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ON EXPONENTIAL CONVERGENCE OF RANDOM VARIABLES

Abstract. Given a discrete-time sequence of nonnegative random variables, general dependencies between the exponential convergence of the expectations, the exponential convergence of the trajectories and the logarithmic growth of the corresponding expected hitting times are analysed. The general results are applied to problems of optimization, stochastic control and estimation.

1. Introduction. Various forms of exponential convergence of random variables appear naturally in many applications of Markov chains [6], supermartingales [11, 12] and more general stochastic processes. Applications often involve various problems from optimization and control theory. This paper compares several definitions of exponential convergence under general assumptions and next applies the results to some problems of optimization, control and estimation. Section 2 presents the general theory. It is assumed that $X_t \in [0, \infty)$ is some discrete-time sequence of nonnegative random variables and most attention is devoted to the exponential convergence of X_t to zero, although some of the results may be applied to the exponential growth of X_t. Section 2 shows, among other results, that the convergence rate of the expectations defined by

$$A := \limsup_{t\to\infty} \sqrt[t]{E[X_t]} \tag{1}$$

bounds from above the convergence rate of the trajectories:

$$\mathbb{P}\Big[\limsup_{t\to\infty} \sqrt[t]{X_t} \le A\Big] = 1, \tag{2}$$

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and if $A \in (0,1)$, the stopping time $\tau_\varepsilon = \inf\{t \in \mathbb{N}\colon X_t < \varepsilon\}$ is integrable and satisfies

$$\limsup_{\varepsilon \to 0^+} \frac{E[\tau_\varepsilon]}{|\log \varepsilon|} \le \frac{-1}{\log A}. \tag{3}$$

The above statements are very general, but the author has not been able to find the proof of (2) or (3) in the literature (the special case of (2) is proved in [18] in the context of continuous optimization). Statement (2) does not assume that $A \le 1$ and thus may be used to bound the exponential growth of the trajectories. It is worth mentioning that in the literature the exponential decrease of $\mathbb{P}[X_n \ge \varepsilon]$ is often studied, for instance in the analysis of large deviations where one usually considers sequences of the form $X_n = |\frac{S_n - E[S_n]}{n}|$ [11, 16]. Generally, if for any $\varepsilon > 0$ the sequence $\mathbb{P}[X_t > \varepsilon]$, $t \in \mathbb{N}$, decreases exponentially fast then $X_t \to 0$ almost surely, but the exponential decrease of the trajectories of X_t is not forced. Note also that by the Chebyshev inequality for any $\varepsilon > 0$ we have $E[X_t] \ge \varepsilon \cdot \mathbb{P}[X_t \ge \varepsilon]$, which implies that for any $\varepsilon > 0$,

$$\limsup_{t \to \infty} \sqrt[t]{E[X_t]} \ge \limsup_{t \to \infty} \sqrt[t]{\mathbb{P}[X_t \ge \varepsilon]},$$

and hence

$$A = \limsup_{t \to \infty} \sqrt[t]{E[X_t]} \ge \lim_{\varepsilon \to 0^+} \limsup_{t \to \infty} \sqrt[t]{\mathbb{P}[X_t \ge \varepsilon]}. \tag{4}$$

Thus, the exponential convergence of the expectations is a very strong convergence mode and the value of A given by (1) bounds the convergence rate for other exponential convergence types (2), (3), (4). Section 2 introduces general definitions precisely and proves (2), (3) and related results. Section 3 presents some applications in optimization and control.

The first motivation for this paper comes from the convergence rate analysis in the context of optimization where all the above mentioned definitions of exponential convergence (linear convergence) are considered [1, 8, 9, 12, 14, 15, 17, 18, 19, 20, 21]. In the context of optimization we often deal with sequences of the form $X_t = f(Z_t)$, where Z_t is some sequence of random variables which represents an optimization process and f represents some real-valued problem function. In that case it is an important question how fast the sequence $f(Z_t)$ approaches the global extremum of f; see Section 3 for more details and for applications of the results from Section 2 to this case. The second motivation comes from the area of stochastic control [2, 3, 4, 5, 10, 13]. If $X_t \in \mathbb{R}$ describes the wealth process of an investor in the context of portfolio optimization then we are rather interested in how fast the process X_t grows and Section 3 shows how the limit of long-run risk sensitive criterion controls the growth rate of the trajectories of X_t under general assumptions (no ergodicity-type assumptions on the log-wealth process are involved). Finally, it is shown, as a simple application of (2), how the convergence rate of the mean squared error bounds from above the convergence rate of the trajectories in estimation problems.

2. Exponential convergence of random variables. In the whole paper we assume that $(\Omega, \Sigma, \mathbb{P})$ is a probability space and $X_t\colon \Omega \to \mathbb{R}^+$, $t \in \mathbb{N}$, is a sequence of random variables, where $\mathbb{R}^+ = [0, \infty)$. We start with the following definition.

Definition 1. Given a sequence $x_t \in \mathbb{R}^+$, $t \in \mathbb{N}$, define

$$\overline{R}(x_t) = \limsup_{t\to\infty} \sqrt[t]{x_t}.$$

We are mostly interested in convergence to 0 and thus we will usually refer to $\overline{R}(x_t)$ as convergence rate. If $x_t \to 0$ then $\overline{R}(x_t) \in [0, 1]$. The condition $\overline{R}(x_t) \in (0, 1)$ means exponential convergence of x_t to zero. We will sometimes slightly abuse the terminology and refer to $\overline{R}(x_t)$ as convergence rate regardless of whether the x_t converges or not. The condition $\overline{R}(x_t) = 1$ excludes exponential convergence and does not determine the convergence of x_t. The condition $\overline{R}(x_t) > 1$ implies that some subsequence of x_t diverges exponentially fast. The following observation presents a general characterisation of exponential convergence and is fairly easy to prove.

Observation 1. *Given $C \in \mathbb{R}^+$ and a sequence $x_t \in \mathbb{R}^+$, the following conditions are equivalent:*

(i) $C \geq \limsup_{t\to\infty} \sqrt[t]{x_t}$,

(ii) $\log C \geq \limsup_{t\to\infty} \frac{1}{t} \log x_t$, *where* $\log 0 := -\infty$,

(iii) $\sup_{t\in\mathbb{N}} \frac{x_t}{R^t} < \infty$ *for any* $R > C$,

(iv) $\lim_{t\to\infty} \frac{x_t}{R^t} = 0$ *for any* $R > C$,

(v) $\sum_{t\in\mathbb{N}} \frac{x_t}{R^t} < \infty$ *for any* $R > C$,

(vi) $\sum_{t\in\mathbb{N}} \sum_{i=t}^{\infty} \frac{x_i}{R^i} < \infty$ *for any* $R > C$.

If additionally $C \in (0, 1)$ and $\tau_\varepsilon := \inf\{t \in \mathbb{N}\colon x_t < \varepsilon\}$, $\varepsilon > 0$, then any of the above conditions implies that

$$\limsup_{\varepsilon\to 0^+} \frac{\tau_\varepsilon}{|\log \varepsilon|} \leq \frac{1}{|\log C|}. \tag{5}$$

Observation 1 implies that $\overline{R}(x_t) = \limsup_{t\to\infty} \sqrt[t]{x_t}$ is the infimum of the constants $C \geq 0$ such that for any $R > C$

$$x_t \le M_R \cdot R^t,\ t \in \mathbb{N}, \quad \text{where} \quad M_R := \sup_{t\in\mathbb{N}} \frac{x_t}{R^t} < \infty. \tag{6}$$

Equation (6) is satisfied for any $R > \overline{R}(x_t)$.

Given any measurable discrete-time sequence $X_t \ge 0$ we may consider the random variable $C\colon \Omega \to [0,\infty]$ given by

$$C := \overline{R}(X_t) = \limsup_{t\to\infty} \sqrt[t]{X_t}. \tag{7}$$

Below, if $E[X_t] = \infty$ then we put $\sqrt[t]{\infty} := \infty$. Theorem 2 does not assume that $X_t \to 0$ and may describe the divergence of X_t to infinity.

THEOREM 2. *Let $X_t \ge 0$. The convergence rate of the trajectories is bounded from above by the convergence rate of the expectations in the sense that*

$$\mathbb{P}\Big[\limsup_{t\to\infty} \sqrt[t]{X_t} \le \limsup_{t\to\infty} \sqrt[t]{E[X_t]}\Big] = 1.$$

Proof. For the proof we can assume that $A := \limsup_{t\to\infty} \sqrt[t]{E[X_t]} < \infty$ and it is enough to show that for any $\varepsilon > 0$,

$$\mathbb{P}\Big[\limsup_{t\to\infty} \sqrt[t]{X_t} > A + \varepsilon\Big] = 0.$$

Fix $\varepsilon > 0$ and note that

$$\limsup_{t\to\infty} \sqrt[t]{X_t} = \lim_{t\to\infty} \sup_{i\ge t} \sqrt[i]{X_i}.$$

Thus

$$\mathbb{P}\Big[\limsup_{t\to\infty} \sqrt[t]{X_t} > A + \varepsilon\Big] = \mathbb{P}\Big[\lim_{t\to\infty} \sup_{i\ge t} \sqrt[i]{X_i} > A + \varepsilon\Big].$$

By the monotonicity $\sup_{i\ge t+1} \sqrt[i]{X_i} \le \sup_{i\ge t} \sqrt[i]{X_i}$, we have

$$\Big\{\lim_{t\to\infty} \sup_{i\ge t} \sqrt[i]{X_i} > A + \varepsilon\Big\} \subset \bigcap_{t\in\mathbb{N}} \Big\{\sup_{i\ge t} \sqrt[i]{X_i} > A + \varepsilon\Big\}$$

and

$$\begin{aligned}
\mathbb{P}\Big[\lim_{t\to\infty} \sup_{i\ge t} \sqrt[i]{X_i} > A + \varepsilon\Big] &\le \lim_{t\to\infty} \mathbb{P}\Big[\sup_{i\ge t} \sqrt[i]{X_i} > A + \varepsilon\Big] \\
&= \lim_{t\to\infty} \mathbb{P}\Big[\bigcup_{i=t}^{\infty} \{X_i > (A+\varepsilon)^i\}\Big] \\
&\le \lim_{t\to\infty} \sum_{i=t}^{\infty} \mathbb{P}[X_i > (A+\varepsilon)^i].
\end{aligned}$$

By the Chebyshev inequality,

$$\lim_{t\to\infty} \sum_{i=t}^{\infty} \mathbb{P}[X_i > (A+\varepsilon)^i] \le \lim_{t\to\infty} \sum_{i=t}^{\infty} \frac{E[X_i]}{(A+\varepsilon)^i},$$

and by Observation 1(v) (with $x_t := E[X_t]$ and $R = A + \varepsilon$),

$$\lim_{t\to\infty} \sum_{i=t}^{\infty} \frac{E[X_i]}{(A+\varepsilon)^i} = 0. \ \blacksquare$$

CONCLUSION 3. *The convergence rate of the expectations $E[X_t]$ is bounded from below by the essential suppremum of $\overline{R}(X_t)$, i.e.*

$$\limsup_{t\to\infty} \sqrt[t]{E[X_t]} \geq \inf\{\hat{C} \in \mathbb{R} \colon \mathbb{P}[\overline{R}(X_t) \leq \hat{C}] = 1\}.$$

Now we will focus on the convergence of the expectations of the hitting times

$$\tau_\epsilon = \inf\{t \in \mathbb{N} \colon X_t < \varepsilon\}, \qquad \varepsilon > 0.$$

Assume from now on that $X_t \to 0$ a.s., so $\overline{R}(X_t) = \limsup_{t\to\infty} \sqrt[t]{X_t} \leq 1$ a.s. Let

$$C := \operatorname{ess\,sup} \overline{R}(X_t).$$

By Observation 1(iii) for every $R > C$ there is some random variable $M_R \colon \Omega \to \mathbb{R}^+$ with

$$X_t \leq M_R \cdot R^t \qquad \text{a.s.}$$

If $C \in (0,1)$ then again by Observation 1,

$$\limsup_{\varepsilon\to 0^+} \frac{\tau_\varepsilon}{|\log \varepsilon|} \leq \frac{-1}{\log C} \qquad \text{a.s.}$$

At the same time the expectation of τ_ϵ may be infinite, as can be seen in the simple example below.

EXAMPLE 1. Let $\Omega = \{0, 1, 2, \dots\}$, $\Sigma = \mathcal{P}(\Omega)$ be the family of all subsets of Ω and let $p_n := \mathbb{P}[n]$, $n \in \Omega$. Assume that X_t is the characteristic function of $\mathbb{N}_t := \{0, 1, \dots, t\}$, i.e.

$$X_t(n) = 1_{\mathbb{N}_t}(n), \qquad n \in \mathbb{N}.$$

Clearly $\limsup_{t\to\infty} \sqrt[t]{X_t} = 0$. At the same time, for $\varepsilon \in (0,1)$ we have

$$\tau_\varepsilon(n) = n + 1, \qquad n \in \Omega,$$

and

$$E[\tau_\varepsilon] = \sum_{t\in\mathbb{N}} \mathbb{P}[\tau_\varepsilon > t] = \sum_{t\in\mathbb{N}} \mathbb{P}[X_t = 1] = \sum_{t\in\mathbb{N}} \mathbb{P}[\mathbb{N}_t] = \sum_{t\in\mathbb{N}} \sum_{n\leq t} p_n.$$

The above sum may be infinite depending on how the probabilities p_n are chosen. At the same time $\overline{R}(X_t) = 0$ and $\lim_{\varepsilon\to 0^+} \frac{\tau_\varepsilon}{|\log \varepsilon|} = 0$ a.s.

If the expectations $E[X_t]$ decrease exponentially fast then $E[\tau_\varepsilon]$ is finite and Theorem 4 shows that the convergence rate of the expectations $\overline{R}(E[X_t])$ controls the convergence of $E[\tau_\varepsilon]$ with $\varepsilon \to 0^+$.

THEOREM 4. *If*

$$A := \overline{R}(E[X_t]) = \limsup_{t\to\infty} \sqrt[t]{E[X_t]} < 1$$

then

$$\limsup_{\varepsilon\to 0^+} \frac{E[\tau_\varepsilon]}{|\log \varepsilon|} \leq \frac{-1}{\log A}.$$

Above, if $A = 0$ then $\frac{-1}{\log A} := 0$. The proof of the above theorem is based on the following lemma which follows from Theorem 2 and Observation 1.

LEMMA 5. *Assume that* $A = \limsup_{t\to\infty} \sqrt[t]{E[X_t]} < 1$. *For any* $R > A$ *the random variable*

$$H_R := \inf\left\{t \in \mathbb{N}\colon \sup_{i\geq t} \frac{X_i}{R^i} \leq 1\right\}$$

has finite expectation.

Proof. Fix $R > A$. By Theorem 2 and Observation 1(iv) we have $\mathbb{P}[H_R < \infty] = 1$. Hence

$$E[H_R] = \sum_{t\in\mathbb{N}} \mathbb{P}[H_R > t].$$

By definition of H_R and by the Chebyshev inequality,

$$\mathbb{P}[H_R > t] = \mathbb{P}\Big[\bigcup_{i=t+1}^{\infty} \{X_i > R^i\}\Big] \leq \sum_{i=t+1}^{\infty} \mathbb{P}[X_i > R^i] \leq \sum_{i=t+1}^{\infty} \frac{E[X_i]}{R^i}.$$

We have thus shown that

$$E[H_R] \leq \sum_{t\in\mathbb{N}} \sum_{i=t+1}^{\infty} \frac{E[X_i]}{R^i}$$

and the above series is finite by Observation 1 applied to the sequence $x_t = E[X_t]$. ∎

Proof of Theorem 4. Fix $C \in (A, 1)$. Under the notation of Lemma 5, for any $t \in \mathbb{N}$ on the set $\{t \geq H_C\}$ we have $X_{t+i} \leq C^{t+i}$ for all $i \in \mathbb{N}$. In other words, for any $t \geq H_C$ we have $X_t \leq C^t$. At the same time, as $C < 1$, note that for

$$\hat{C} := \frac{-1}{\log C}$$

the ceiling $t = \lceil \hat{C} \cdot |\log \varepsilon| \rceil$ is the smallest natural number with $C^t \leq \varepsilon$. This implies that if $t \geq \max\{H_C, \lceil \hat{C} \cdot |\log \varepsilon|\rceil\}$ then $X_t \leq \varepsilon$ and $X_{t+1} < \varepsilon$. Hence,

$$\tau_\varepsilon \leq \max\{H_C, \lceil \hat{C} \cdot |\log \varepsilon|\rceil\} + 1$$

and

$$\frac{\tau_\varepsilon}{|\log \varepsilon|} \leq \frac{1}{|\log \varepsilon|}(\max\{H_C, \lceil \hat{C} \cdot |\log \varepsilon|\rceil\} + 1). \tag{8}$$

Thus

$$E\left[\frac{\tau_\varepsilon}{|\log\varepsilon|}\right] \le \frac{1}{|\log\varepsilon|}E[\max\{H_C, \lceil\hat{C}\cdot|\log\varepsilon|\rceil\}] + \frac{1}{|\log\varepsilon|}. \tag{9}$$

Lemma 5 allows us to use Lebesgue's dominated convergence theorem:

$$\frac{1}{|\log\varepsilon|}E[\max\{H_C, \lceil\hat{C}\cdot|\log\varepsilon|\rceil\}] = E\left[\max\left\{\frac{H_C}{|\log\varepsilon|}, \hat{C}\right\}\right] \to \hat{C} \text{ as } \varepsilon\to 0^+.$$

Equation (9) and the above imply that $\limsup_{\varepsilon\to 0^+}\frac{E[\tau]}{|\log\varepsilon|} \le \hat{C}$ for any $C \in (A,1)$ and thus

$$\limsup_{\varepsilon\to 0^+}\frac{E[\tau_\varepsilon]}{|\log\varepsilon|} \le \hat{A} = \frac{-1}{\log A}. \ \blacksquare$$

3. Applications. The results of the previous sections will be applied to the areas of optimization and stochastic control.

3.1. Optimization. Let (K,d) be some metric space and $f\colon K\to\mathbb{R}^+$ be a Borel-measurable function which attains its global minimum $f_{\min}$. Assume that $X_t\in K$, $t\in\mathbb{N}$, represents an optimization process. The hitting time of the ε-optimal sublevel set $\{x\in A\colon f(x) < f_{\min}+\varepsilon\}$ is defined by

$$\tau_\varepsilon = \inf\{t\in\mathbb{N}\colon f(X_t) < f_{\min}+\epsilon\}.$$

Theorems 2 and 4 give us relations between the convergence rate of the trajectories $f(X_t)$, the convergence rate of the expectations $E[|f(X_t)-f_{\min}|]$ and the convergence behaviour of $E[\tau_\varepsilon]$ under general assumptions on the sequence $f(X_t)$. In particular, if the constant

$$A = \overline{R}(E[f(X_t)-f_{\min}], t\in\mathbb{N}) = \limsup_{t\to\infty}\sqrt[t]{E[f(X_t)-f_{\min}]}$$

satisfies $A<1$ then by Theorem 4 we have the upper bound

$$\limsup_{t\to\infty}\frac{E[\tau_\varepsilon]}{|\log\varepsilon|} \le \frac{-1}{\log A}$$

and the following control of the asymptotic behaviour of $E[\tau_\varepsilon]$ with $\varepsilon\to 0^+$:

$$E[\tau_\varepsilon] \le C(\varepsilon)\cdot|\log\varepsilon|,$$

where $C\colon(0,1)\to\mathbb{R}^+$ satisfies $\limsup_{\varepsilon\to 0^+}C(\varepsilon) \le \hat{A} = \frac{-1}{\log A}$. This significantly strengthens [18, Theorem 7]. We also have, by (2) and (4),

$$\mathbb{P}\left[\limsup_{t\to\infty}\sqrt[t]{f(X_t)-f_{min}} \le A\right] = 1,$$

$$\lim_{\varepsilon\to 0^+}\limsup_{t\to\infty}\sqrt[t]{\mathbb{P}[f(X_t)\ge f_{\min}+\varepsilon]} \le A.$$

3.2. Risk sensitive stochastic control. Let $W_t\colon \Omega \to \mathbb{R}^+$, $t \in \mathbb{N}$, represent the wealth process of an investor (see [2, 4, 5, 13] for details). Let $\gamma \neq 0$ represent the risk-averse parameter of an investor so we are interested in the limit of the long-run risk-sensitive criterion of the following log wealth growth:

$$C = \liminf_{t\to\infty} \frac{1}{t\gamma} \log(E[(W_t)^\gamma]). \tag{10}$$

We will assume that $\gamma < 0$, which is a common investment criterion in the context of long run optimization. This section shows how the value of the risk-sensitive criterion determines the exponential growth of the trajectories of the wealth process W_t and the logarithmic growth of the expected barrier-hitting time T_b with $b \nearrow \infty$. This type of convergence behaviour is natural under suitable ergodicity-type assumptions on the log wealth process – see for instance the asymptotic optimality principle of [2]. This section brings to attention that the log growth rate of the trajectories of W_t and the logarithmic growth of the corresponding expected hitting times are determined by the limit (10) under general assumptions on W_t (no ergodicity involved), which follows from the results of Section 2.

THEOREM 6. *Let $\gamma < 0$ and let $W_t\colon \Omega \to (0,\infty)$ satisfy $E[(W_t)^\gamma] < \infty$ for all $t \in \mathbb{N}$. The constant*

$$C = \liminf_{t\to\infty} \frac{1}{t}\frac{1}{\gamma} \log(E[(W_t)^\gamma])$$

determines the log-growth of W_t in the sense that

$$\liminf_{t\to\infty} \frac{1}{t} \log W_t \geq C \qquad a.s.$$

Additionally, if $C > 0$ then the hitting times $T_b = \inf\{t \in \mathbb{N}\colon W_t > b\}$, $b \in \mathbb{R}^+$, are integrable and satisfy

$$\limsup_{b\to\infty} \frac{E[T_b]}{\log b} \leq \frac{1}{C}.$$

Let $X_t := \frac{1}{W_t}$. Recall that

$$E[(X_t)^{p_1}]^{1/p_1} \leq E[(X_t)^{p_2}]^{1/p_2} \qquad \text{for } 0 < p_1 < p_2 < \infty, \tag{11}$$

assuming that the above expectations exist. For $\gamma < 0$ let

$$C_\gamma = \liminf_{t\to\infty} \frac{1}{t}\frac{1}{\gamma} \log(E[(W_t)^\gamma]).$$

By elementary calculations and by (11) one may show that

$$C_{\gamma_1} \leq C_{\gamma_2} \qquad \text{for } \gamma_1 < \gamma_2 < 0. \tag{12}$$

See also [10, Lemma 2.1] for the above. Theorem 6 and (12) immediately lead to the following.

Conclusion 7. *Assume that $W_t\colon \Omega \to (0,\infty)$ satisfy $E[(W_t)^\gamma] < \infty$ for all $t \in \mathbb{N}$, for any $\gamma < 0$ close to zero. The limit*

$$C = \lim_{\gamma \to 0^-} \liminf_{t\to\infty} \frac{1}{t}\frac{1}{\gamma} \log(E[(W_t)^\gamma])$$

bounds the log-growth of W_t in the sense that $\liminf_{t\to\infty} \frac{1}{t}\log W_t \geq C$ a.s. Additionally, if $C > 0$ then the hitting times $T_b = \inf\{t \in \mathbb{N}\colon W_t > b\}$ are integrable and satisfy

$$\limsup_{b\to\infty} \frac{E[T_b]}{\log b} \leq \frac{1}{C}.$$

Proof of Theorem 6. As $\gamma < 0$, equation (10) implies that

$$C \cdot \gamma = \limsup_{t\to\infty} \frac{1}{t} \log(E[(W_t)^\gamma]).$$

Hence,

$$\exp(C\gamma) = \limsup_{t\to\infty} (E[(W_t)^\gamma])^{1/t}. \tag{13}$$

By Theorem 2,

$$\limsup_{t\to\infty} \left((W_t)^\gamma\right)^{1/t} \leq \exp(C\cdot\gamma) \quad \text{a.s.}$$

As $\gamma < 0$, the above controls the growth of W_t according to

$$\liminf_{t\to\infty} \left((W_t)^{|\gamma|}\right)^{1/t} \geq \exp(C\cdot|\gamma|) \quad \text{a.s.}$$

Hence,

$$\liminf_{t\to\infty} (W_t)^{1/t} \geq \exp(C) \quad \text{a.s.} \tag{14}$$

and

$$\liminf_{t\to\infty} \frac{1}{t}\log W_t \geq C \quad \text{a.s.,}$$

which proves the first part of the theorem.

Now we assume that $C > 0$ and we will discuss the hitting times $T_b = \inf\{t \in \mathbb{N}\colon W_t > b\}$. Let

$$\tau_\varepsilon = \inf\{t \in \mathbb{N}\colon (W_t)^\gamma < \varepsilon\} = \inf\{t \in \mathbb{N}\colon W_t > (1/\varepsilon)^{|\gamma|^{-1}}\}.$$

The condition $C > 0$ implies $\exp(\gamma \cdot C) < 1$. By (13) and Theorem 4, we have

$$\limsup_{\varepsilon\to 0^+} \frac{E[\tau_\varepsilon]}{|\log\varepsilon|} \leq \frac{1}{C\cdot|\gamma|}. \tag{15}$$

Let

$$T(\varepsilon) = \inf\{t \in \mathbb{N}\colon W_t > 1/\varepsilon\}$$

so we have $\tau_\varepsilon = \inf\{t \in \mathbb{N} \colon W_t > 1/\varepsilon^{|\gamma|^{-1}}\} = T(\varepsilon^{|\gamma|^{-1}})$ and

$$\limsup_{\varepsilon\to 0^+} \frac{E[\tau_\varepsilon]}{|\log\varepsilon|} = \limsup_{\varepsilon\to 0^+} \frac{E[T(\varepsilon^{1/|\gamma|})]}{|\log\varepsilon|} = \limsup_{\varepsilon\to 0^+} \frac{E[T(\varepsilon)]}{|\log(\varepsilon^{|\gamma|})|} = \limsup_{\varepsilon\to 0^+} \frac{E[T(\varepsilon)]}{|\gamma|\cdot|\log\varepsilon|}.$$

Hence, by (15),

$$\limsup_{\varepsilon\to 0^+} \frac{E[T(\varepsilon)]}{|\gamma|\cdot|\log\varepsilon|} \le \frac{1}{|\gamma|\cdot C},$$

and thus

$$\limsup_{\varepsilon\to 0^+} \frac{E[T(\varepsilon)]}{|\log\varepsilon|} \le \frac{1}{C}.$$

For $b = 1/\varepsilon$ we have $T_b = T(\varepsilon)$ and $|\log\varepsilon| = \log(1/\varepsilon) = \log b$. We thus have

$$\limsup_{b\to\infty} \frac{E[T_b]}{\log b} \le \frac{1}{C}. \ \blacksquare$$

3.3. Estimation. Assume that a sequence $Z_t \in \mathbb{R}$ approaches the unknown parameter $\theta \in \Theta \subset \mathbb{R}$ and the mean squared error $E|Z_t - \theta|^2$ satisfies

$$C = \limsup_{t\to\infty} (E|Z_t - \theta|^2)^{1/t}.$$

By Theorem 2 applied to the sequence $X_t = |Z_t - \theta|^2$ we bound the convergence rate of the trajectories

$$\mathbb{P}\Big[\limsup_{t\to\infty} (|Z_t - \theta|^2)^{1/t} \le C\Big] = 1 \text{ and so } \mathbb{P}\Big[\limsup_{t\to\infty} (|Z_t - \theta|)^{1/t} \le \sqrt{C}\Big] = 1.$$

More generally, if (K, d) is a metric space, $\theta \in K$ and $Z_t \in K$ converges in mean to θ, i.e. $E[d(Z_t, \theta)] \to 0$, then the results of Section 2 may be applied to the sequence $X_t := d(Z_t, \theta)$.

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