,t) + u_2(x,t)$. Combining (H1), (H3), (3.11) and (3.12) gives

$$\begin{aligned} \tilde{u}_x(x,t) &= \int_0^b \dot{\varphi}(\xi)\mathcal{N}(x,t,\xi,0)\,d\xi - 2\int_0^t \dot{f}(\tau)\mathcal{K}(x,t-\tau)\,d\tau, \\ D^{1-\alpha}_{0+}\tilde{u}_x(x,t) &= \int_0^b \dot{\varphi}(\xi)N(x,t,\xi,0)\,d\xi - 2\int_0^t \dot{f}(\tau)K(x,t-\tau)\,d\tau. \end{aligned} \tag{5.9}$$

Moreover, it can be easily verified that the function $\tilde{u}(x,t)$ is a solution of the problem

$$\begin{aligned} \tilde{u}_t(x,t) &= (D^{1-\alpha}_{0+}\tilde{u}(x,t)_x)_x, \qquad x > 0,\ t \in (0,\sigma), \\ \tilde{u}(0,t) &= f(t),\ t \in (0,\sigma), \qquad \tilde{u}(x,0) = \overline{\varphi}(x),\ x > 0, \end{aligned} \tag{5.10}$$

where

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) & \text{for } x \in [0,b), \\ 0 & \text{for } x \geq b. \end{cases}$$

By (5.6)–(5.10) we obtain

$$\begin{aligned} u_t(x,t) &= (D^{1-\alpha}_{0+}u(x,t)_x)_x \quad \text{in } Q^{\varrho}_{\sigma} \cup \hat{Q}^{\varrho}_{\sigma}, \\ u(0,t) &= f(t),\ t \in (0,\sigma), \qquad u(x,0) = \overline{\varphi}(x),\ x > 0, \end{aligned} \tag{5.11}$$

and in addition

$$\begin{aligned} D^{1-\alpha}_{0+}u_x(x,t) &= \int_0^b \dot{\varphi}(\xi)N(x,t,\xi,0)\,d\xi - 2\int_0^t \dot{f}(\tau)K(x,t-\tau)\,d\tau \\ &\quad + \int_0^t v(\tau)G_x(x,t,\varrho(\tau),\tau)\,d\tau. \end{aligned} \tag{5.12}$$

Lemma 3.2 implies that

$$\lim_{x\to\varrho(t)-} D_{0+}^{1-\alpha} u_x(x,t) = \int_0^b \dot{\varphi}(\xi) N(\varrho(t),t,\xi,0)\,d\xi \\ - 2\int_0^t \dot{f}(\tau) K(\varrho(t), t-\tau)\,d\tau + \int_0^t v(\tau) G_x(\varrho(t),t,\varrho(\tau),\tau)\,d\tau + \frac{1}{2}v(t). \tag{5.13}$$

It is unclear whether $\lim_{x\to\varrho(t)-} D_{l(x)+}^{1-\alpha} u(x,t)_x(x,t) = v(t)$ as in (1.8). That is why we formally put $\lim_{x\to\varrho(t)-} D_{0+}^{1-\alpha} u(x,t)_x(x,t) = v(t)$. Then we return to the original problem.

6. Integral equation

6.1. Statement of the problem. In view of (5.13), (1.8) we consider the integral equation

$$\begin{aligned} \mathrm{v}(t) &= 2\int_0^b \dot{\varphi}(\xi) N(\rho(t),t,\xi,0)\,d\xi - 4\int_0^t \dot{f}(\tau) K(\rho(t),t-\tau)\,d\tau \\ &\quad + 2\int_0^t \mathrm{v}(\tau) G_x(\rho(t),t,\varrho(\tau),\tau)\,d\tau \\ &= \mathcal{A}_1[\mathrm{v}] + \mathcal{A}_2[\mathrm{v}] + \mathcal{A}_3[\mathrm{v}], \end{aligned} \tag{6.1}$$

where

$$\rho(t) = b - \int_0^t \mathrm{v}(\tau)\,d\tau. \tag{6.2}$$

Consider the set (see (H1))

$$S_{\sigma,r} = \left\{ \mathrm{v} \in C_{1-\alpha}([0,\sigma]) : \left| t^{1-\alpha}\mathrm{v} - \frac{\dot{\varphi}(b)}{\Gamma(\alpha)} \right|_\sigma \le \frac{r}{2} \right\},$$

where $r = |\dot{\varphi}(b)|/\Gamma(\alpha) > 0$. Note that

$$\mathrm{v}_0(t) = \frac{\dot{\varphi}(b)}{\Gamma(\alpha)} t^{\alpha-1} \in C_{1-\alpha}([0,\sigma]) \quad \text{and} \quad \left| t^{1-\alpha}\mathrm{v} - \frac{\dot{\varphi}(b)}{\Gamma(\alpha)} \right|_\sigma = |\mathrm{v} - \mathrm{v}_0|_{\sigma,1-\alpha}.$$

On the set $S_{\sigma,r}$ we define a mapping $\hat{\mathrm{v}} = \mathcal{A}[\mathrm{v}] := \mathcal{A}_1[\mathrm{v}] + \mathcal{A}_2[\mathrm{v}] + \mathcal{A}_3[\mathrm{v}]$, where ρ is defined by (6.2).

If $\mathrm{v} \in S_{\sigma,r}$ then

$$-\frac{3r}{2} \le t^{1-\alpha}\mathrm{v}(t) \le -\frac{r}{2}, \qquad t \in [0,\sigma]. \tag{6.3}$$

This guarantees monotonicity of ρ:

$$0 < \frac{r}{2\alpha}(t^\alpha - \tau^\alpha) \le \rho(t) - \rho(\tau) \le \frac{3r}{2\alpha}(t-\tau)^\alpha, \qquad 0 \le \tau < t \le \sigma. \tag{6.4}$$

The main purpose is to show that $\mathcal{A}$ is a contractive mapping with $\mathcal{A}(S_{\sigma,r}) \subseteq S_{\sigma,r}$.

6.2. $\mathcal{A}$ maps $S_{\sigma,r}$ into itself. Denote

$$\hat{\mathrm{v}}_i = \mathcal{A}_i[\mathrm{v}], \qquad \varUpsilon_\kappa = \int_{-\infty}^{\infty} (1+|\eta|)\exp(-\kappa|\eta^{2/(2-\alpha)}|)\, d\eta.$$

Let us decompose the difference $t^{1-\alpha}\hat{\mathrm{v}}_1(t) - \frac{\dot\varphi(b)}{\varGamma(\alpha)}$ onto the sum of the following integrals:

$$\begin{aligned}(6.5)\quad t^{1-\alpha}\hat{\mathrm{v}}_1(t) - \frac{\dot\varphi(b)}{\varGamma(\alpha)} &= 2\int_0^b \dot\varphi(\xi)[t^{1-\alpha}N(\rho(t),t,\xi,0) - t^{1-\alpha}N(b,t,\xi,0)]\, d\xi \\ &\quad + 2\int_0^b [\dot\varphi(\xi) - \dot\varphi(b)]t^{1-\alpha}N(b,t,\xi,0)\, d\xi \\ &\quad + \dot\varphi(b)\left[2\int_0^b t^{1-\alpha}N(b,t,\xi,0)\, d\xi - \frac{1}{\varGamma(\alpha)}\right] \\ &= P_1 + P_2 + P_3.\end{aligned}$$

We have

$$P_1 = 2\int_0^b \dot\varphi(\xi)\, d\xi \int_0^1 tN_x(\mu\rho(t) + (1-\mu)b, t, \xi, 0)\frac{\rho(t)-b}{t^\alpha}\, d\mu.$$

Denote $\zeta_\mu = \mu\rho(t) + (1-\mu)b$. By virtue of (6.3), (3.15) and (3.17) with $n=0$, we have

$$\begin{aligned}(6.6)\qquad |P_1| &\le CM_1\frac{3r}{2\alpha}\int_0^b d\xi \int_0^1 \left[\frac{|\zeta_\mu - \xi|}{t^{\alpha/2}} \exp\left(-\kappa\left[\frac{|\zeta_\mu - \xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right. \\ &\quad \left. + \frac{|\zeta_\mu + \xi|}{t^{\alpha/2}} \exp\left(-\kappa\left[\frac{|\zeta_\mu + \xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right] d\mu \\ &\le C\int_0^1 d\mu \int_{-\infty}^{\infty} |\eta|\exp(-\kappa|\eta|^{\frac{2}{2-\alpha}})\, d\eta\, t^{\alpha/2} \le C\varUpsilon_\kappa t^{\alpha/2} \le \tilde{C}_1 t^{\alpha/2}.\end{aligned}$$

For P_2 we take into account that $|\xi - b| \le |\xi + b|$ since both ξ and b are positive. Applying (3.13) gives

$$\begin{aligned}(6.7)\qquad |P_2| &\le CM_2\int_0^b |\xi - b| t^{1-\alpha}\left[t^{\alpha/2-1}\exp\left(-\kappa\left[\frac{|\xi - b|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right. \\ &\quad \left. + t^{\alpha/2-1}\exp\left(-\kappa\left[\frac{|\xi + b|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right] d\xi\end{aligned}$$

$$\leq CM_2\int_0^b\bigg(\frac{|\xi-b|}{t^{\alpha/2}}\exp\bigg(-\kappa\bigg[\frac{|\xi-b|}{t^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)$$
$$+\frac{|\xi+b|}{t^{\alpha/2}}\exp\bigg(-\kappa\bigg[\frac{|\xi+b|}{t^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)\bigg)\,d\xi$$
$$\leq C\Upsilon_\kappa t^{\alpha/2}\leq \tilde{C}_2 t^{\alpha/2}.$$

By (3.1) and (3.2) we derive

$$P_3=\dot{\varphi}(b)\bigg[\int_0^b t^{-\alpha/2}\big(\phi(-\alpha/2,\alpha/2,-(b-\xi)t^{-\alpha/2})$$
$$+\phi(-\alpha/2,\alpha/2,-(b+\xi)t^{-\alpha/2})\big)\,d\xi-\frac{1}{\Gamma(\alpha)}\bigg]$$
$$=\dot{\varphi}(b)\bigg[\int_0^{2bt^{-\alpha/2}}\phi(-\alpha/2,\alpha/2,-\zeta)\,d\zeta-\int_0^\infty\phi(-\alpha/2,\alpha/2,-\zeta)\,d\zeta\bigg]$$
$$=-\dot{\varphi}(b)\int_{2bt^{-\alpha/2}}^\infty\phi(-\alpha/2,\alpha/2,-\zeta)\,d\zeta$$
$$=-\dot{\varphi}(b)\phi(-\alpha/2,\alpha,-2bt^{-\alpha/2}). \tag{6.8}$$

Thus

$$|P_3|\leq CM_1\frac{t^{\alpha/2}}{b}\,\frac{b}{t^{\alpha/2}}\exp\bigg(-\kappa\bigg(\frac{2b}{t^{\alpha/2}}\bigg)^{\frac{2}{2-\alpha}}\bigg)\leq\tilde{C}_3t^{\alpha/2}. \tag{6.9}$$

Summing (6.6), (6.7), (6.9) yields

$$\bigg|t^{1-\alpha}\hat{\mathrm{v}}_1(t)-\frac{\dot{\varphi}(b)}{\Gamma(\alpha)}\bigg|\leq C_1t^{\alpha/2}. \tag{6.10}$$

Combining (H2), (3.13), (3.15)–(3.17) and (6.4) we get

$$\begin{aligned}
&|t^{1-\alpha}\hat{\mathrm{v}}_2(t)|\leq t^{1-\alpha}C|f|^{(1)}_{\sigma,1-\alpha}\int_0^t\frac{\tau^{\alpha-1}}{|t-\tau|^{1-\alpha/2}}\,d\tau\leq C_2\sigma^{\alpha/2},\\
&|t^{1-\alpha}\hat{\mathrm{v}}_3(t)|\leq t^{1-\alpha}C|v|_{\sigma,1-\alpha}\int_0^t\tau^{\alpha-1}\frac{|\rho(t)-\rho(\tau)|}{|t-\tau|^{1+\alpha/2}}\,d\tau\\
&\qquad\leq Ct^{1-\alpha}r^2\int_0^t\tau^{\alpha-1}(t-\tau)^{\alpha/2-1}\,d\tau\leq C_3\sigma^{\alpha/2}.
\end{aligned} \tag{6.11}$$

By virtue of (6.10) and (6.11) we obtain

$$\bigg|t^{1-\alpha}\hat{\mathrm{v}}(t)-\frac{\dot{\varphi}(b)}{\Gamma(\alpha)}\bigg|_\sigma\leq C_*\sigma^{\alpha/2} \tag{6.12}$$

and $\mathcal{A}(S_{\sigma,r}) \subseteq S_{\sigma,r}$ for all $\sigma \in (0, \sigma_*]$ where

$$C_* \sigma_*^{\alpha/2} = r/2. \tag{6.13}$$

6.3. $\mathcal{A}$ is a contraction. Let $\mathrm{v}' \in S_{\sigma,r}$ and $\rho'(t) = b - \int_0^t \mathrm{v}'(\tau)\, d\tau$. We will establish that for sufficiently small σ,

$$|\mathcal{A}[\mathrm{v}] - \mathcal{A}[\mathrm{v}']|_{\sigma,1-\alpha} \le \frac{1}{2} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha} \quad \text{for all } \mathrm{v}, \mathrm{v}' \in S_{\sigma,r}. \tag{6.14}$$

It is clear that

$$|\rho(t) - \rho'(t)| \le \int_0^t |\mathrm{v}(\tau) - \mathrm{v}'(\tau)|\, d\tau \le \frac{t^\alpha}{\alpha} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha}. \tag{6.15}$$

Denote $\hat{\mathrm{v}}_i' = \mathcal{A}_i[\mathrm{v}']$, $i = 1, 2, 3$. We claim that

$$\begin{aligned} &|t^{1-\alpha}(\hat{\mathrm{v}}_1(t) - \hat{\mathrm{v}}_1'(t))| \le C M_1 t^{\alpha/2} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha}, \\ &|t^{1-\alpha}(\hat{\mathrm{v}}_2(t) - \hat{\mathrm{v}}_2'(t))| \le C t^{1+\alpha} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha}. \end{aligned} \tag{6.16}$$

Indeed, the first estimate is obtained in a very similar way to (6.6) with $\zeta_\mu = \mu\rho(t) + (1-\mu)\rho'(t)$. Combine (6.15), (3.15) and (3.17) to deduce that

$$\begin{aligned} &|t^{1-\alpha}(\hat{\mathrm{v}}_1(t) - \hat{\mathrm{v}}_1'(t))| \\ &\qquad \le 2 \int_0^b |\dot{\varphi}(\xi)|\, d\xi \int_0^1 t |N_x(\mu\rho(t) + (1-\mu)\rho'(t), t, \xi, 0)|\, d\mu\, \frac{|\rho(t) - \rho'(t)|}{t^\alpha} \\ &\qquad \le C M_1 \int_0^b d\xi \int_0^1 \bigg[\frac{|\zeta_\mu - \xi|}{t^{\alpha/2}} \exp\bigg(-\kappa \bigg[\frac{|\zeta_\mu - \xi|}{t^{\alpha/2}} \bigg]^{\frac{2}{2-\alpha}} \bigg) \\ &\qquad\quad + \frac{|\zeta_\mu + \xi|}{t^{\alpha/2}} \exp\bigg(-\kappa \bigg[\frac{|\zeta_\mu + \xi|}{t^{\alpha/2}} \bigg]^{\frac{2}{2-\alpha}} \bigg) \bigg] d\mu \\ &\qquad \le C \Upsilon_\kappa t^{\alpha/2} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha} \le C |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha} \\ &\qquad \le C M_1 t^{\alpha/2} |\mathrm{v} - \mathrm{v}'|_{\sigma,1-\alpha}. \end{aligned}$$

In order to prove the second estimate in (6.16) we observe that $\xi_\mu = \mu\rho(t) + (1-\mu)\rho'(t) \ge b$. By virtue of (6.15), (3.15) and (3.17) with $n = 0$, we have

$$\begin{aligned} &|t^{1-\alpha}(\hat{\mathrm{v}}_2(t) - \hat{\mathrm{v}}_2'(t))| \\ &\qquad \le C \int_0^t |\dot{f}(\tau)|\, d\tau \int_0^1 t |K_x(\mu\rho(t) + (1-\mu)\rho'(t), t - \tau)|\, d\mu\, \frac{|\rho(t) - \rho'(t)|}{t^\alpha} \end{aligned}$$

$$
\begin{aligned}
&\le C|f|^{(1)}_{\sigma,1-\alpha} t \int_0^t d\tau \int_0^1 \frac{|\xi_\mu|\tau^{\alpha-1}}{(t-\tau)^{\alpha/2+1}} \exp\left(-\kappa\left[\frac{|\xi_\mu|}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\mu \, |\mathrm{v}-\mathrm{v}'|_{\sigma,1-\alpha} \\
&\le C|f|^{(1)}_{\sigma,1-\alpha} t \int_0^t \frac{\tau^{\alpha-1}}{(t-\tau)} \, d\tau \int_0^1 \exp\left(-\frac{\kappa}{2}\left[\frac{|\xi_\mu|}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\mu \, |\mathrm{v}-\mathrm{v}'|_{\sigma,1-\alpha} \\
&\le C|f|^{(1)}_{\sigma,1-\alpha} t \int_0^t \frac{\tau^{\alpha-1}}{(t-\tau)} \exp\left(-\frac{\kappa}{2}\left[\frac{b}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\tau \, |\mathrm{v}-\mathrm{v}'|_{\sigma,1-\alpha} \\
&\le C|f|^{(1)}_{\sigma,1-\alpha} t b^{-2/\alpha} \int_0^t \tau^{\alpha-1} \, d\tau \, |\mathrm{v}-\mathrm{v}'|_{\sigma,1-\alpha} \\
&\le C t^{1+\alpha} |\mathrm{v}-\mathrm{v}'|_{\sigma,1-\alpha}.
\end{aligned}
$$

Here we have used the following simple inequalities:

$$
\frac{|\zeta_\mu|}{(t-\tau)^{\alpha/2}} \exp\left(-\kappa\left[\frac{|\xi_\mu|}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) \le C \exp\left(-\frac{\kappa}{2}\left[\frac{|\xi_\mu|}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right),
$$

$$
\begin{aligned}
\frac{\exp\left(-\frac{\kappa}{2}\left[\frac{b}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)}{(t-\tau)} &= b^{-2/\alpha}\left(\frac{b}{(t-\tau)^{\alpha/2}}\right)^{2/\alpha} \exp\left(-\frac{\kappa}{2}\left[\frac{b}{(t-\tau)^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) \\
&\le C b^{-2/\alpha}.
\end{aligned}
$$

This completes the proof of (6.16).

Next we rewrite the difference $\hat{\mathrm{v}}_3(t) - \hat{\mathrm{v}}'_3(t)$ as

$$
\begin{aligned}
\text{(6.17)} \qquad \hat{\mathrm{v}}_3(t) - \hat{\mathrm{v}}'_3(t) &= R_1 + R_2 \\
&= 2\int_0^t \big(\mathrm{v}(\tau)K_x(\rho(t)-\rho(\tau), t-\tau) - \mathrm{v}'(\tau)K_x(\rho'(t)-\rho'(\tau), t-\tau)\big) \, d\tau \\
&\quad + 2\int_0^t \big(\mathrm{v}'(\tau)K_x(\rho'(t)+\rho'(\tau), t-\tau) - \mathrm{v}(\tau)K_x(\rho(t)+\rho(\tau), t-\tau)\big) \, d\tau.
\end{aligned}
$$

By (3.15) we further decompose R_1 into the sum of three integrals:

$$
\begin{aligned}
R_1 = 2\int_0^t \Bigg(&\mathrm{v}'(\tau)\frac{\rho'(t)-\rho'(\tau)}{(t-\tau)^\alpha} L(\rho'(t)-\rho'(\tau), t-\tau) \\
&- \mathrm{v}(\tau)\frac{\rho(t)-\rho(\tau)}{(t-\tau)^\alpha} L(\rho(t)-\rho(\tau), t-\tau)\Bigg) \, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^t ((\mathrm{v}'(\tau)-\mathrm{v}(\tau))\frac{\rho'(t)-\rho'(\tau)}{(t-\tau)^\alpha}L(\rho'(t)-\rho'(\tau),t-\tau)\,d\tau\\
&\quad+2\int_0^t \mathrm{v}(\tau)\frac{\rho'(t)-\rho'(\tau)-(\rho(t)-\rho(\tau))}{(t-\tau)^\alpha}L(\rho'(t)-\rho'(\tau),t-\tau)\,d\tau\\
&\quad+2\int_0^t \mathrm{v}(\tau)\frac{\rho(t)-\rho(\tau)}{(t-\tau)^\alpha}\big(L(\rho'(t)-\rho'(\tau),t-\tau)-L(\rho(t)-\rho(\tau),t-\tau)\big)\,d\tau\\
&= R_1^{(1)}(t)+R_1^{(2)}(t)+R_1^{(3)}(t).
\end{aligned}
$$

From (3.17) and (6.4) one gets

$$
\begin{aligned}
|t^{1-\alpha}R_1^{(1)}(t)| &\le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}\int_0^t \frac{t^{1-\alpha}\tau^{\alpha-1}}{(t-\tau)^{1-\alpha/2}}\,d\tau\\
&\le Ct^{\alpha/2}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}.
\end{aligned}\tag{6.18}
$$

An easy computation shows that

$$
|\rho'(t)-\rho'(\tau)-(\rho(t)-\rho(\tau))| \le \frac{(t-\tau)^\alpha}{\alpha}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}\tag{6.19}
$$

so that

$$
\begin{aligned}
|t^{1-\alpha}R_1^{(2)}(t)| &\le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}\int_0^t \frac{t^{1-\alpha}\tau^{\alpha-1}}{(t-\tau)^{1-\alpha/2}}\,d\tau\\
&\le Ct^{\alpha/2}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}.
\end{aligned}\tag{6.20}
$$

Using (3.17), (6.19) and denoting

$$
\eta = \mu(\rho'(t)-\rho'(\tau))+(1-\mu)(\rho(t)-\rho(\tau)),
$$

we obtain

$$
\begin{aligned}
&|t^{1-\alpha}R_1^{(3)}(t)|\\
&\quad\le Ct^{1-\alpha}\int_0^t \tau^{\alpha-1}\,d\tau\int_0^1 |L_x(\eta,t-\tau)(\rho'(t)-\rho'(\tau)-\rho(t)+\rho(\tau))|\,d\mu\\
&\quad\le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}t^{1-\alpha}\int_0^t \tau^{\alpha-1}(t-\tau)^{\alpha-1}\,d\tau\\
&\quad\le Ct^{\alpha}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}.
\end{aligned}\tag{6.21}
$$

By construction

$$
\rho(t)+\rho(\tau)\ge 2b,\qquad \rho'(t)+\rho'(\tau)\ge 2b.
$$

The second integral R_2 is estimated in the following way:

$$R_2(t) = 2\int_0^t (\mathrm{v}'(\tau) - \mathrm{v}(\tau))K_x(\rho'(t) + \rho'(\tau), t-\tau)\,d\tau \qquad (6.22)$$
$$+ \int_0^t \mathrm{v}(\tau)\,d\tau \int_{\rho(t)+\rho(\tau)}^{\rho'(t)+\rho'(\tau)} K_{xx}(x, t-\tau))\,dx$$
$$= R_2^{(1)}(t) + R_2^{(2)}(t),$$

where

$$|t^{1-\alpha}R_2^{(1)}(t)| \qquad (6.23)$$
$$\le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}\int_0^t \frac{t^{1-\alpha}\tau^{\alpha-1}}{t-\tau}\exp\left[-\kappa\left(\frac{2b}{(t-\tau)^{\alpha/2}}\right)^{\frac{2}{2-\alpha}}\right]d\tau$$
$$\le Ct^{\alpha/2}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}$$

and

$$|t^{1-\alpha}R_2^{(2)}(t)| \le Ct^{1-\alpha}\int_0^t \frac{\tau^{\alpha-1}}{(t-\tau)^{\alpha/2+1}}\exp\left[-\kappa\left(\frac{2b}{(t-\tau)^{\alpha/2}}\right)^{\frac{2}{2-\alpha}}\right] \qquad (6.24)$$
$$\times (t-\tau)^{\alpha}|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}\,d\tau$$
$$\le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}t^{1-\alpha}\int_0^t \frac{\tau^{\alpha-1}}{(t-\tau)^{1-\alpha/2}}\,d\tau \le C|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}t^{\alpha/2}.$$

Summing up the estimates (6.16), (6.18), (6.20), (6.21), (6.23) and (6.24) gives

$$|\hat{\mathrm{v}}'-\hat{\mathrm{v}}|_{\sigma,1-\alpha} \le [C\sigma^{\alpha/2} + C\sigma^{\alpha}]|\mathrm{v}'-\mathrm{v}|_{\sigma,1-\alpha}. \qquad (6.25)$$

Pick σ_2 such that the expression in square brackets is less than $1/2$ and finally set $\sigma = \min\{\sigma_1, \sigma_2\}$. Thus we have proved the following result.

THEOREM 6.1. *Let assumptions* (H1)–(H4) *hold. Then for sufficiently small σ there exists a unique solution of equation* (6.1) *in the ball $S_{\sigma,r}$.*

7. Proof of Theorem 2.3. We return to formula (4.21) and define

$$\mathrm{u}(x,t) = \int_0^b \varphi(\xi)\mathcal{G}(x,t,\xi,0)\,d\xi - 2\int_0^t f(\tau)K_x(x,t-\tau)\,d\tau \qquad (7.1)$$
$$+ \int_0^t \mathrm{v}(\tau)\mathcal{G}(x,t,\rho(\tau),\tau)\,d\tau$$
$$= \mathrm{u}_1(x,t) + \mathrm{u}_2(x,t) + \mathrm{u}_3(x,t),$$

with v, $\rho(t)$ being the solution of (6.1)–(6.2).

By definition of v, ρ and Lemma 3.2 we obtain

$$\lim_{x\to\varrho(t)-} D_{0+}^{1-\alpha}\mathrm{u}_x(x,t) = D_{0+}^{1-\alpha}\mathrm{u}_{1,x}(\rho(t),t) + D_{0+}^{1-\alpha}\mathrm{u}_{2,x}(\rho(t),t) + \frac{1}{2}\mathrm{v}(t) + \int_0^t \mathrm{v}(\tau)G_x(\rho(t),t,\rho(\tau),\tau)\,d\tau = \mathrm{v}(t),$$

$$\lim_{x\to\varrho(t)+} D_{0+}^{1-\alpha}\mathrm{u}_x(x,t) = D_{0+}^{1-\alpha}\mathrm{u}_{1,x}(\rho(t),t) + D_{0+}^{1-\alpha}\mathrm{u}_{2,x}(\rho(t),t) - \frac{1}{2}\mathrm{v}(t) + \int_0^t \mathrm{v}(\tau)G_x(\rho(t),t,\rho(\tau),\tau)\,d\tau.$$

This implies

$$\lim_{x\to\varrho(t)+0} D_{0+}^{1-\alpha}\mathrm{u}_x(x,t) = 0, \qquad t>0.$$

Repeating the arguments of Section 5 leads to the conclusion that u is a solution of the "inner" problem

$$\begin{aligned} \mathrm{u}_t(x,t) &= (D_{0+}^{1-\alpha}\mathrm{u}_x)_x(x,t) \quad \text{in } Q_\sigma^\rho,\\ \mathrm{u}(0,t) &= f(t) > 0, \quad 0\le t\le\sigma,\\ \mathrm{u}(x,0) &= \varphi(x) \quad \text{for } 0\le x\le b,\\ \dot\rho(t) &= -\lim_{x\to\varrho(t)-} D_{0+}^{1-\alpha}\mathrm{u}_x(x,t) \quad \text{for } 0<t\le\sigma, \end{aligned} \tag{7.2}$$

and "outer" problem

$$\begin{aligned} \mathrm{u}_t(x,t) &= (D_{0+}^{1-\alpha}\mathrm{u}_x)_x(x,t) \quad \text{in } \hat{Q}_\sigma^\rho,\\ \mathrm{u}(x,0) &= 0 \quad \text{for } x>b,\\ \lim_{x\to\varrho(t)+} D_{0+}^{1-\alpha}\mathrm{u}_x(x,t) &= 0 \quad \text{for } 0<t\le\sigma. \end{aligned} \tag{7.3}$$

Let us prove that

$$\mathrm{u}(x,t) = 0 \quad \text{in } \hat{Q}_\sigma^\rho. \tag{7.4}$$

To do so we have to study the behaviour of u and its derivatives as $x\to\infty$. We have

$$\begin{aligned} x-\xi &\ge x/2 \quad \text{for any } \xi \text{ such that } \xi\in(0,b),\\ x-\rho(t) &\ge x/2 \quad \text{for all } t\in[0,\sigma], \end{aligned} \tag{7.5}$$

whenever $x \ge 2\sup_{(0,\sigma)}|\rho(t)| = 2\rho(\sigma)$. Combining estimates (7.5), (3.14), (3.17) and

$$|y|^\gamma \exp(-\kappa|y|^{\frac{2}{2-\alpha}}) \le C(\gamma,\kappa), \qquad y,\gamma>0,$$

we obtain

$$|\mathrm{u}_x(x,t)| \le C(\sigma,\alpha,\gamma)(M_0+M_f+|\mathrm{v}|_{\sigma,1-\alpha})x^{-\gamma} \tag{7.6}$$

for any $\gamma \geq 1$ and $x \geq 2\rho(\sigma)$. We need the following assertion:

$$D^{1-\alpha}_{0+}(\mathrm{u}_x)^2(x,t) \leq 2\mathrm{u}_x(x,t) D^{1-\alpha}_{0+}\mathrm{u}_x(x,t) \quad \text{in } Q^\rho_\sigma, \tag{7.7}$$

based on the "fundamental identity"

$$\begin{aligned} D^{1-\alpha}_{0+}(\mathrm{u}_x)^2(x,t) = 2\mathrm{u}_x(x,t) D^{1-\alpha}_{0+}\mathrm{u}_x(x,t) - \frac{t^{\alpha-1}}{\Gamma(2-\alpha)}(\mathrm{u}_x)^2(x,t) \\ - \frac{1-\alpha}{\Gamma(2-\alpha)} \int_0^t \frac{|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2}{(t-\tau)^{2-\alpha}}\, d\tau, \end{aligned} \tag{7.8}$$

(see [7, Lemma 18.4.4], [27], [1]). The results in [7, 27] are more general but the approach in [1] is more suitable in our case. We set

$$D^{1-\alpha}_{0+}(\mathrm{u}_x)^2(x,t) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(I^\alpha_{0+}u(x,t+\epsilon) - I^\alpha_{0+}u(x,t)),$$

and after some calculations we reach the right hand side of (7.8) provided that the last integral is finite (for more details see [1, Lemma 1]). Let us make sure about that. We consider u_x as a sum of the following form (cf. (5.1), (5.9)):

$$\begin{aligned} \mathrm{u}_x(x,t) &= \int_0^b \dot\varphi(\xi)\mathcal{N}(x,t,\xi,0)\, d\xi \\ &\quad - 2\int_0^t \dot f(\tau)\mathcal{K}(x,t-\tau)\, d\tau + \int_0^t v(\tau)\mathcal{G}_x(x,t,\varrho(\tau),\tau)\, d\tau \\ &= w_1(x,t) + w_2(x,t) + w_3(x,t). \end{aligned}$$

Let $0<\tau<t\leq\sigma$. Define $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_{+,\sigma} = \{(x,t) : x\in\mathbb{R},\ t\in(0,\sigma)\}$. By routine calculations we obtain

$$\sup_{\mathbb{R}_{+,\sigma}} |\mathrm{u}_x(x,t)| \leq C(M_1 + M_f\sigma^{\alpha/2} + |v|_{\sigma,1-\alpha}) \equiv \mathcal{M}. \tag{7.9}$$

In order to estimate the difference $\mathrm{u}_x(x,t) - \mathrm{u}_x(x,\tau)$ let us first consider the integral w_3. We have

$$\begin{aligned} &w_3(x,t) - w_3(x,\tau) \\ &\quad = \int_\tau^t v(s)\mathcal{G}_x(x,t,\varrho(s),s)\, ds + \int_0^\tau v(s)(\mathcal{G}_x(x,t,\varrho(s),s) - \mathcal{G}_x(x,\tau,\varrho(s),s))\, ds \\ &\quad = W_1(x,t) + W_2(x,t). \end{aligned}$$

Estimate (3.14) gives

$$|W_1(x,t)| \leq C|v|_{\sigma,1-\alpha}\int_\tau^t s^{\alpha-1}(t-s)^{-\alpha}\, ds \leq C|v|_{\sigma,1-\alpha}\left(\frac{t-\tau}{\tau}\right)^{1-\alpha}.$$

Then

$$W_2(x,t) = \int_0^\tau v(s)\,ds \int_0^1 \mathcal{G}_{x\lambda}(x, \lambda(t-\tau)+\tau, \varrho(s), s)\,d\lambda$$
$$= \int_0^{\tau/2} \ldots ds + \int_{\tau/2}^{\tau} \ldots ds = W'(x,t) + W''(x,t),$$

Applying (3.14) yields

$$|W'(x,t)| \le C|v|_{\sigma,1-\alpha} \int_0^{\tau/2} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)\,d\lambda}{(\lambda(t-\tau)+\tau-s)^{1+\alpha}}$$
$$\le C|v|_{\sigma,1-\alpha} \int_0^{\tau/2} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)\,d\lambda}{(\lambda(t-\tau))^{\alpha}(\tau-s)}$$
$$\le C|v|_{\sigma,1-\alpha} \left(\frac{t-\tau}{\tau}\right)^{1-\alpha}.$$

Let $\vartheta \in \left(0, \frac{1-\alpha}{2}\right)$. Then

$$|W''(x,t)| \le C|v|_{\sigma,1-\alpha} \int_{\tau/2}^{\tau} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)\,d\lambda}{(\lambda(t-\tau))^{\frac{1+\alpha}{2}-\vartheta}(\tau-s)^{\frac{1+\alpha}{2}+\vartheta}}$$
$$\le C|v|_{\sigma,1-\alpha}\tau^{\alpha-1} \int_{\tau/2}^{\tau} ds \int_0^1 \lambda^{\vartheta-\frac{1+\alpha}{2}}(t-\tau)^{1+\vartheta-\frac{1+\alpha}{2}}(\tau-s)^{-\vartheta-\frac{1+\alpha}{2}}\,d\lambda.$$

The inequalities

$$1+\vartheta-\frac{1+\alpha}{2} > 0, \qquad 1-\vartheta-\frac{1+\alpha}{2} = \frac{1-\alpha}{2}-\vartheta > 0$$

and routine calculations give

$$|W''(x,t)| \le C|v|_{\sigma,1-\alpha}\left(\frac{t-\tau}{\tau}\right)^{\frac{1-\alpha}{2}+\vartheta}.$$

Estimating $|w_1(x,t) - w(x,\tau)|$ is much simpler:

$$|w_1(x,t) - w_1(x,\tau)| = \left|\int_0^b \dot\varphi(\xi) \int_\tau^t \mathcal{N}_s(x,s,\xi,0)\,ds\right|$$
$$\le CM_1 b \int_\tau^t s^{-\alpha/2-1}\,ds \le CM_1 b\,\frac{t-\tau}{\tau^{\alpha/2+1}}.$$

The method of estimating $|w_3(x,t) - w_3(x,\tau)|$ can be adapted to the estimation of $|w_2(x,t) - w_2(x,\tau)|$:

$$
\begin{aligned}
w_2(x,t) - w_2(x,\tau) &= -2\int_{\tau}^{t} \dot f(s)\mathcal{K}(x,t-s)\,ds \\
&\quad + 2\int_{0}^{\tau} \dot f(s)(\mathcal{K}(x,\tau-s) - \mathcal{K}(x,t-s))\,ds \\
&= W_2^{(1)}(x,t) + W_2^{(2)}(x,t).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
W_2^{(1)}(x,t) &\le CM_f \int_{\tau}^{t} s^{\alpha-1}(t-s)^{-\alpha/2}\,ds \\
&\le CM_f\tau^{\alpha-1}(t-\tau)^{1-\alpha/2} \le CM_f\sigma^{\alpha/2}\Big(\frac{t-\tau}{\tau}\Big)^{1-\alpha}.
\end{aligned}
$$

Split $W_2^{(2)}(x,t)$ into the sum of two integrals

$$
\begin{aligned}
W_2^{(2)}(x,t) &= -2\int_0^{\tau} \dot f(s)\,ds \int_0^1 \mathcal{K}_t(x,\lambda(t-\tau)+\tau-s)(t-\tau)\,d\lambda \\
&= \int_0^{\tau/2} \ldots ds + \int_{\tau/2}^{\tau} \ldots ds = W_2^{(2,1)}(x,t) + W_2^{(2,2)}(x,t).
\end{aligned}
$$

In virtue of (3.14) we get

$$
\begin{aligned}
|W_2^{(2,1)}(x,t)| &\le CM_f \int_0^{\tau/2} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)d\lambda}{(\lambda(t-\tau)+\tau-s)^{1+\alpha/2}} \\
&\le CM_f \int_0^{\tau/2} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)^{1-\alpha/2}}{\lambda^{\alpha/2}(\tau-s)}\,d\lambda \\
&\le CM_f\tau^{\alpha-1}(t-\tau)^{1-\alpha/2} \le CM_f\sigma^{\alpha/2}\Big(\frac{t-\tau}{\tau}\Big)^{1-\alpha}.
\end{aligned}
$$

We take the same $\vartheta \in (0, \frac{1-\alpha}{2})$ as in W'':

$$
\begin{aligned}
|W_2^{(2,2)}(x,t)| &\le CM_f \int_{\tau/2}^{\tau} s^{\alpha-1}\,ds \int_0^1 \frac{(t-\tau)d\tau}{(\lambda(t-\tau))^{\frac{1+\alpha}{2}-\vartheta}(\tau-s)^{\frac{1}{2}+\vartheta}} \\
&\le CM_f\tau^{\alpha-1} \int_{\tau/2}^{\tau} ds \int_0^1 \lambda^{\vartheta-\frac{1+\alpha}{2}}(t-\tau)^{\vartheta+\frac{1-\alpha}{2}}(\tau-s)^{-\frac{1}{2}-\vartheta}\,d\lambda \\
&\le CM_f\tau^{\alpha-1+\frac{1}{2}-\vartheta}(t-\tau)^{\vartheta+\frac{1-\alpha}{2}} \le CM_f\sigma^{\alpha/2}\Big(\frac{t-\tau}{\tau}\Big)^{\vartheta+\frac{1-\alpha}{2}}.
\end{aligned}
$$

Thus, we conclude that

$$
(7.10) \qquad |\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)| \le C(M_1, M_f, |v|_{\sigma,1-\alpha},\sigma) \times \left[\frac{1}{\tau^{\alpha/2}}\,\frac{t-\tau}{\tau} + \left(\frac{t-\tau}{\tau}\right)^{1-\alpha} + \left(\frac{t-\tau}{\tau}\right)^{\vartheta+\frac{1-\alpha}{2}}\right].
$$

In virtue of (7.9), we infer

$$
\begin{aligned}
&|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2 \\
&\quad \le C\Bigg[\frac{1}{\tau^{\alpha}}\left(\frac{t-\tau}{\tau}\right)^{2} + \left(\frac{t-\tau}{\tau}\right)^{2-2\alpha} + \left(\frac{t-\tau}{\tau}\right)^{2\vartheta+1-\alpha} \\
&\qquad + \frac{1}{\tau^{\alpha/2}}\left(\frac{t-\tau}{\tau}\right)^{2-\alpha} + \frac{1}{\tau^{\alpha/2}}\left(\frac{t-\tau}{\tau}\right)^{1+\vartheta+\frac{1-\alpha}{2}} + \left(\frac{t-\tau}{\tau}\right)^{1-\alpha+\vartheta+\frac{1-\alpha}{2}}\Bigg].
\end{aligned}
$$

The last estimate and (7.9) give

$$
\begin{aligned}
&\int_0^t \frac{|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2}{(t-\tau)^{2-\alpha}}\,d\tau \\
&\qquad = \int_0^{t/2} \frac{|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2}{(t-\tau)^{2-\alpha}}\,d\tau + \int_{t/2}^{t} \frac{|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2}{(t-\tau)^{2-\alpha}}\,d\tau \\
&\qquad \le [2\mathcal{M}]^2\left(\frac{t}{2}\right)^{\alpha-1} + C(M_1, M_f, |v|_{\sigma,1-\alpha},\sigma)\left[\frac{1}{t^{\alpha}} + \frac{1}{t^{\alpha/2}} + 1\right]t^{\alpha-1}.
\end{aligned}
$$

Thus we have

$$
(7.11) \qquad \int_0^t \frac{|\mathrm{u}_x(x,t)-\mathrm{u}_x(x,\tau)|^2}{(t-\tau)^{2-\alpha}}\,d\tau \le C(\mathcal{M}, M_1, M_f, |v|_{\sigma,1-\alpha},\sigma)t^{-1} \quad \text{for any } t>0.
$$

Estimate (7.11) implies identity (7.8), which in turn yields (7.7). Multiplying (7.3) by $\mathrm{u}(x,t)$ and integrating over $\hat{Q}^{\rho}_{\sigma}$ gives

$$
(7.12) \qquad \int_0^t d\tau \int_{\rho(\tau)}^{\infty} \frac{(\mathrm{u}^2)_\tau(x,\tau)}{2}\,dx + \frac{1}{2}\int_0^t d\tau \int_{\rho(\tau)}^{\infty} D^{1-\alpha}_{0+}\mathrm{u}_x(x,\tau)\mathrm{u}_x(x,\tau)\,dx = 0.
$$

By (7.6), all integrals in (7.12) are finite. Applying (7.7), we find that

$$
\begin{aligned}
&\int_0^t \left[\frac{\partial}{\partial\tau}\int_{\rho(\tau)}^{\infty} \mathrm{u}^2(x,\tau)dx + \dot\rho(\tau)\mathrm{u}^2(\rho(\tau),\tau)\right]d\tau \\
&\qquad + \int_0^t \left[\frac{\partial}{\partial\tau}\int_{\rho(\tau)}^{\infty} I^{\alpha}_{0+}(\mathrm{u}_x)^2(x,\tau)\,dx + \dot\rho(\tau)I^{\alpha}_{0+}(\mathrm{u}_x)^2(\rho(\tau),\tau)\,d\tau\right] \le 0.
\end{aligned}
$$

By monotonicity of ρ and initial conditions at zero we conclude

$$\int_{\rho(t)}^{\infty} \mathrm{u}^2(x,t)\,dx + 2\int_{\rho(t)}^{\infty} I^{\alpha}_{0+}(\mathrm{u}_x)^2(x,t)\,dx \le 0. \tag{7.13}$$

Inequality (7.13) entails (7.4).

Firstly we see that

$$\mathrm{u}(\rho(t),t) = 0, \qquad t > 0. \tag{7.14}$$

Secondly, we deduce

$$\begin{aligned} D^{1-\alpha}_{0+}\mathrm{u}_x(x,t) &= \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\mathrm{u}_x(x,\tau)\,d\tau \\ &= \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\Big[\int_0^{l(x)} (t-\tau)^{\alpha-1}\mathrm{u}_x(x,\tau)\,d\tau + \int_{l(x)}^{t} (t-\tau)^{\alpha-1}\mathrm{u}_x(x,\tau)\,d\tau\Big] \\ &= \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\Big[\int_{l(x)}^{t} (t-\tau)^{\alpha-1}\mathrm{u}_x(x,\tau)\,d\tau\Big] \\ &= D^{1-\alpha}_{l(x)+}\mathrm{u}_x(x,t). \end{aligned} \tag{7.15}$$

From (7.2), (7.14) and (7.15), the functions u, ρ satisfy (1.8). Therefore one should verify that they have the regularity properties (A1)–(A7). In view of Theorem 6.1 we see that (A1) holds for $\rho(t)$. As for $\mathrm{u}(x,t)$, let us consider u_x, u_t in more detail.

The function $\mathrm{u}(x,t)$ defined by (7.1) becomes zero in $\hat{Q}^{\rho}_{\sigma}$. Mainly basing on Lemma 3.4 we find that $\mathrm{u}_x(x,t)$ is continuous on $(0,\infty)\times(0,\sigma)$ and hence for $t>0$ not only $\mathrm{u}(\rho(t),t)=0$, but also

$$\mathrm{u}_x(\rho(t),t) = 0 \qquad \text{for any } t>0.$$

REMARK 7.1. The last identity agrees with the one given in [15].

In addition, one can give more precise information about the behaviour of $\mathrm{u}_x(x,t)$ near $t=0$. Assume $x\in[0,b)$. Let $\varepsilon\in(0,b)$ be such that $x\in[0,b-\varepsilon]$. We have

$$\begin{aligned} w_1(x,t) - \dot{\varphi}(x) &= \int_0^b (\dot{\varphi}(\xi)-\dot{\varphi}(x))\mathcal{N}(x,t,\xi,0)\,d\xi - \dot{\varphi}(x)\int_b^{\infty}\mathcal{N}(x,t,\xi,0)\,d\xi \\ &= w_1'(x,t) + w_1''(x,t). \end{aligned}$$

We obtain

$$|w_1'(x,t)| \le CM_2 \int_0^b |\xi - x| t^{-\alpha/2} \exp\left(-\kappa\left[\frac{|x-\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\xi \le CM_2 t^{\alpha/2},$$

$$\begin{aligned}|w_1''(x,t)| &\le CM_1 \int_b^\infty t^{-\alpha/2} \exp\left(-\kappa\left[\frac{|x-\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\xi \\ &\le CM_1 \int_b^\infty t^{-\alpha/2} \exp\left(-\frac{\kappa}{2}\left[\frac{|x-\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) \exp\left(-\frac{\kappa}{2}\left[\frac{\varepsilon}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\xi \\ &\le C\frac{M_1}{\varepsilon} t^{\alpha/2}.\end{aligned}$$

Applying (7.9) yields

$$|w_2(x,t)| \le CM_f t^{\alpha/2}.$$

Then

$$\begin{aligned}|w_3(x,t)| &\le C|v|_{\sigma,1-\alpha} \int_0^t \tau^{\alpha-1}(t-\tau)^{-\alpha} \exp\left(-\kappa\left[\frac{|x-\rho(\tau)|}{t^{\alpha/2}}\right]^{2/2-\alpha}\right) d\tau \\ &\le C|v|_{\sigma,1-\alpha} \int_0^t \tau^{\alpha-1}(t-\tau)^{-\alpha} \exp\left(-\kappa\left[\frac{\varepsilon}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\tau \\ &\le C\frac{|v|_{\sigma,1-\alpha}}{\varepsilon} t^{\alpha/2}.\end{aligned}$$

Finally, we get

$$|\mathrm{u}_x(x,t) - \dot{\varphi}(x)| \le C\left(\frac{M_1 + |v|_{\sigma,1-\alpha}}{\varepsilon} + M_2 + M_f\right) t^{\alpha/2} \quad \text{for any } x \in [0, b-\varepsilon].$$

The arguments above confirm that the derivative u_x satisfies (A3).

Let us now study $\mathrm{u}_t(x,t)$. Assume that $x, t > 0$ and consider separately each term on the right side of (7.1).

In virtue of (3.2) we have

$$\int_0^\infty \mathcal{G}(x,\xi,t,0)\, d\xi = 1, \qquad \mathrm{u}_{1,t}(x,t) = \int_0^\infty (\overline{\varphi}(\xi) - \overline{\varphi}(x)) \mathcal{G}_t(x,\xi,t,0)\, d\xi.$$

By (3.14) we deduce that

$$\begin{aligned}|\mathrm{u}_{1,t}(x,t)| &\le C\frac{M_1}{t} \int_0^\infty \frac{|x-\xi|}{t^{\alpha/2}} \left[\exp\left(-\kappa\left[\frac{|x-\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right. \\ &\qquad \left. + \exp\left(-\kappa\left[\frac{|x+\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\right] d\xi \\ &\le 2C\frac{M_1}{t} \int_0^\infty \frac{|x-\xi|}{t^{\alpha/2}} \exp\left(-\kappa\left[\frac{|x-\xi|}{t^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right) d\xi \le CM_1 t^{\alpha/2-1},\end{aligned}$$

and

$$\sup_{(x,t)\in\mathbb{R}_{+,\sigma}} |t^{1-\alpha/2}\mathrm{u}_{1,t}(x,t)| \le CM_1. \tag{7.16}$$

Next

$$\mathrm{u}_{2,t}(x,t) = -2\int_0^t \dot f(\tau)K_x(x,t-\tau)\,d\tau = -2\Big(\int_0^{t/2}\ldots d\tau + \int_{t/2}^t \ldots d\tau\Big).$$

In virtue of (3.15) and (3.17) we get

$$\begin{aligned} |\mathrm{u}_{2,t}(x,t)| &\le CM_f\bigg[\int_{t/2}^t \frac{(t-\tau)^{\alpha-1}}{t}\,\frac{|x|}{\tau^{\alpha/2}}\exp\bigg(-\kappa\bigg[\frac{|x|}{\tau^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)\,d\tau \\ &\quad + t^{\alpha-1}\int_0^{t/2}\exp\bigg(-\kappa\bigg[\frac{|x|}{\tau^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)\frac{|x|}{\tau^{\alpha/2+1}}\,d\tau\bigg] \\ &\le CM_f t^{\alpha-1}\Big[1+\int_0^\infty \exp(-\kappa|\eta|^{\frac{2}{2-\alpha}})\,d\eta\Big], \end{aligned} \tag{7.17}$$

and

$$\sup_{(x,t)\in\mathbb{R}_{+,\sigma}} |t^{1-\alpha}\mathrm{u}_{2,t}(x,t)| \le CM_f. \tag{7.18}$$

Further we fix any $\omega\in(0,b)$ and assume that $|x-\rho(t)|\ge\omega$. By (5.5) we have

$$\mathrm{u}_{3,t}(x,t) = \int_0^t \mathrm{v}(\tau)\mathcal{G}_t(x,t,\rho(\tau),\tau)\,d\tau.$$

Note that

$$\begin{aligned} &x+\rho(t)\ge b, \\ &-|x-\rho(\tau)| \le -|x-\rho(t)| + |\rho(t)-\rho(\tau)| \le -\omega + \frac{3r}{2\alpha}(t-\tau)^\alpha. \end{aligned} \tag{7.19}$$

Looking ahead we say that the estimate of $\mathrm{u}_{3,t}$ is similar to the estimate of $\mathrm{u}_{2,t}$. By applying (3.14) and (7.19), we infer that

$$\begin{aligned} |\mathrm{u}_{3,t}(x,t)| &\le C(r,\sigma)|v|_{\sigma,1-\alpha}\bigg[\int_0^t \frac{(t-\tau)^{\alpha-1}}{\tau^{\alpha/2+1}}\exp\bigg(-\kappa\bigg[\frac{\omega}{\tau^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)\,d\tau \\ &\quad + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\tau^{\alpha/2+1}}\exp\bigg(-\kappa\bigg[\frac{b}{\tau^{\alpha/2}}\bigg]^{\frac{2}{2-\alpha}}\bigg)\,d\tau\bigg] \\ &= U'(x,t)+U''(x,t). \end{aligned}$$

We check at once that (cf. (7.17))

$$U'(x,t) \le C(r,\sigma)\left[\int\limits_{t/2}^{t} \frac{(t-\tau)^{\alpha-1}}{t\omega}\,\frac{\omega}{\tau^{\alpha/2}}\exp\left(-\kappa\left[\frac{\omega}{\tau^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)d\tau \right. \tag{7.20}$$
$$\left. + \frac{t^{\alpha-1}}{\omega}\int\limits_{0}^{t/2}\exp\left(-\kappa\left[\frac{\omega}{\tau^{\alpha/2}}\right]^{\frac{2}{2-\alpha}}\right)\frac{\omega}{\tau^{\alpha/2+1}}\,d\tau\right]$$
$$\le C(r,\sigma)\frac{t^{\alpha-1}}{\omega},$$

and

$$U''(x,t) \le C(r,\sigma)\frac{t^{\alpha-1}}{b}. \tag{7.21}$$

Denote $\mathbb{R}^{\rho}_{+,\sigma}(\omega) = \{(x,t) : x \in \mathbb{R}_+,\ |x-\rho(t)| \ge \omega,\ t \in (0,\sigma)\}$. Combining (7.20) with (7.21) yields

$$\sup_{(x,t)\in\mathbb{R}^{\rho}_{+,\sigma}(\omega)} |t^{1-\alpha}\mathrm{u}_{3,t}(x,t)| \le C(r,\sigma,\omega). \tag{7.22}$$

Now, assumption (A8) holds for u_t due to the estimates (7.16), (7.18), (7.22). The first equation in (7.2) gives (A8) for

$$(D^{1-\alpha}_{l(x)+}\mathrm{u}_x)_x(x,t) = (D^{1-\alpha}_{0+}\mathrm{u}_x)_x(x,t) \quad \text{in } Q^{\rho}_{\sigma}.$$

Thus the pair (u,ρ) forms a solution of the fractional Stefan problem (1.8) in the sense of Definition 2.2. This finishes the proof of Theorem 2.3.

Remark 7.2. We cannot yet confirm that

$$\mathrm{u}(x,t) > 0 \quad \text{for } 0 < x < \rho(t), \tag{7.23}$$

since the maximum principle has not been proven. If (7.23) holds then one can show the uniqueness of the solution following the methods of [16, Chap. V, §9].

Acknowledgements. The author would like to express his gratitude to the referee for careful reading and valuable suggestions. This work was partially supported by a grant from the Simons Foundation (Award 1160640, Presidential Discretionary-Ukraine Support Grants, Mykola V. Krasnoshchok).

References

[1] A. Alsaedi, B. Ahmad and M. Kirane, *Maximum principle for certain generalized time and space fractional diffusion equations*, Quart. Appl. Math. 73 (2015), 163–175.

[2] C. Atkinson, *Moving boundary problems for time fractional and composition dependent diffusion*, Frac. Calc. Appl. Anal. 15 (2012), 207–221.

[3] B. V. Bazaliy and S. P. Degtyarev, *On the classical solvability of a multidimensional Stefan problem with convective movement*, Math. USSR-Sb. 60 (1988), 1–17.

[4] E. Bazhlekova, *Subordination principle for fractional evolution equations*, Fract. Calc. Appl. Anal. 3 (2000), 213–230.

[5] J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford Univ. Press, New York, 1987.

[6] A. Friedman, *Free boundary problems for parabolic equations. I. Melting of solids*, J. Math. Mech. 8 (1959), 499–517.

[7] G. Gripenberg, S. O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge Univ. Press, Cambridge, 1990.

[8] S. C. Gupta, *The Classical Stefan Problem*, Elsevier, Amsterdam, 2018.

[9] J. Kemppainen, *Boundary behavior of the layer potentials for the time fractional diffusion equation in Lipschitz domains*, J. Integral Equations Appl. 23 (2011), 541–563.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Fractional Integrals and Derivatives. Theory and Applications*, Elsevier, Amsterdam, 2006.

[11] M. Krasnoshchok, *Application of the integral equation method to the fractional Stefan problem*, Mat. Stud. 44 (2015), 56–66.

[12] M. Krasnoshchok, *Solvability in Hölder space of an initial boundary value problem for the time-fractional diffusion equation*, J. Math. Phys. Anal. Geom. 12 (2016), 48–77.

[13] M. Krasnoshchok and N. Vasylyeva, *On a solvability of a nonlinear fractional reaction-diffusion system in the Hölder spaces*, Nonlinear Stud. 20 (2013), 591–621.

[14] A. Kubica, P. Rybka and K. Ryszewska, *Weak solutions of fractional differential equations in non cylindrical domains*, Nonlinear Anal. Real World Appl. 36 (2017), 154–182.

[15] A. Kubica and K. Ryszewska, *A self-similar solution to time-fractional Stefan problem*, Math. Methods Appl. Sci. 6 (2021), 4245–4275.

[16] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence, RI, 1968.

[17] J. Liu and M. Xu, *Some exact solutions to Stefan problems with fractional differential equations*, J. Math. Anal. Appl. 351 (2009), 536–542.

[18] A. M. Meirmanov, *The Stefan Problem*, de Gruyter, Berlin, 1992.

[19] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. 339 (2000), no. 1, 77 pp.

[20] Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*, Springer, Cham, 2015.

[21] J. Prüss and G. Simonett, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, Birkhäuser/Springer, 2016.

[22] A. V. Pskhu, *The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain*, Izv. Math. 81 (2017), 1212–1233.

[23] A. V. Pskhu, *The fundamental solution of a diffusion-wave equation of fractional order*, Izv. Math. 73 (2009), 351–392.

[24] S. D. Roscani, J. Bollati and D. A. Tarzia, *A new mathematical formulation for a phase change problem*, Chaos Solitons Fractals 116 (2018), 340–347.

[25] S. D. Roscani, N. D. Caruso and D. A. Tarzia, *Explicit solutions to fractional Stefan-like problems for Caputo and Riemann–Liouville derivatives*, Comm. Nonlinear Sci. Numer. Simul. 90 (2020), art. 105361, 17 pp.

[26] V. A. Solonnikov, *Lectures on evolution free boundary problems: classical solutions*, in: Mathematical Aspects of Evolving Interfaces, Lecture Notes in Math. 1812, Springer, Berlin, 2003, 123–175.

[27] V. Vergara and R. Zacher, *Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods*, SIAM J. Math. Anal. 47 (2015), 210–239.

[28] V. A. Volpert and A. A. Nepomnyashchy, *Similarity solution of a Stefan drug-release subdiffusion problem*, Phys. Scripta 91 (2016), art. 044005, 6 pp.

Mykola Krasnoshchok
Institute of of Applied Mathematics and Mechanics
National Academy of Sciences of Ukraine
84116, Sloviansk, Ukraine
E-mail: iamm012@ukr.net