STUDIA MATHEMATICA 280 (1) (2025)

On some properties of modulation spaces as Banach algebras

by

HANS G. FEICHTINGER, MASAHARU KOBAYASHI and ENJI SATO

Abstract. We give some properties of the modulation spaces $M_s^{p,1}(\mathbf{R}^n)$ as commutative Banach algebras. In particular, we prove the Wiener–Lévy theorem for $M_s^{p,1}(\mathbf{R}^n)$, and clarify the sets of spectral synthesis for $M_s^{p,1}(\mathbf{R}^n)$ by using the "ideal theory for Segal algebras" developed by Reiter. The inclusion relationship between the modulation space $M_0^{p,1}(\mathbf{R})$ and the Fourier Segal algebra $\mathcal{F}A_p(\mathbf{R})$ is also determined.

1. Introduction. A commutative Banach algebra is a Banach space $(X, \|\cdot\|_X)$ which is a commutative algebra, i.e., with an associative, distributive and commutative multiplication satisfying $\|fg\|_X \le \|f\|_X\|g\|_X$ for all $f, g \in X$. There are many examples of commutative Banach algebras. For this article an important example is the Fourier algebra $A(\mathbf{R})$, i.e., the set of all functions on $\mathbf{R}$ which are the Fourier transforms of functions in $L^1(\mathbf{R})$, which is a commutative Banach algebra with pointwise multiplication, denoted by $\mathcal{F}L^1(\mathbf{R})$ elsewhere. In addition, the Wiener algebra $A(\mathbf{T})$, which corresponds to a periodic version of $A(\mathbf{R})$, and the Fourier–Beurling algebra $\mathcal{F}L_s^1(\mathbf{R}^n)$ $(s \ge 0)$, which can be regarded as a generalization of $A(\mathbf{R})$, are commutative Banach algebras (see Section 2.3 for the definition of $\mathcal{F}L_s^1(\mathbf{R}^n)$). As is well-known, those function spaces play important roles in various fields such as commutative Banach algebras, operating functions, and spectral synthesis. We refer the reader to Kahane [20], Katznelson [21], Reiter [31], Reiter–Stegeman [32] and Rudin [33] for more details.

On the other hand, modulation spaces $M_s^{p,q}(\mathbf{R}^n)$ form a family of function spaces introduced by Feichtinger [9] (see Definition 2.1). In some sense, they behave like the Besov spaces $B_s^{p,q}(\mathbf{R}^n)$, but they appear to be better suited for the description of problems in the area of time-frequency analysis and are often a good substitute for the usual spaces $L^p(\mathbf{R}^n)$ or

2020 *Mathematics Subject Classification*: Primary 42B35; Secondary 43A45.
Key words and phrases: modulation spaces, Wiener–Lévy theorem, set of spectral synthesis, Segal algebra.
Received 16 March 2024; revised 28 August 2024.
Published online 13 December 2024 in Open Access (under CC-BY license).

DOI: 10.4064/sm240316-9-9

$B^{p,q}_s(\mathbf{R}^n)$ (see [15, 18] for more details). It is important for our considerations that $M^{p,q}_s(\mathbf{R}^n)$ is a commutative Banach algebra if $s > n(1-1/q)$, or $q = 1$ and $s \geq 0$ (see [13, Theorem 10]), and that it appears naturally in the study of certain partial differential equations. For example, we can solve the nonlinear Schrödinger equations $iu_t + \Delta u = |u|^{2k}u$ in $M^{2,1}_s(\mathbf{R}^n)$ for all $k \in \mathbf{N}$ by using the algebraic properties of $M^{p,q}_s(\mathbf{R}^n)$ (see [1, 38]). Furthermore, as a commutative Banach algebra, $M^{p,q}_s(\mathbf{R}^n)$ has many interesting properties similar to those of $A(\mathbf{R})$, $A(\mathbf{T})$ and also $\mathcal{F}L^1_s(\mathbf{R}^n)$ (see [2, 3, 22, 23, 24, 30]). Therefore, it is of interest to further clarify the properties of $M^{p,1}_s(\mathbf{R}^n)$.

In this paper we study several properties of $M^{p,1}_s(\mathbf{R}^n)$ as a commutative Banach algebra. In particular, we will establish the validity of the Wiener–Lévy theorem for $M^{p,1}_s(\mathbf{R}^n)$, and see that the sets of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^n)$ coincide with those for $M^{p,1}_s(\mathbf{R}^n)$, independently of p (given the general restrictions). Moreover, we will consider the inclusion relation between the modulation space $M^{p,1}_0(\mathbf{R})$ and the Fourier Segal algebra $\mathcal{F}A_p(\mathbf{R})$. We refer to [14] for further study of modulation spaces $M^{p,q}_s(\mathbf{R}^n)$ as Banach algebras with $q > 1$ and $s > n(1-1/q)$.

The organization of this paper is as follows. After a preliminary section devoted to the definition and basic properties of $M^{p,q}_s(\mathbf{R}^n)$ and $\mathcal{F}L^1_s(\mathbf{R}^n)$ we will demonstrate in Section 3 that there exist approximate units for $M^{p,q}_s(\mathbf{R}^n)$ (see Theorem 3.1). In Section 4 we consider the closed ideals in $M^{p,1}_s(\mathbf{R}^n)$ and prove that closed modulation invariant subspaces of $M^{p,1}_s(\mathbf{R}^n)$ coincide with closed ideals in $M^{p,1}_s(\mathbf{R}^n)$. Section 5 is devoted to the Wiener–Lévy theorem, which was originally proved by Wiener [39] and by Lévy [27] for $A(\mathbf{T})$. In particular, we will show that the Wiener–Lévy theorem and also Wiener's general Tauberian theorem hold for $M^{p,1}_s(\mathbf{R}^n)$ if $1 \leq p < \infty$ and $s \geq 0$ (see Theorems 5.1 and 5.6). In Section 6 we will consider the set of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$. By using the "ideal theory for Segal algebras" developed in Reiter [31] we will clarify the relation between the sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$ and the sets of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^n)$ (see Theorem 6.3). In Section 7 we will prove that single points of $\mathbf{R}^n$ are sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$ without using Theorem 6.3. Finally, in Section 8 we will determine the inclusion relation between $M^{p,1}_0(\mathbf{R}^n)$ and the Fourier Segal algebras $\mathcal{F}A_p(\mathbf{R}^n)$.

2. Preliminaries. The following notation will be used throughout this article. We use C to denote various positive constants which may change from line to line. We use the notation $I \lesssim J$ if I is bounded by a constant times J and we write $I \approx J$ if $I \lesssim J$ and $J \lesssim I$. The closed ball with center $x_0 \in \mathbf{R}^n$ and radius $r > 0$ is defined by $B_r(x_0) = \{x \in \mathbf{R}^n \mid |x - x_0| \leq r\}$.

For $x \in \mathbf{R}^n$, we write $\langle x\rangle = (1+|x|^2)^{1/2}$. We define for $1 \le p < \infty$ and $s \in \mathbf{R}$

$$\|f\|_{L^p_s} = \Big(\int_{\mathbf{R}^n} (\langle x\rangle^s |f(x)|)^p\,dx\Big)^{1/p},$$

and $\|f\|_{L^\infty_s} = \operatorname{ess\,sup}_{x\in\mathbf{R}^n} \langle x\rangle^s |f(x)|$. We simply write $L^p(\mathbf{R}^n)$ instead of $L^p_0(\mathbf{R}^n)$. For $1 \le p < \infty$, we denote by p' the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. We write $C_c^\infty(\mathbf{R}^n)$ to denote the set of all complex-valued infinitely differentiable functions on $\mathbf{R}^n$ with compact support. We write $\mathcal{S}(\mathbf{R}^n)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$ to denote the space of tempered distributions on $\mathbf{R}^n$. We use $\langle F, G\rangle$ to denote the extension of the inner product $\langle F, G\rangle = \int F(t)\overline{G(t)}\,dt$ on L^2 to $\mathcal{S}' \times \mathcal{S}$ or $M^{p,q}_s \times M^{p',q'}_{-s}$ (see Lemma 2.2(ii) below). The *Fourier transform* of $f \in L^1(\mathbf{R}^n)$ is defined by

$$\mathcal{F}f(\xi) = f^\wedge(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\xi}\,dx.$$

Similarly, if $h \in L^1(\mathbf{R}^n)$ then the inverse Fourier transform of h is defined by $\mathcal{F}^{-1}h(x) = (2\pi)^{-n}\widehat{h}(-x)$. We note that $\langle f, g\rangle_{L^2(\mathbf{R}^n)} = (2\pi)^{-n}\langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbf{R}^n)}$, $(f * g)^\wedge = \widehat{f}\widehat{g}$ and $(fg)^\wedge = (2\pi)^{-n}(\widehat{f} * \widehat{g})$, where the convolution $f * g$ is defined by $(f * g)(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy$. Moreover, $M(\mathbf{R}^n)$ denotes the sets of all bounded regular Borel measures μ on $\mathbf{R}^n$ with the norm

$$\|\mu\|_{M(\mathbf{R}^n)} = \sup_{f\in C_c(\mathbf{R}^n),\, \|f\|_{L^\infty}\le 1} \Big|\int_{\mathbf{R}^n} f(x)\,d\mu(x)\Big|.$$

The Fourier–Stieltjes transform of μ is defined by $\widehat{\mu}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi}\,d\mu(x)$. Here $C_c(\mathbf{R}^n)$ denotes the set of all $f \in C(\mathbf{R}^n)$ with compact support. Let δ_a ($a \in \mathbf{R}^n$) be the unit mass concentrated at the point $x = a$, i.e., $\delta_a(E) = 1$ if $a \in E$ and $\delta_a(E) = 0$ otherwise. Then $\delta_a \in M(\mathbf{R}^n)$, $\|\delta_a\|_{M(\mathbf{R}^n)} = 1$, $\widehat{\delta_a}(\xi) = e^{-ia\xi}$ and $\delta_a * \delta_b = \delta_{a+b}$ ($a, b \in \mathbf{R}^n$) (see [33, Ch. 1.3] for more details). For two Banach spaces B_1 and B_2, $B_1 \hookrightarrow B_2$ means that B_1 is continuously embedded into B_2.

2.1. Short-time Fourier transform. For $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$, the short-time Fourier transform V_ϕ of f with respect to the window ϕ is defined by the duality

$$V_\phi f(x,\xi) = \langle f, M_\xi T_x \phi\rangle = \langle f(t), \phi(t-x)e^{it\xi}\rangle = \int_{\mathbf{R}^n} f(t)\overline{\phi(t-x)}e^{-it\xi}\,dt,$$

where the translation operator T_x and the modulation operator M_ξ are defined by $(T_x h)(t) = h(t-x)$ and $(M_\xi h)(t) = e^{it\xi}h(t)$, respectively.

It is known that $V_\phi f \in C(\mathbf{R}^n \times \mathbf{R}^n)$ (see [18, Lemma 11.2.3]) and

$$V_\phi f(x,\xi) = (2\pi)^{-n} e^{-ix\xi} V_{\widehat{\phi}}\widehat{f}(\xi, -x) = (2\pi)^{-n} e^{-ix\xi}(f * M_\xi \phi^*)(x), \tag{2.1}$$

where $\phi^*(x) = \overline{\phi(-x)}$ (see [18, Lemma 3.1.1]). We also note that if $f \in \mathcal{S}(\mathbf{R}^n)$, then $V_\phi f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$ (see [18, Theorem 11.2.5]). Moreover, we have Moyal's equation:

$$\langle V_\phi f, V_\psi g\rangle_{L^2(\mathbf{R}^{2n})} = (2\pi)^n \langle \psi, \phi\rangle_{L^2(\mathbf{R}^n)} \langle f, g\rangle_{L^2(\mathbf{R}^n)}$$

for all $f, g, \phi, \psi \in L^2(\mathbf{R}^n)$ (see [18, Theorem 3.2.1]).

2.2. Modulation spaces

Definition 2.1 ([9]). Let $1 \le p, q \le \infty$, $s \in \mathbf{R}$ and $\phi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$. The modulation space $M^{p,q}_s(\mathbf{R}^n) = M^{p,q}_s$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{M^{p,q}_s} = \Big(\int_{\mathbf{R}^n} \langle \xi\rangle^{sq} \Big(\int_{\mathbf{R}^n} |V_\phi f(x,\xi)|^p \, dx\Big)^{q/p} d\xi\Big)^{1/q}$$

is finite (with the usual modification if $p = \infty$ or $q = \infty$). We simply write $M^{p,q}(\mathbf{R}^n)$ instead of $M^{p,q}_0(\mathbf{R}^n)$.

We collect basic properties of $M^{p,q}_s(\mathbf{R}^n)$ in the following lemma:

Lemma 2.2. *Let* $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$ *and* $s, s_1, s_2 \in \mathbf{R}$.

(i) $M^{p,q}_s(\mathbf{R}^n)$ *is a Banach space, and different windows define equivalent norms.*
(ii) (Density and duality) *If* $p, q < \infty$, *then* $\mathcal{S}(\mathbf{R}^n)$ *is dense in* $M^{p,q}_s(\mathbf{R}^n)$ *and* $(M^{p,q}_s(\mathbf{R}^n))' = M^{p',q'}_{-s}(\mathbf{R}^n)$.
(iii) *If* $p_1 \le p_2$, $q_1 \le q_2$ *and* $s_1 \ge s_2$, *then* $M^{p_1,q_1}_{s_1}(\mathbf{R}^n) \hookrightarrow M^{p_2,q_2}_{s_2}(\mathbf{R}^n)$.
(iv) *If* $s > n/q'$, *or* $q = 1$ *and* $s \ge 0$, *then* $M^{p,q}_s(\mathbf{R}^n) \subset C(\mathbf{R}^n)$ *and* $M^{p,q}_s(\mathbf{R}^n)$ *are multiplication algebras, i.e.,*

$$\|fg\|_{M^{p,q}_s} \le c\|f\|_{M^{p,q}_s}\|g\|_{M^{p,q}_s}, \qquad f, g \in M^{p,q}_s(\mathbf{R}^n),$$

for some $c \ge 1$.
(v) *If* $s \ge 0$, *then there exists* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{\infty,1}_s} \le C\|f\|_{M^{\infty,1}_s} \quad \forall f \in M^{\infty,1}_s(\mathbf{R}^n)$$

for all $0 < \lambda \le 1$, *where* $f_\lambda(x) = f(\lambda x)$.

(i)–(v) are proved in [6, Theorem 3.2], [9, Section 6], [17, 36, 37], and summaries are given in [1, 7], [18, Chapter 11].

We also note that there is another characterization of the modulation spaces using BUPUs on the Fourier transform side (see [11, 13, 38]): Assume that $\varphi \in \mathcal{S}(\mathbf{R}^n)$ satisfies

$$\operatorname{supp}\varphi \subset [-1,1]^n \quad \text{and} \quad \sum_{k\in\mathbf{Z}^n} \varphi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbf{R}^n.$$

Then we have, writing $\varphi(D-k)f = \mathcal{F}^{-1}(T_k\varphi \cdot \widehat{f}\,)$,

$$\|f\|_{M_s^{p,q}} \approx \Big(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq}\Big(\int_{\mathbf{R}^n} |\varphi(D-k)f(x)|^p\,dx\Big)^{q/p}\Big)^{1/q}$$

with obvious modifications if p or q is ∞.

REMARK 2.3. We may assume that $0 \le \varphi(\xi) \le 1$ ($\xi \in \mathbf{R}^n$) and $\varphi(\xi) = 1$ on the box $[-1/10, 1/10]^n$.

REMARK 2.4. Let $1 \le q \le \infty$, $s \in \mathbf{R}$ and $(B, \|\cdot\|_B)$ be a Banach space of tempered distributions on $\mathbf{R}^n$ such that $\mathcal{S}\cdot B \subset B$. Using BUPUs the *Wiener amalgam space* $W(B, \ell_s^q)(\mathbf{R}^n)$ is defined by the norm

$$\|f\|_{W(B,\ell_s^q)} = \Big(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq}\|f\cdot T_k\varphi\|_B^q\Big)^{1/q} < \infty.$$

We note that $W(\mathcal{F}L^p, \ell_s^q)(\mathbf{R}^n) = \mathcal{F}^{-1}(M_s^{p,q}(\mathbf{R}^n))$, where $\mathcal{F}L^p(\mathbf{R}^n)$ ($=\mathcal{F}L^p$) denotes the Fourier–Lebesgue space with the norm $\|f\|_{\mathcal{F}L^p} = \|\widehat{f}\|_{L^p}$ (see also Section 2.3 below).

The following lemma seems to be known to many people, but for the reader's convenience, we give the proof (cf. [13, 19, 35]).

LEMMA 2.5. *Let $1 \le p, q < \infty$. Suppose that $s > n/q'$, or $q = 1$ and $s \ge 0$. Then*

$$\|fg\|_{M_s^{p,q}} \lesssim \|f\|_{M_s^{\infty,q}}\|g\|_{M_s^{p,q}}, \qquad f, g \in \mathcal{S}(\mathbf{R}^n).$$

If $f \in M_s^{p,1}(\mathbf{R}^n)$ and $g \in M_{-s}^{p',\infty}(\mathbf{R}^n)$, then $fg \in M_{-s}^{1,\infty}(\mathbf{R}^n)$ ($\hookrightarrow M_{-s}^{p',\infty}(\mathbf{R}^n)$).

Proof. We start the proof by observing that the pointwise relationships for modulation spaces correspond to convolution relations for their (inverse) Fourier transforms in $W(\mathcal{F}L^p, \ell_s^q)(\mathbf{R}^n)$.

The claim made is thus equivalent to the statement that (writing f, g for $\widehat{f}$, $\widehat{g}$ respectively)

$$\|f * g\|_{W(\mathcal{F}L^p,\ell_s^q)} \lesssim \|f\|_{W(\mathcal{F}L^\infty,\ell_s^q)}\|g\|_{W(\mathcal{F}L^p,\ell_s^q)}, \tag{2.2}$$

which follows from the coordinatewise convolution result established in [12] and the fact that

$$\mathcal{F}L^p * \mathcal{F}L^\infty \subset \mathcal{F}L^p \quad \text{because} \quad L^p\cdot L^\infty \subseteq L^p,$$

as well as the fact that $(\ell_s^q(\mathbf{Z}^n), \|\cdot\|_{\ell_s^q})$ is a Banach algebra with respect to the natural convolution, because the weight function $x \mapsto \langle x\rangle^s$ is weakly subadditive for any $s \ge 0$ (i.e. $\langle x+y\rangle^s \lesssim \langle x\rangle^s\langle y\rangle^s$) and consequently $\ell^1(\mathbf{Z}^n)\cap\ell_s^q(\mathbf{Z}^n)$ is a Banach algebra with respect to convolution. This observation has been made explicit in [4]. But for $s > n/q'$, it follows from the Hölder inequality that $\ell_s^q(\mathbf{Z}^n) \hookrightarrow \ell^1(\mathbf{Z}^n)$ (with corresponding continuous embedding), and thus $(\ell_s^q(\mathbf{Z}^n), \|\cdot\|_{\ell_s^q})$ is a Banach algebra with respect to (discrete) convolution.

In a similar way we use the local convolution relation

$$\mathcal{F}L^p * \mathcal{F}L^{p'} \subset \mathcal{F}L^\infty \quad \text{due to the relation} \quad L^p \cdot L^{p'} \subseteq L^\infty,$$

combined with the global condition

$$\ell^p_s * \ell^{p'}_{-s} \subseteq \ell^\infty_{-s},$$

using the submultiplicativity condition (writing $x = (x-y)+y$), we deduce that

$$\langle x+y\rangle^{-s} \le 2^{|s|}\langle x\rangle^{-s}\langle y\rangle^{s}. \ \blacksquare$$

Let us also observe another consequence of the reasoning used in the last proof, which provides some insight into the pointwise multiplier algebra of Sobolev algebras. We formulate it as another lemma. Let us briefly recall that the classical Sobolev spaces $(H^s(\mathbf{R}^n), \|\cdot\|_{H^s})$ are just the inverse images of (polynomially) weighted L^2-spaces, and thus appear in the family of modulation spaces as

$$H^s = \mathcal{F}^{-1}(L^2_s) = \mathcal{F}^{-1}(W(\mathcal{F}^{-1}(L^2), \ell^2_s)) = \mathcal{F}^{-1}(W(L^2, \ell^2_s))$$

so that one has $\ell^2_s(\mathbf{Z}^n) \hookrightarrow \ell^1(\mathbf{Z}^n)$ for $s > n/2$ via the Cauchy–Schwarz inequality, and consequently by the Hausdorff–Young version for Wiener amalgams (see [10]) one has the following chain of continuous embeddings:

$$H^s \hookrightarrow \mathcal{F}^{-1}(W(L^2, \ell^1)) \hookrightarrow W(\mathcal{F}L^1, \ell^2) \hookrightarrow W(C_0, \ell^2).$$

LEMMA 2.6. *For $s > n/2$ we have $M^{\infty,2}_s \cdot H^s \subseteq H^s$, meaning that $M^{\infty,2}_s(\mathbf{R}^n)$ is a subset of the pointwise multiplier algebra of the Sobolev algebra $H^s(\mathbf{R}^n)$.*

Proof. We have already noted that $s > n/2$ implies $H^s(\mathbf{R}^n) \hookrightarrow W(C_0, \ell^2)(\mathbf{R}^n)$. The following reasoning implies the classical fact that $H^s(\mathbf{R}^n)$ is a Banach algebra with respect to pointwise multiplication (referred to as the *Sobolev algebra*).

We have to verify that the corresponding convolution relation is valid on the Fourier transform side. In fact, we may invoke the main result of [12]:

$$W(\mathcal{F}L^\infty, \ell^2_s) * W(\mathcal{F}L^2, \ell^2_s) \subset W(L^2, \ell^2_s), \tag{2.3}$$

using again the fact that $\ell^2_s(\mathbf{Z}^n) = \ell^2_s(\mathbf{Z}^n) \cap \ell^1(\mathbf{Z}^n)$ is a Banach algebra with respect to convolution, and

$$\mathcal{F}L^\infty * \mathcal{F}L^2 \subseteq \mathcal{F}L^2 = L^2, \quad \text{due to} \quad L^\infty \cdot L^2 \subseteq L^2. \ \blacksquare$$

In order to make our article self-contained let us recall the following result which makes use of elementary inclusion results for Wiener amalgams and the Hausdorff–Young theorem for generalized amalgam spaces as described in [13, Theorem 9]. It states (in the unweighted version) that

$$\mathcal{F}(W(\mathcal{F}L^p, \ell^q)) \hookrightarrow W(\mathcal{F}L^q, \ell^p) \quad \text{for } 1 \le q \le p \le \infty.$$

In particular, it implies the Fourier invariance of the space $W(\mathcal{F}L^p,\ell^p)$ for $1\le p\le\infty$ (the unweighted version of [13, Theorem 6]).

LEMMA 2.7. *For $1\le p\le 2$ we have the chain of continuous embeddings*

$$W(\mathcal{F}L^p,\ell^p)\hookrightarrow \mathcal{F}L^p\hookrightarrow W(\mathcal{F}L^p,\ell^{p'}),$$

or by taking inverse Fourier transforms,

$$M^{p,p}(\mathbf{R}^n)\hookrightarrow L^p(\mathbf{R}^n)\hookrightarrow M^{p,p'}(\mathbf{R}^n).$$

In particular, $M^{2,2}(\mathbf{R}^n)=L^2(\mathbf{R}^n)$.

Proof. We start with the observation that the classical Hausdorff–Young estimate implies for $1\le p\le 2$, with $1/p'+1/p=1$,

$$\mathcal{F}L^p\hookrightarrow L^{p'}\quad\text{or equivalently}\quad L^p\hookrightarrow\mathcal{F}L^{p'},$$

and consequently

$$W(\mathcal{F}L^p,\ell^p)\hookrightarrow W(L^{p'},\ell^p)\hookrightarrow W(L^p,\ell^p)=L^p,$$

which implies

$$W(\mathcal{F}L^p,\ell^p)=\mathcal{F}(W(\mathcal{F}L^p,\ell^p))\hookrightarrow\mathcal{F}L^p\qquad\text{for } 1\le p\le 2.$$

Thus the first continuous embedding is verified. For the second inclusion recall that [13] (see above, due to the obvious fact that $p\le 2\le p'$) implies

$$\mathcal{F}L^p=\mathcal{F}(W(L^p,\ell^p))\hookrightarrow\mathcal{F}(W(\mathcal{F}L^{p'},\ell^p))\hookrightarrow W(\mathcal{F}L^p,\ell^{p'}).\ ■$$

2.3. Fourier–Beurling algebra. For $s\ge 0$ the Fourier–Beurling algebra $\mathcal{F}L^1_s(\mathbf{R}^n)=\mathcal{F}L^1_s$ is the set of all $f\in\mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{\mathcal{F}L^1_s}=\int_{\mathbf{R}^n}\langle\xi\rangle^s|\widehat{f}(\xi)|\,d\xi$$

is finite. It is well-known that $\mathcal{F}L^1_s(\mathbf{R}^n)$ is a multiplication algebra, because Beurling algebras are Banach convolution algebras [32]. We also recall the following result (see, e.g., [32, Proposition 1.6.14]):

LEMMA 2.8. *Let $s\ge 0$ and $f\in\mathcal{F}L^1_s(\mathbf{R}^n)$. Then for any $\varepsilon>0$, there exists $\phi\in C_c^\infty(\mathbf{R}^n)$ such that $\|f-\phi f\|_{\mathcal{F}L^1_s}<\varepsilon$.*

For the inclusion relation between $\mathcal{F}L^1_s(\mathbf{R}^n)$ and $M^{p,1}_s(\mathbf{R}^n)$, we have the following (cf. [29, 30]).

LEMMA 2.9. *For $1\le p\le 2$ and $s\ge 0$ one has $M^{p,1}_s(\mathbf{R}^n)\hookrightarrow\mathcal{F}L^1_s(\mathbf{R}^n)$.*

Proof. Let $f\in M^{p,1}_s(\mathbf{R}^n)$ and $\varphi\in C_c^\infty(\mathbf{R}^n)$ be such that $\operatorname{supp}\varphi\subset[-1,1]^n$ and $\sum_{k\in\mathbf{Z}^n}\varphi(\xi-k)=1$. Since $\operatorname{supp}\varphi(\cdot-k)\subset k+[-1,1]^n$, it follows from the Minkowski inequality for integrals, the Hölder inequality and the Hausdorff–Young inequality that

$$\|f\|_{\mathcal{F}L^1_s} \le \Big\| \sum_{k\in\mathbf{Z}^n} \langle\cdot\rangle^s |\varphi(\cdot-k)\widehat{f}(\cdot)| \Big\|_{L^1} \lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^s \|\varphi(\cdot-k)\widehat{f}(\cdot)\|_{L^1}$$
$$\lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^s \|\varphi(\cdot-k)\widehat{f}(\cdot)\|_{L^{p'}} \lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^s \|\varphi(D-k)f\|_{L^p} = \|f\|_{M^{p,1}_s},$$

which yields the desired result. ∎

Lemma 2.10. *For $s \ge 0$ we define*

$$(\mathcal{F}L^1_s)_c = \{f \in \mathcal{F}L^1_s(\mathbf{R}^n) \mid \operatorname{supp} f \text{ is compact}\}.$$

Then $(\mathcal{F}L^1_s)_c \hookrightarrow M^{p,1}_s(\mathbf{R}^n)$ for $1 \le p < \infty$.

Proof. Let $f \in \mathcal{F}L^1_s(\mathbf{R}^n)$ with $\operatorname{supp} f$ compact. Then for $\phi \in \mathcal{S}(\mathbf{R}^n)$ with $\phi(x) = 1$ on $\operatorname{supp} f$, by Lemma 2.5 we have

$$\|f\|_{M^{p,1}_s} = \|f\phi\|_{M^{p,1}_s} \lesssim \|f\|_{M^{\infty,1}_s} \|\phi\|_{M^{p,1}_s}.$$

Moreover, for $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\sum_{k\in\mathbf{Z}^n} \varphi(\xi-k) = 1$,

$$\|f\|_{M^{\infty,1}_s} = \sum_{k\in\mathbf{Z}^n} \langle k\rangle^s \|\varphi(D-k)f\|_{L^\infty} \lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^s \|\varphi(\cdot-k)\widehat{f}\|_{L^1} \lesssim \|f\|_{\mathcal{F}L^1_s},$$

which implies the desired result. ∎

3. Approximate units. In this section, we prove the following result, which corresponds to the $M^{p,q}_s$-version of Bhimani–Ratnakumar's [3, Proposition 3.14].

Theorem 3.1. *Let $1 \le p, q < \infty$ and $s > n/q'$, or $q = 1$ and $s \ge 0$. Then for any $f \in M^{p,q}_s(\mathbf{R}^n)$, and $\varepsilon > 0$ there exists $\phi \in C^\infty_c(\mathbf{R}^n)$ such that*

$$\|f - \phi f\|_{M^{p,q}_s} < \varepsilon.$$

We remark that Bhimani [2, Proposition 4.8] considers the case $q = 1$ and $s = 0$.

To prove Theorem 3.1 we first prepare another lemma.

Lemma 3.2. *Let $s \ge 0$ and $\psi \in C^\infty_c(\mathbf{R}^n)$ be such that $\psi(0) = 1$. For $0 < \lambda < 1$, define $\psi_\lambda(x) = \psi(\lambda x)$. Then, for any $g \in \mathcal{S}(\mathbf{R}^n)$ and $\varepsilon > 0$, there exists $0 < \lambda_0 < 1$ such that*

$$\|(1-\psi_\lambda)g\|_{M^{1,1}_s} < \varepsilon \quad \forall 0 < \lambda < \lambda_0.$$

Proof. Let $\phi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$. We first note that by (2.1),

$$V_\phi((1-\psi_\lambda)g)(x,\xi) = (2\pi)^{-n} e^{-ix\xi} V_{\widehat{\phi}}(\widehat{g} - (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{g}))(\xi, -x).$$

Moreover, since $\widehat{\psi_\lambda}(\eta) = \frac{1}{\lambda^n}\widehat{\psi}(\eta/\lambda)$ and $1 = \psi(0) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{\psi_\lambda}(\eta)\,d\eta$, we have

$$\widehat{g}(t) - (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{g})(t) = \frac{1}{(2\pi\lambda)^n}\int_{\mathbf{R}^n} (\widehat{g}(t) - \widehat{g}(t-\eta))\widehat{\psi}\Big(\frac{\eta}{\lambda}\Big)\,d\eta$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} (\widehat{g}(t) - (T_{\lambda\eta}\widehat{g})(t))\widehat{\psi}(\eta)\,d\eta$$

and thus

$$V_{\widehat{\phi}}(\widehat{g} - (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{g}))(\xi,-x) = \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n} V_{\widehat{\phi}}(\widehat{g} - T_{\lambda\eta}\widehat{g})(\xi,-x)\widehat{\psi}(\eta)\,d\eta.$$

Therefore, by the Fubini theorem,

$$\|(1-\psi_\lambda)g\|_{M_s^{1,1}} \approx \iint_{\mathbf{R}^{2n}} \langle\xi\rangle^s |V_{\widehat{\phi}}(\widehat{g} - (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{g}))(\xi,-x)|\,dx\,d\xi$$
$$\lesssim \int_{\mathbf{R}^n} \|\langle\xi\rangle^s V_{\widehat{\phi}}(\widehat{g} - T_{\lambda\eta}\widehat{g})(\xi,-x)\|_{L^1(\mathbf{R}^{2n}_{(x,\xi)})} |\widehat{\psi}(\eta)|\,d\eta.$$

On the other hand, since $0<\lambda<1$, $s \ge 0$ and

$$V_{\widehat{\phi}}(T_{\lambda\eta}\widehat{g})(\xi,-x) = e^{-i\lambda\eta(-x)}(V_{\widehat{\phi}}\widehat{g})(\xi-\lambda\eta,-x),$$

we have

$$\|\langle\xi\rangle^s V_{\widehat{\phi}}(\widehat{g} - T_{\lambda\eta}\widehat{g})(\xi,-x)\|_{L^1(\mathbf{R}^{2n}_{(x,\xi)})}$$
$$\lesssim \iint_{\mathbf{R}^{2n}} \langle\xi\rangle^s |V_{\widehat{\phi}}\widehat{g}(\xi,-x)|\,dx\,d\xi$$
$$+ \langle\lambda\eta\rangle^s \iint_{\mathbf{R}^{2n}} \langle\xi-\lambda\eta\rangle^s |V_{\widehat{\phi}}\widehat{g}(\xi-\lambda\eta,-x)|\,dx\,d\xi$$
$$\lesssim \langle\eta\rangle^s \|\langle\xi\rangle^s V_{\widehat{\phi}}\widehat{g}(\xi,-x)\|_{L^1(\mathbf{R}^{2n}_{(x,\xi)})}.$$

We note that since $\widehat{\phi}, \widehat{g} \in \mathcal{S}(\mathbf{R}^n)$, we have $V_{\widehat{\phi}}\widehat{g} \in \mathcal{S}(\mathbf{R}^{2n})$, and also

$$V_{\widehat{\phi}}(\widehat{g} - T_{\lambda\eta}\widehat{g})(\xi,-x) = \int_{\mathbf{R}^n} (\widehat{g}(t) - \widehat{g}(t-\lambda\eta))\overline{\widehat{\phi}(t-\xi)}e^{ixt}\,dt \to 0 \quad (\lambda \to 0)$$

for all $\eta, \xi, x \in \mathbf{R}^n$. Thus it follows from the Lebesgue convergence theorem that

$$\|(1-\psi_\lambda)g\|_{M_s^{1,1}} \lesssim \iiint_{\mathbf{R}^{3n}} \langle\xi\rangle^s |V_{\widehat{\phi}}(\widehat{g} - T_{\lambda\eta}\widehat{g})(\xi,-x)|\,|\widehat{\psi}(\eta)|\,dx\,d\xi\,d\eta \to 0$$

as $\lambda \to 0$, which implies the desired result. ∎

Proof of Theorem 3.1. Let $\varepsilon > 0$. Since $\mathcal{S}(\mathbf{R}^n)$ is dense in $M_s^{p,q}(\mathbf{R}^n)$, there exists $g \in \mathcal{S}(\mathbf{R}^n)$ such that $\|f-g\|_{M_s^{p,q}} < \varepsilon$. Moreover by Lemma 3.2, there exist $\psi \in C_c^\infty(\mathbf{R}^n)$ and $0 < \lambda_0 < 1$ such that $\|(1-\psi_\lambda)g\|_{M_s^{1,1}} < \varepsilon$ for all $0 < \lambda < \lambda_0$, where $\psi_\lambda(x) = \psi(\lambda x)$. Thus by Lemmas 2.2(v) and 2.5,

$$\begin{aligned}\|f-\psi_\lambda f\|_{M_s^{p,q}} &\lesssim \|f-g\|_{M_s^{p,q}} + \|\psi_\lambda(f-g)\|_{M_s^{p,q}} + \|(1-\psi_\lambda)g\|_{M_s^{p,q}} \\ &\lesssim (1+\|\psi_\lambda\|_{M_s^{\infty,q}})\|f-g\|_{M_s^{p,q}} + \|(1-\psi_\lambda)g\|_{M_s^{p,q}} \\ &\lesssim (1+\|\psi_\lambda\|_{M_s^{\infty,1}})\|f-g\|_{M_s^{p,q}} + \|(1-\psi_\lambda)g\|_{M_s^{1,1}} \\ &\le (2+C\|\psi\|_{M_s^{\infty,1}})\varepsilon,\end{aligned}$$

which implies the desired result. ■

4. Closed ideals in $M_s^{p,1}(\mathbf{R}^n)$.

In this section, we consider the closed ideals in $M_s^{p,1}(\mathbf{R}^n)$, which play an important role in Section 6. Throughout this section, X stands for $M_s^{p,1}(\mathbf{R}^n)$ $(1 \le p < \infty,\ s \ge 0)$ or $\mathcal{F}L_s^1(\mathbf{R}^n)$ $(s \ge 0)$.

DEFINITION 4.1. Let I be a linear subspace of X. Then I is called an *ideal* in X if $fg \in I$ whenever $f \in X$ and $g \in I$. Moreover, if an ideal I in X is a closed subset of X, then I is called a *closed ideal* in X. For a subset S of X, the set $\bigcap_{\lambda\in\Lambda} I_\lambda$ is called the ideal *generated by* S, where $\{I_\lambda\}_{\lambda\in\Lambda}$ denotes the set of all ideals in X containing S.

It is easy to see that the closed ideal generated by S coincides with the closure in X of the set

$$\Big\{\sum_{j=1}^N f_j g_j \;\Big|\; f_j \in X,\ g_j \in S,\ N \in \mathbf{N}\Big\}.$$

DEFINITION 4.2. For a closed ideal I of X the *zero-set* of I is defined by $Z(I) = \bigcap_{f\in I} f^{-1}(\{0\})$ with $f^{-1}(\{0\}) = \{x \in \mathbf{R}^n \mid f(x) = 0\}$.

We note that $x \in Z(I)$ if and only if $f(x) = 0$ for all $f \in I$. Moreover, $Z(I)$ is a closed subset of X whenever I is a closed ideal in X. In fact, if $f \in X$, then f is continuous on $\mathbf{R}^n$ and thus $f^{-1}(\{0\})$ is a closed subset of $\mathbf{R}^n$.

The following lemma seems to be known to many people, but for the reader's convenience, we give the proof. We will write $f|_E = 0$ if $f(x) = 0$ for all $x \in E$.

LEMMA 4.3. *Let E be a closed subset of $\mathbf{R}^n$. Then*

$$I(E) = \{f \in X \mid f|_E = 0\}$$

is a closed ideal in X with $E = Z(I(E))$.

Proof. We give the proof only for the case $X = M_s^{p,1}(\mathbf{R}^n)$; the other case is similar. It is clear that $I(E)$ is an ideal in $M_s^{p,1}(\mathbf{R}^n)$. To see that $I(E)$ is closed, let $f \in M_s^{p,1}(\mathbf{R}^n)$, $\{f_m\}_{m=1}^\infty \subset I(E)$ and $\|f_m - f\|_{M_s^{p,1}} \to 0$

$(m \to \infty)$. Since

$$\|f_m - f\|_{L^\infty} \lesssim \|f_m - f\|_{M^{\infty,1}} \lesssim \|f_m - f\|_{M^{p,1}} \lesssim \|f_m - f\|_{M_s^{p,1}},$$

we see that $\{f_m\}_{m=1}^\infty$ converges pointwise to f on $\mathbf{R}^n$. Since $f_m|_E = 0$, we have $f|_E = 0$, and thus $f \in I(E)$. Hence $I(E)$ is closed. Next we prove $E = Z(I(E))$. Since $E \subset Z(I(E))$ is clear, we show $Z(I(E)) \subset E$. Suppose $x_0 \notin E$. Since E is closed and $C_c^\infty(\mathbf{R}^n) \subset M_s^{p,1}(\mathbf{R}^n)$, there exists $f \in M_s^{p,1}(\mathbf{R}^n)$ such that $f(x_0) = 1$ and $f|_E = 0$. Then $f \in I(E)$ and $f(x_0) \neq 0$. Thus $x_0 \notin Z(I(E))$, which implies the desired result. ∎

We also prepare the following lemma, which characterizes closed ideals in $M_s^{p,1}(\mathbf{R}^n)$. We will say a subspace M of $M_s^{p,1}(\mathbf{R}^n)$ is *modulation invariant* if $M_\eta f(x) = e^{ix\eta} f(x) \in M$ whenever $f \in M$ and $\eta \in \mathbf{R}^n$.

LEMMA 4.4. *Let $1 \le p < \infty$ and $s \ge 0$. Then for any closed ideal I in $M_s^{p,1}(\mathbf{R}^n)$:*

(i) *$g \in M_{-s}^{p',\infty}(\mathbf{R}^n)$ satisfies*

$$\iint_{\mathbf{R}^{2n}} V_\phi f(x,\xi)\overline{V_\psi g(x,\xi)}\,dx\,d\xi = 0 \quad \text{for all } f \in I \tag{4.1}$$

for all pairs $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle \psi, \phi\rangle \neq 0$ if and only if

$$\int_{\mathbf{R}^n} [V_\phi f(x,\cdot) * V_{\overline{\psi}}\overline{g}(x,\cdot)](\xi)\,dx = 0 \quad \text{for all } f \in I,\ \xi \in \mathbf{R}^n \tag{4.2}$$

for all pairs $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle \psi, \phi\rangle \neq 0$.

(ii) *Every closed ideal in $M_s^{p,1}(\mathbf{R}^n)$ coincides with a closed modulation invariant subspace of $M_s^{p,1}(\mathbf{R}^n)$.*

Proof. We first assume that (4.2) holds. Then

$$0 = \int_{\mathbf{R}^n} [V_\phi f(x,\cdot) * V_{\overline{\psi}}\overline{g}(x,\cdot)](\xi)\,dx = (2\pi)^n \langle \psi, \phi\rangle \langle M_{-\xi} f, g\rangle$$

for all $f \in I$, $\xi \in \mathbf{R}^n$ and $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle \psi, \phi\rangle \neq 0$, and thus $\langle f, g\rangle = 0$. Hence

$$\iint_{\mathbf{R}^{2n}} V_\phi f(x,\xi)\overline{V_\psi g(x,\xi)}\,dx\,d\xi = (2\pi)^{-n}\langle \psi, \phi\rangle\langle f, g\rangle = 0,$$

which gives (4.1).

Next we assume that (4.1) holds. We note that Lemma 2.5 implies that $f\overline{g} \in M_{-s}^{1,\infty}(\mathbf{R}^n) \hookrightarrow M_{-s}^{p',\infty}(\mathbf{R}^n)$ for all $f \in I$. Then for all $h \in M_s^{p,1}(\mathbf{R}^n)$ and $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle \psi, \phi\rangle \neq 0$, we have

$$\langle \psi, \phi\rangle\langle f\overline{g}, h\rangle = \langle \psi, \phi\rangle\langle f\overline{h}, g\rangle = (2\pi)^{-n}\langle V_\phi(f\overline{h}), V_\psi g\rangle.$$

On the other hand, since $f \in I$ and I is a closed ideal in $M_s^{p,1}(\mathbf{R}^n)$, we see that $f\overline{h} \in I$. Therefore, (4.1) implies that $\langle V_\phi(f\overline{h}), V_\psi g\rangle = 0$, and thus

$\langle f\overline{g}, h\rangle = 0$. By duality, we obtain $f\overline{g} = 0$. Hence, for $x, \xi \in \mathbf{R}^n$,

$$\begin{aligned} 0 &= V_{\phi\overline{\psi}}(f\overline{g})(x,\xi) \\ &= (2\pi)^{-n} \int_{\mathbf{R}^n} V_\phi f(x, \xi-\eta) V_{\overline{\psi}}\overline{g}(x,\eta)\,d\eta \\ &= (2\pi)^{-n} \int_{\mathbf{R}^n} V_\phi(M_{-\xi} f)(x,-\eta)\overline{V_\psi g(x,-\eta)}\,d\eta, \end{aligned}$$

and thus $\langle\psi,\phi\rangle\langle M_{-\xi} f, g\rangle = 0$, which implies (4.2).

Finally, we give the characterization of closed ideals in $M^{p,1}_s(\mathbf{R}^n)$. Let I be a closed ideal in $M^{p,1}_s(\mathbf{R}^n)$. We first note that if $g \in M^{p',\infty}_{-s}(\mathbf{R}^n)$ satisfies (4.2), then

$$0 = \int_{\mathbf{R}^n} [V_\phi f(x,\cdot) * V_{\overline{\psi}}\overline{g}(x,\cdot)](\xi)\,dx = \langle V_\phi(M_{-\xi} f), V_\psi g\rangle \tag{4.3}$$

for all $f \in I$, $\xi \in \mathbf{R}^n$ and $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle\psi,\phi\rangle \neq 0$. Suppose that I is not modulation invariant. Then there exist $f_0 \in I$ and $\eta_0 \in \mathbf{R}^n$ such that $M_{-\eta_0} f_0 \notin I$. On the other hand, $(M^{p,1}_s(\mathbf{R}^n))' = M^{p',\infty}_{-s}(\mathbf{R}^n)$, i.e., for any $\ell \in (M^{p,1}_s(\mathbf{R}^n))'$ and $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$ with $\langle\psi,\phi\rangle \neq 0$, there exists $g \in M^{p',\infty}_{-s}(\mathbf{R}^n)$ such that

$$\ell(f) = \frac{1}{(2\pi)^n\langle\psi,\phi\rangle}\langle V_\phi f, V_\psi g\rangle.$$

Therefore, the Hahn–Banach theorem and duality imply that there exists $g_0 \in M^{p',\infty}_{-s}(\mathbf{R}^n)$ such that

$$\langle f, g_0\rangle = \frac{1}{(2\pi)^n\langle\psi,\phi\rangle}\langle V_\phi f, V_\psi g_0\rangle = 0$$

for all $f \in I$, and

$$\langle M_{-\eta} f, g_0\rangle = \frac{1}{(2\pi)^n\langle\psi,\phi\rangle}\langle V_\phi(M_{-\eta} f_0), V_\psi g_0\rangle = 1.$$

The above two equalities imply that g_0 satisfies (4.2) but not (4.3), which yields a contradiction.

Conversely, we assume that I is a closed modulation invariant subspace of $M^{p,1}_s(\mathbf{R}^n)$. If I is not an ideal in $M^{p,1}_s(\mathbf{R}^n)$, then there exist $f_0 \in I$ and $h_0 \in M^{p,1}_s(\mathbf{R}^n)$ such that $f_0\overline{h_0} \notin I$. The Hahn–Banach theorem and duality imply that there exists $g_0 \in M^{p',\infty}_{-s}(\mathbf{R}^n)$ such that

$$\langle f, g_0\rangle = \frac{1}{(2\pi)^n\langle\psi,\phi\overline{\phi}\rangle}\langle V_{\phi\overline{\phi}} f, V_\psi g_0\rangle = 0 \tag{4.4}$$

for all $f \in I$, and

$$\langle f_0\overline{h_0}, g_0\rangle = \frac{1}{(2\pi)^n\langle\psi,\phi\overline{\phi}\rangle}\langle V_{\phi\overline{\phi}}(f_0\overline{h_0}), V_\psi g_0\rangle = 1. \tag{4.5}$$

We note that

$$V_{\phi\overline{\phi}}(f_0\overline{h_0})(x,\xi) = (2\pi)^{-n}[V_\phi f_0(x,\cdot) * V_{\overline{\phi}}\overline{h_0}(x,\cdot)](\xi)$$

and thus

$$\begin{aligned}\langle V_{\phi\overline{\phi}}(f_0\overline{h_0}), V_\psi g_0\rangle &= (2\pi)^{-n} \int_{\mathbf{R}^n} [(V_\phi f_0(x,\cdot) * V_{\overline{\phi}}\overline{h_0}(x,\cdot)) * V_{\overline{\psi}}\overline{g_0}(x,\cdot)](0)\,dx \\ &= (2\pi)^{-n} \int_{\mathbf{R}^n} [(V_\phi f_0(x,\cdot) * V_{\overline{\psi}}\overline{g_0}(x,\cdot)) * V_{\overline{\phi}}\overline{h_0}(x,\cdot)](0)\,dx.\end{aligned}$$

Moreover for all $x, \xi \in \mathbf{R}^n$, by the Parseval identity we have

$$\begin{aligned}&[V_\phi f_0(x,\cdot) * V_{\overline{\psi}}\overline{g_0}(x,\cdot)](\xi) \\ &\quad= \int_{\mathbf{R}^n} V_\phi(M_{-\xi}f_0)(x,\eta)\overline{V_\psi g_0(x,\eta)}\,d\eta \\ &\quad= \int_{\mathbf{R}^n} \mathcal{F}_{y\to\eta}[(M_{-\xi}f_0)(y)\cdot\overline{\phi(y-x)}](\eta)\overline{\mathcal{F}_{y\to\eta}[g_0(y)\overline{\psi(y-x)}](\eta)}\,d\eta \\ &\quad= (2\pi)^n \int_{\mathbf{R}^n} (M_{-\xi}f_0)(y)\cdot\overline{\phi(y-x)}\,\overline{g_0(y)\overline{\psi(y-x)}}\,dy \\ &\quad= \langle M_{-\xi}f_0\cdot T_x(\overline{\phi}\psi), g_0\rangle.\end{aligned}$$

Since I is modulation invariant, we have $M_{-\xi}f_0 \in I$, and thus $M_{-\xi}f_0 \cdot T_x(\overline{\phi}\psi) \in I$ by $T_x(\overline{\phi}\psi) \in M_s^{p,1}(\mathbf{R}^n)$. Therefore by (4.4) we have

$$[V_\phi f_0(x,\cdot) * V_{\overline{\psi}}\overline{g_0}(x,\cdot)](\xi) = 0.$$

Hence $\langle V_{\phi\overline{\phi}}(f_0\overline{h_0}), V_\psi g_0\rangle = 0$, which contradicts (4.5). ∎

5. Wiener–Lévy theorem for $M_s^{p,1}(\mathbf{R}^n)$. In this section, we consider the Wiener–Lévy theorem for $M_s^{p,1}(\mathbf{R}^n)$ (see [32, Theorem 1.3.1]). More precisely, we show the following result.

THEOREM 5.1. *Let $1 \le p < \infty$, $s \ge 0$, $f \in M_s^{p,1}(\mathbf{R}^n)$ and a compact subset $K \subset \mathbf{R}^n$ be given. Suppose that F is an analytic function on a neighborhood of $f(K) = \{f(x) \mid x \in K\}$ in $\mathbf{C}$. Then there exists $g \in M_s^{p,1}(\mathbf{R}^n)$ such that $g(x) = F(f(x))$ for all $x \in K$.*

To prove Theorem 5.1, we prepare the following lemmas.

LEMMA 5.2. *Let $s \ge 0$, $f, h \in M_s^{1,1}(\mathbf{R}^n)$ and $x_0 \in \mathbf{R}^n$. Set*

$$H_{x_0}^\lambda(x) = \left(f\left(x_0 + \frac{x}{\lambda}\right) - f(x_0)\right)h(x) \quad \text{for } \lambda > 0.$$

Then

$$\lim_{\lambda\to\infty} \|H_{x_0}^\lambda\|_{M_s^{1,1}} = 0. \tag{5.1}$$

Proof. With $g_0(t) = e^{-|t|^2/2}$, by (2.1) for $\lambda > 1$ we have

$$\begin{aligned} V_{g_0}(H_{x_0}^{\lambda})(x,\xi) &= (2\pi)^{-n} e^{ix\xi} V_{\widehat{g_0}}(H_{x_0}^{\lambda})^{\wedge}(\xi,-x) \\ &= (2\pi)^{-n}\big((H_{x_0}^{\lambda})^{\wedge} * M_{-x}(\widehat{g_0})^*\big)(\xi) \\ &= (2\pi)^{-n/2}\big((H_{x_0}^{\lambda})^{\wedge} * M_{-x}g_0\big)(\xi). \end{aligned}$$

We note that

$$\left(f\left(x_0 + \frac{\cdot}{\lambda}\right)\right)^{\wedge}(\eta) = (T_{-\lambda x_0} D_\lambda f)^{\wedge}(\eta) = \lambda^n e^{i\lambda x_0 \eta} \widehat{f}(\lambda\eta)$$

with $D_\lambda f(\eta) = f(\eta/\lambda)$. Moreover, since $f, \widehat{f} \in M^{1,1}(\mathbf{R}^n) \hookrightarrow L^1(\mathbf{R}^n)$, by the Fourier inversion formula we have

$$f(x_0) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \lambda^n e^{i\lambda x_0\eta} \widehat{f}(\lambda\eta)\, d\eta.$$

Thus, by putting $H(x,\xi) = [\widehat{h} * M_{-x}g_0](\xi)$, we see that

$$\begin{aligned} &((H_{x_0}^{\lambda})^{\wedge} * M_{-x}g_0)(\xi) \\ &= \frac{1}{(2\pi)^n}\left[\left(f\left(x_0 + \frac{\cdot}{\lambda}\right)\right)^{\wedge} * \widehat{h} * M_{-x}g_0\right](\xi) - f(x_0)[\widehat{h} * M_{-x}g_0](\xi) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \lambda^n e^{i\lambda x_0\eta} \widehat{f}(\lambda\eta)(H(x,\xi-\eta) - H(x,\xi))\, d\eta \\ &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix_0\eta} \widehat{f}(\eta)\left(H\left(x,\xi - \frac{\eta}{\lambda}\right) - H(x,\xi)\right) d\eta. \end{aligned}$$

Therefore, by the Fubini theorem,

$$\|H_{x_0}^{\lambda}\|_{M_s^{1,1}} \lesssim \int_{\mathbf{R}^n} |\widehat{f}(\eta)| \bigg(\iint_{\mathbf{R}^{2n}} \langle\xi\rangle^s \left| H\left(x, \xi - \frac{\eta}{\lambda}\right) - H(x,\xi)\right| dx\, d\xi \bigg)\, d\eta.$$

Moreover, by the weak submultiplicity of $\langle\cdot\rangle^s$ for $s \ge 0$,

$$\langle\xi\rangle^s \lesssim \left\langle \xi - \frac{\eta}{\lambda}\right\rangle^s \left\langle \frac{\eta}{\lambda}\right\rangle^s \quad \text{and} \quad \left\langle \frac{\eta}{\lambda}\right\rangle \approx 1 + \left|\frac{\eta}{\lambda}\right| \le 1 + |\eta| \approx \langle\eta\rangle$$

for large λ, and thus

$$|\widehat{f}(\eta)| \iint_{\mathbf{R}^{2n}} \langle\xi\rangle^s \left| H\left(x, \xi - \frac{\eta}{\lambda}\right) - H(x,\xi)\right| dx\, d\xi \lesssim \langle\eta\rangle^s |\widehat{f}(\eta)|\, \|h\|_{M_s^{1,1}}.$$

Assume now that (5.1) is not valid, i.e., there exists $\varepsilon_0 > 0$ such that for some sequence $\{\lambda_m\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} \lambda_m = \infty$, one has $\|H_{x_0}^{\lambda_m}\|_{M_s^{1,1}} \ge \varepsilon_0$. But this is not possible since the Lebesgue dominated convergence theorem implies that $\|H_{x_0}^{\lambda_m}\|_{M_s^{1,1}} \to 0$ $(m \to \infty)$. This implies (5.1). ■

Lemma 5.3. *Let $1 \le p < \infty$, $s \ge 0$, $f \in M_s^{p,1}(\mathbf{R}^n)$, $\tau \in C_c^\infty(\mathbf{R}^n)$ and $x_0 \in \mathbf{R}^n$. For $\lambda > 0$, define*

$$G_{x_0}^\lambda(x) = \left(f\left(x_0 + \frac{x}{\lambda}\right) - f(x_0)\right)\tau(x).$$

Then

$$\lim_{\lambda\to\infty} \|G_{x_0}^\lambda\|_{M_s^{p,1}} = 0.$$

Proof. Suppose that $\lambda > 0$ and $\operatorname{supp}\tau \subset B_\lambda(0)$. Take $\psi \in C_c^\infty(\mathbf{R}^n)$ such that $\psi(x) = 1$ on $B_2(x_0)$. Since $\psi(x_0) = 1$ and $\psi(x_0 + x/\lambda) = 1$ on $\operatorname{supp}\tau$, we see that

$$\left(f\left(x_0 + \frac{x}{\lambda}\right) - f(x_0)\right)\tau(x) = \left((f\psi)\left(x_0 + \frac{x}{\lambda}\right) - (f\psi)(x_0)\right)\tau(x)$$

for all $x \in \mathbf{R}^n$. On the other hand, Lemma 2.5 implies

$$\|f\psi\|_{M_s^{1,1}} \lesssim \|f\|_{M_s^{\infty,1}}\|\psi\|_{M_s^{1,1}} \lesssim \|f\|_{M_s^{p,1}}\|\psi\|_{M_s^{1,1}},$$

and thus $f\psi \in M_s^{1,1}(\mathbf{R}^n)$. Hence, it follows from Lemma 5.2 that

$$\|G_{x_0}^\lambda\|_{M_s^{p,1}} \lesssim \|G_{x_0}^\lambda\|_{M_s^{1,1}} = \left\|\left((f\psi)\left(x_0 + \frac{\cdot}{\lambda}\right) - (f\psi)(x_0)\right)\tau\right\|_{M_s^{1,1}} \to 0$$

as $\lambda \to \infty$. ∎

Lemma 5.4. *Let f, K and F be as in Theorem 5.1, and $x_0 \in K$. Then there exists $g_{x_0} \in M_s^{p,1}(\mathbf{R}^n)$ such that $g_{x_0}(x) = F(f(x))$ on some neighborhood of x_0.*

Proof. Set $z_0 = f(x_0)$. Since F is analytic at z_0, there exists $0 < \varepsilon_0 < 1$ such that $F(z)$ has the power series expansion

$$F(z) = F(z_0) + \sum_{j=1}^\infty c_j(z - z_0)^j \qquad (|z - z_0| < \varepsilon_0).$$

Let $\tau \in C_c^\infty(\mathbf{R}^n)$ be a function which is 1 near the origin. Then it follows from Lemma 5.3 that there exists $\lambda_{x_0} > 0$ such that $\left\|(f(x_0 + \frac{\cdot}{\lambda_{x_0}}) - f(x_0))\tau\right\|_{M_s^{p,1}} < \varepsilon_0/c$, where the constant c is as in Lemma 2.2(iv). Thus we see that

$$b_{x_0}(x) = F(f(x_0))\tau(x) + \sum_{j=1}^\infty c_j\left(\left(f\left(x_0 + \frac{x}{\lambda_{x_0}}\right) - f(x_0)\right)\tau(x)\right)^j$$

converges in $M_s^{p,1}(\mathbf{R}^n)$. Hence

$$\begin{aligned} g_{x_0}(x) &= b_{x_0}(\lambda_{x_0}(x - x_0)) \\ &= F(f(x_0))\tau_{x_0}(x) + \sum_{j=1}^\infty c_j\big((f(x) - f(x_0))\tau_{x_0}(x)\big)^j \end{aligned}$$

is in $M_s^{p,1}(\mathbf{R}^n)$, where we denote $\tau_{x_0}(x) = \tau(\lambda_{x_0}(x - x_0))$. Since $g_{x_0}(x) = F(f(x))$ in some neighborhood of x_0, we obtain the desired result. ■

Now we prove Theorem 5.1 by extending the local result above to the compact set K.

Proof of Theorem 5.1. We first note that by Lemma 5.4, for each $x_0 \in K$, there exist $\tau_{x_0} \in C_c^\infty(\mathbf{R}^n)$ and a neighborhood U_{x_0} of x_0 such that g_{x_0} defined by

$$g_{x_0}(x) = F(f(x_0))\tau_{x_0}(x) + \sum_{j=1}^{\infty} c_j((f(x) - f(x_0))\tau_{x_0}(x))^j$$

satisfies $g_{x_0}(x) = F(f(x))$ on U_{x_0}. Since $K \subset \bigcup_{x_0 \in K} U_{x_0}$ and K is compact, there exist $\{x_j\}_{j=1}^N \subset K$ such that the corresponding neighborhoods $\{U_{x_j}\}_{j=1}^N$ cover K. Now we set $h_1(x) = \tau_{x_1}(x)$ and

$$h_j(x) = \tau_{x_j}(x)(1 - \tau_{x_1}(x)) \cdots (1 - \tau_{x_{j-1}}(x)) \quad (j = 2, \ldots, N).$$

Since $\tau_{x_j}(x) = 1$ and $g_{x_j}(x) = F(f(x))$ on U_{x_j} $(j = 1, \ldots, N)$, we easily see that $\sum_{j=1}^N h_j(x) = 1$ on K. Therefore $g \in M_s^{p,1}(\mathbf{R}^n)$ defined by $g(x) = \sum_{j=1}^N h_j(x) g_{x_j}(x)$ satisfies

$$g(x) = \sum_{j=1}^{N} h_j(x) F(f(x)) = F(f(x)) \sum_{j=1}^{N} h_j(x) = F(f(x))$$

for all $x \in K$. ■

Under a mild extra condition on f (respectively F), one can extend the local result to the following global result.

THEOREM 5.5. *Let $1 \le p < \infty$, $s \ge 0$, $f \in M_s^{p,1}(\mathbf{R}^n)$ and F be analytic on an open neighborhood of $f(\mathbf{R}^n) \cup \{0\}$ with $F(0) = 0$. Then there exists $g \in M_s^{p,1}(\mathbf{R}^n)$ such that $g(x) = F(f(x))$.*

Proof. Since F is analytic on a neighborhood of 0 with $F(0) = 0$, there exists $\varepsilon_0 > 0$ such that $F(z)$ has the power series representation

$$F(z) = \sum_{j=1}^{\infty} c_j z^j \quad (|z| < \varepsilon_0).$$

It follows from Theorem 3.1 that for any ε with $0 < \varepsilon < \varepsilon_0$, there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that

$$\|f - \phi f\|_{L^\infty} \lesssim \|f - \phi f\|_{M_s^{p,1}} < \varepsilon / c,$$

where the constant c is as in Lemma 2.2(iv). Now we set

$$g_0(x) = \sum_{j=1}^{\infty} c_j \big(f(x) - \phi(x) f(x)\big)^j.$$

Then $g_0 \in M_s^{p,1}(\mathbf{R}^n)$ and $g_0(x) = F(f(x) - \phi(x)f(x)) = F(f(x))$ for $x \notin \operatorname{supp}\phi$. On the other hand, let $\tau_0 \in C_c^\infty(\mathbf{R}^n)$ be such that $\tau_0(x) = 1$ on $\operatorname{supp}\phi$. Then Theorem 5.1 yields $g_1 \in M_s^{p,1}(\mathbf{R}^n)$ such that $g_1(x) = F(f(x))$ on $\operatorname{supp}\tau_0$. Now we set

$$g(x) = (1 - \tau_0(x))g_0(x) + \tau_0(x)g_1(x).$$

We note that $g \in M_s^{p,1}(\mathbf{R}^n)$. If $x \in \operatorname{supp}\phi$, then $\tau_0(x) = 1$ and $g_1(x) = F(f(x))$, and thus $g(x) = F(f(x))$. Moreover, if $x \in \operatorname{supp}\tau_0 \setminus \operatorname{supp}\phi$, then $g_0(x) = F(f(x) - \phi(x)f(x)) = F(f(x))$ and $g_1(x) = F(f(x))$, and thus $g(x) = F(f(x))$. Finally, if $x \notin \operatorname{supp}\tau_0$, then $\tau_0(x) = 0$ and $g_0(x) = F(f(x))$, and thus $g(x) = F(f(x))$. ∎

As an application of Theorem 5.1, we obtain the following version of Wiener's general Tauberian theorem for $M_s^{p,1}(\mathbf{R}^n)$ (cf. [32, Theorem 1.4.1], [21, Ch. VIII. 6.4]).

Theorem 5.6. *Let $1 \le p < \infty$, $s \ge 0$, $f \in M_s^{p,1}(\mathbf{R}^n)$ and I be a closed ideal in $M_s^{p,1}(\mathbf{R}^n)$. If I is generated by one function f in $M_s^{p,1}(\mathbf{R}^n)$, then $I = M_s^{p,1}(\mathbf{R}^n)$ if and only if $f(x) \neq 0$ $(x \in \mathbf{R}^n)$.*

Proof. Suppose that $I = M_s^{p,1}(\mathbf{R}^n)$. Since I is the closed ideal generated by $f \in M_s^{p,1}(\mathbf{R}^n)$, we see that I is equal to the closure in $M_s^{p,1}(\mathbf{R}^n)$ of the set

$$\Big\{ \sum_{j=1}^{N} \lambda_j \phi_j f \;\Big|\; \lambda_j \in \mathbf{C},\ \phi_j \in M_s^{p,1}(\mathbf{R}^n),\ N \in \mathbf{N} \Big\}.$$

If $f(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$, then $g(x_0) = 0$ for all $g \in I$. Since $\mathcal{S}(\mathbf{R}^n) \subset M_s^{p,1}(\mathbf{R}^n)$, this contradicts $I = M_s^{p,1}(\mathbf{R}^n)$.

Conversely, suppose that $f(x) \neq 0$ for all $x \in \mathbf{R}^n$. Let K be a compact subset of $\mathbf{R}^n$ and $\phi \in C_c^\infty(\mathbf{R}^n)$ be such that $\operatorname{supp}\phi \subset K$. Since $F(z) = 1/z$ is analytic on $\mathbf{C} \setminus \{0\}$ and $f(x) \neq 0$ for all $x \in K$, it follows from Theorem 5.1 that there exists $g \in M_s^{p,1}(\mathbf{R}^n)$ such that $fg = 1$ on K. Since $f \in I$ and I is an ideal in $M_s^{p,1}(\mathbf{R}^n)$, we have $\phi = \phi g \cdot f \in I$. Moreover, since $C_c^\infty(\mathbf{R}^n)$ is dense in $M_s^{p,1}(\mathbf{R}^n)$ (see Lemma 2.2 and Theorem 3.1), we obtain $I = M_s^{p,1}(\mathbf{R}^n)$. ∎

Remark 5.7. By Lemma 4.4, the ideal I in Theorem 5.6 is a closed modulation invariant subspace of $M_s^{p,1}(\mathbf{R}^n)$. Thus I is equal to the closure in $M_s^{p,1}(\mathbf{R}^n)$ of the set

$$A = \Big\{ \sum_{j=1}^{N} \lambda_j e^{i\eta_j x} f(x) \;\Big|\; \lambda_j \in \mathbf{C},\ \eta_j \in \mathbf{R}^n,\ N \in \mathbf{N} \Big\}.$$

As a corollary of Theorem 5.6, we also obtain a variant of Wiener's approximation theorem for the Wiener amalgam space $W(\mathcal{F}L^p, \ell_s^1)(\mathbf{R}^n)$ (cf. [32, Theorem 1.4.8]).

Corollary 5.8. *Let* $1 \le p < \infty$, $s \ge 0$ *and* $W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$ *be the Wiener amalgam space consisting of all* $\widehat{f} \in \mathcal{S}'(\mathbf{R}^n)$ *such that* $f \in M^{p,1}_s(\mathbf{R}^n)$ *with the norm* $\|\widehat{f}\|_{W(\mathcal{F}L^p,\ell^1_s)} = \|f\|_{M^{p,1}_s}$. *Then for any* $\widehat{f} \in W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$, *we have* $f(x) \neq 0$ *for all* $x \in \mathbf{R}^n$ *if and only if the set*

$$\mathcal{F}(A) = \Big\{ \sum_{j=1}^{N} \lambda_j \widehat{f}(\xi - \eta_j) \;\Big|\; \lambda_j \in \mathbf{C},\ \eta_j \in \mathbf{R}^n,\ N \in \mathbf{N} \Big\}$$

is dense in $W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$.

Proof. Let $\widehat{f} \in W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$. We first assume that $f(x) \neq 0$ for all $x \in \mathbf{R}^n$. Then Remark 5.7 implies that for any $\widehat{g} \in W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$ and $\varepsilon > 0$, there exist $\{\lambda_j\}_{j=1}^N \subset \mathbf{C}$ and $\{\eta_j\}_{j=1}^N \subset \mathbf{R}^n$ such that

$$\Big\|\widehat{g} - \sum_{j=1}^{N} \lambda_j \widehat{f}(\xi - \eta_j)\Big\|_{W(\mathcal{F}L^p,\ell^1_s)} = \Big\|g - \sum_{j=1}^{N} \lambda_j e^{i n_j x} f(x)\Big\|_{M^{p,1}_s} < \varepsilon.$$

Therefore, $\mathcal{F}(A)$ is dense in $W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$. Conversely, assume that $\mathcal{F}(A)$ is dense in $W(\mathcal{F}L^p, \ell^1_s)(\mathbf{R}^n)$. Then the set A in Remark 5.7 is dense in $M^{p,1}_s(\mathbf{R}^n)$. Hence Theorem 5.6 implies that $f(x) \neq 0$ for all $x \in \mathbf{R}^n$. ∎

Remark 5.9. The same ideas apply to the case of e.g. finitely many functions $f_1, \ldots, f_\ell$ without common zero.

6. Sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$. In this section, we consider the sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$. Throughout this section, X stands for $M^{p,1}_s(\mathbf{R}^n)$ ($1 \le p < \infty$, $s \ge 0$) or $\mathcal{F}L^1_s(\mathbf{R}^n)$ ($s \ge 0$).

Definition 6.1. Let E be a closed subset of $\mathbf{R}^n$ and $I(E)$ be the set defined in Lemma 4.3. Define $J(E)$ as the closed ideal in X generated by

$$J_0(E) = \{f \in X \mid f(x) = 0 \text{ in a neighborhood of } E\}.$$

Then E is called a *set of spectral synthesis* for X if $I(E) = J(E)$.

Remark 6.2. We can easily see that E is a set of spectral synthesis for X if and only if $x + E$ is a set of spectral synthesis for X for some/any $x \in \mathbf{R}^n$.

We shall prove the following result.

Theorem 6.3. *Let* $1 \le p \le 2$, $s \ge 0$ *and* K *be a closed subset of* $\mathbf{R}^n$. *Then* K *is a set of spectral synthesis for* $M^{p,1}_s(\mathbf{R}^n)$ *if and only if* K *is a set of spectral synthesis for* $\mathcal{F}L^1_s(\mathbf{R}^n)$.

6.1. Technical lemmas. To prove Theorem 6.3, we use the "ideal theory for Segal algebras" developed by Reiter [31, Ch. 6, §2] (see also [5, 32]). In the following, we denote by $(\mathcal{F}L^1_s)_c$ the space defined in Lemma 2.10.

LEMMA 6.4. *Let $s \geq 0$ and I be a closed ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$. Then I is the closure of $I \cap (\mathcal{F}L^1_s)_c$ in $\mathcal{F}L^1_s(\mathbf{R}^n)$.*

Proof. Since $I \cap (\mathcal{F}L^1_s)_c \subset I$ and I is closed in $\mathcal{F}L^1_s(\mathbf{R}^n)$, the closure of $I \cap (\mathcal{F}L^1_s)_c$ in $\mathcal{F}L^1_s(\mathbf{R}^n)$ is contained in I. We next show that I is contained in the closure of $I \cap (\mathcal{F}L^1_s)_c$ in $\mathcal{F}L^1_s(\mathbf{R}^n)$. Let $f \in I$. Then, by Lemma 2.8, for any $\varepsilon > 0$, there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|f - \phi f\|_{\mathcal{F}L^1_s} < \varepsilon$. Since I is an ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$, we have $\phi f \in I \cap (\mathcal{F}L^1_s)_c$. Hence we obtain the desired result. ∎

LEMMA 6.5. *Let $1 \leq p \leq 2$ and $s \geq 0$. Suppose that I and I' are closed ideals in $\mathcal{F}L^1_s(\mathbf{R}^n)$. Then*

(i) *$I \cap M^{p,1}_s(\mathbf{R}^n)$ is a closed ideal in $M^{p,1}_s(\mathbf{R}^n)$.*
(ii) *If $I \cap M^{p,1}_s(\mathbf{R}^n) = I' \cap M^{p,1}_s(\mathbf{R}^n)$, then $I = I'$.*

Proof. (i) We first prove that $I \cap M^{p,1}_s(\mathbf{R}^n)$ is an ideal in $M^{p,1}_s(\mathbf{R}^n)$. Let $f \in I \cap M^{p,1}_s(\mathbf{R}^n)$ and $g \in M^{p,1}_s(\mathbf{R}^n)$. Since $M^{p,1}_s(\mathbf{R}^n)$ is a multiplication algebra, we see that $fg \in M^{p,1}_s(\mathbf{R}^n)$. Moreover, since $M^{p,1}_s(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^1_s(\mathbf{R}^n)$ and I is an ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$, we have $fg \in I$. Hence $fg \in I \cap M^{p,1}_s(\mathbf{R}^n)$ for any $f \in I \cap M^{p,1}_s(\mathbf{R}^n)$ and $g \in M^{p,1}_s(\mathbf{R}^n)$, as desired.

We next prove $I \cap M^{p,1}_s(\mathbf{R}^n)$ is closed in $M^{p,1}_s(\mathbf{R}^n)$. Let f be in the closure of $I \cap M^{p,1}_s(\mathbf{R}^n)$ in $M^{p,1}_s(\mathbf{R}^n)$. Then there exists $\{f_m\}_{m=1}^\infty \subset I \cap M^{p,1}_s(\mathbf{R}^n)$ such that $\|f_m - f\|_{M^{p,1}_s} \to 0$ $(m \to \infty)$. We note that $M^{p,1}_s(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^1_s(\mathbf{R}^n)$, and thus $\|f_m - f\|_{\mathcal{F}L^1_s} \to 0$ $(m \to \infty)$. Since I is closed in $\mathcal{F}L^1_s(\mathbf{R}^n)$ and $M^{p,1}_s(\mathbf{R}^n)$ is complete, we obtain $f \in I \cap M^{p,1}_s(\mathbf{R}^n)$, as desired.

(ii) Since $I \cap M^{p,1}_s \cap (\mathcal{F}L^1_s)_c = I' \cap M^{p,1}_s \cap (\mathcal{F}L^1_s)_c$ and $(\mathcal{F}L^1_s)_c \hookrightarrow M^{p,1}_s(\mathbf{R}^n)$, we have $I \cap (\mathcal{F}L^1_s)_c = I' \cap (\mathcal{F}L^1_s)_c$. Thus it follows from Lemma 6.4 that $I = I'$. ∎

PROPOSITION 6.6. *Let $1 \leq p \leq 2$ and $s \geq 0$. Then for any closed ideal I_M in $M^{p,1}_s(\mathbf{R}^n)$, the ideal I_F in $\mathcal{F}L^1_s(\mathbf{R}^n)$ defined by the closure of I_M in $\mathcal{F}L^1_s(\mathbf{R}^n)$ satisfies $I_M = I_F \cap M^{p,1}_s(\mathbf{R}^n)$.*

Proof. Let I_M be a closed ideal in $M^{p,1}_s(\mathbf{R}^n)$ and define I'_F as the closure of $I_M \cap (\mathcal{F}L^1_s)_c$ in $\mathcal{F}L^1_s(\mathbf{R}^n)$. Then I'_F is a closed ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$. To see that I'_F is an ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$, let $f \in I'_F$ and $g \in \mathcal{F}L^1_s(\mathbf{R}^n)$. Then there exists $\{f_m\}_{m=1}^\infty \subset I_M \cap (\mathcal{F}L^1_s)_c$ such that $\|f - f_m\|_{\mathcal{F}L^1_s} \to 0$ $(m \to \infty)$. Since $f_m \in (\mathcal{F}L^1_s)_c$, there exists $\psi_m \in C_c^\infty(\mathbf{R}^n)$ such that $\psi_m(x) = 1$ on $\operatorname{supp} f_m$. Then $\psi_m g \in (\mathcal{F}L^1_s)_c \hookrightarrow M^{p,1}_s(\mathbf{R}^n)$. Therefore $\psi_m g \cdot f_m \in I_M$, and thus $f_m g = f_m \cdot \psi_m g \in I_M \cap (\mathcal{F}L^1_s)_c$. Moreover, since

$$\|fg - f_m g\|_{\mathcal{F}L^1_s} \lesssim \|f - f_m\|_{\mathcal{F}L^1_s} \|g\|_{\mathcal{F}L^1_s} \to 0 \quad (m \to \infty),$$

we have $fg \in I'_F$. Hence, I'_F is an ideal in $\mathcal{F}L^1_s(\mathbf{R}^n)$.

Next, we prove that I_F is equal to I'_F. Let $f \in I_F$. Then for any $\varepsilon > 0$, there exists $g \in I_M$ such that $\|f - g\|_{\mathcal{F}L^1_s} < \varepsilon$. On the other hand, Theorem 3.1 implies that there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|g - \phi g\|_{M_s^{p,1}} < \varepsilon$. Hence $\phi g \in I_M \subset M_s^{p,1}(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^1_s(\mathbf{R}^n)$. Thus $\phi g \in I_M \cap (\mathcal{F}L^1_s)_c$ and

$$\begin{aligned}\|f - \phi g\|_{\mathcal{F}L^1_s} &\le \|f - g\|_{\mathcal{F}L^1_s} + \|\phi g - g\|_{\mathcal{F}L^1_s} \\ &\lesssim \|f - g\|_{\mathcal{F}L^1_s} + \|\phi g - g\|_{M_s^{p,1}} \lesssim \varepsilon,\end{aligned}$$

which yields $I_F \subset I'_F$. The reverse inclusion is clear.

Finally, we prove $I_M = I_F \cap M_s^{p,1}(\mathbf{R}^n)$. Since $I_M \subset I_F$ and $I_M \subset M_s^{p,1}(\mathbf{R}^n)$, we see that $I_M = I_M \cap M_s^{p,1}(\mathbf{R}^n) \subset I_F \cap M_s^{p,1}(\mathbf{R}^n)$. On the other hand, let $f \in I_F \cap M_s^{p,1}(\mathbf{R}^n)$ and $\varepsilon > 0$. By Theorem 3.1, there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|f - \phi f\|_{M_s^{p,1}} < \varepsilon$. Now we take $\varphi \in C_c^\infty(\mathbf{R}^n)$ with $\varphi(x) = 1$ on $\operatorname{supp}\phi$. Since $f \in I_F = I'_F$, there exists $h \in I_M \cap (\mathcal{F}L^1_s)_c$ such that $\|f - h\|_{\mathcal{F}L^1_s} < \frac{\varepsilon}{\|\varphi\|_{M_s^{p,1}} \|\phi\|_{\mathcal{F}L^1_s}}$. Then $\phi h \in I_M$. Since $\mathcal{F}L^1_s(\mathbf{R}^n) \hookrightarrow M_s^{\infty,1}(\mathbf{R}^n)$ (see the proof of Lemma 2.10), we obtain

$$\begin{aligned}\|f - \phi h\|_{M_s^{p,1}} &\le \|f - \phi f\|_{M_s^{p,1}} + \|\varphi\phi(f - h)\|_{M_s^{p,1}} \\ &\lesssim \|f - \phi f\|_{M_s^{p,1}} + \|\varphi\|_{M_s^{p,1}} \|\phi(f - h)\|_{M_s^{\infty,1}} \\ &\lesssim \|f - \phi f\|_{M_s^{p,1}} + \|\varphi\|_{M_s^{p,1}} \|\phi(f - h)\|_{\mathcal{F}L^1_s} \\ &\lesssim \|f - \phi f\|_{M_s^{p,1}} + \|\varphi\|_{M_s^{p,1}} \|\phi\|_{\mathcal{F}L^1_s} \|f - h\|_{\mathcal{F}L^1_s}.\end{aligned}$$

Therefore f is in the closure of I_M in $M_s^{p,1}(\mathbf{R}^n)$. Since I_M is closed in $M_s^{p,1}(\mathbf{R}^n)$, we get the desired result. ∎

Remark 6.7. Let I_M and I'_M be closed ideals in $M_s^{p,1}(\mathbf{R}^n)$, and I_F be the closure of I_M in $\mathcal{F}L^1_s(\mathbf{R}^n)$. If the closure of I'_M in $\mathcal{F}L^1_s(\mathbf{R}^n)$ is equal to I_F, then Proposition 6.6 implies that $I_M = I'_M$.

Combining Lemma 6.5 and Proposition 6.6, we obtain the following result.

Theorem 6.8 (The ideal theory for Segal algebras). *Let $1 \le p \le 2$, $s \ge 0$, $\mathcal{I}_F$ be the set of all closed ideals in $\mathcal{F}L^1_s(\mathbf{R}^n)$, and $\mathcal{I}_M$ be the set of all closed ideals in $M_s^{p,1}(\mathbf{R}^n)$. Define $\iota : \mathcal{I}_F \to \mathcal{I}_M$ by $\iota(I_F) = I_F \cap M_s^{p,1}(\mathbf{R}^n)$. Then ι is bijective. More precisely,*

$$\iota^{-1}(I_M) = \overline{I_M}^{\|\cdot\|_{\mathcal{F}L^1_s}} \qquad (I_M \in \mathcal{I}_M)$$

and

$$\iota\big(\overline{I_M}^{\|\cdot\|_{\mathcal{F}L^1_s}}\big) = \overline{I_M}^{\|\cdot\|_{\mathcal{F}L^1_s}} \cap M_s^{p,1}(\mathbf{R}^n) = I_M,$$

where $\overline{I_M}^{\|\cdot\|_{\mathcal{F}L^1_s}}$ denotes the closure of I_M in $\mathcal{F}L^1_s(\mathbf{R}^n)$.

6.2. The proof of Theorem 6.3. For a closed subset K of $\mathbf{R}^n$, we set $I_F(K) = \{f \in \mathcal{F}L^1_s(\mathbf{R}^n) \mid f|_K = 0\}$ and $I_M(K) = \{f \in M^{p,1}_s(\mathbf{R}^n) \mid f|_K = 0\}$. Moreover, we define $J_F(K)$ as the closure of $\{f \in \mathcal{F}L^1_s(\mathbf{R}^n) \mid f(x) = 0$ in a neighborhood of $K\}$ in $\mathcal{F}L^1_s(\mathbf{R}^n)$, and similarly $J_M(K)$ as the closure of $\{f \in M^{p,1}_s(\mathbf{R}^n) \mid f(x) = 0$ in a neighborhood of $K\}$ in $M^{p,1}_s(\mathbf{R}^n)$. Then $J_F(K)$ is the smallest closed ideal I in $\mathcal{F}L^1_s(\mathbf{R}^n)$ with $\bigcap_{f\in I} f^{-1}(\{0\}) = K$ (see [32, Proposition 2.4.5]). Similarly, $J_M(K)$ is the smallest closed ideal I in $M^{p,1}_s(\mathbf{R}^n)$ with $\bigcap_{f\in I} f^{-1}(\{0\}) = K$ (see Theorems 5.1 and 5.5). Thus Proposition 6.6 implies that $I_F(K) = J_M(K)$ if and only if $I_F(K) = J_F(K)$. Therefore, K is a set of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$ if and only if K is a set of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^n)$.

6.3. Examples. As an application of Theorem 6.3, we show some concrete examples of sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$.

Example 6.9 (cf. [32, Theorem 2.7.6]). Let $1 \le p \le 2$. Then a circle in $\mathbf{R}^2$ is a set of spectral synthesis for $M^{p,1}_s(\mathbf{R}^2)$ if $0 \le s < 1/2$, but not if $s \ge 1/2$.

Example 6.10 (cf. [32, Theorem 2.7.7]). Let $1 \le p \le 2$ and $s \ge 0$. Then the sphere $S^{n-1} \subset \mathbf{R}^n$ is not a set of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$ if $n \ge 3$.

Example 6.11 (cf. [32, Theorem 2.7.9]). Let $1 \le p \le 2$. Single points of $\mathbf{R}^n$ are sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$, if $0 \le s < 1$.

Example 6.12 (cf. [32, Theorem 2.7.10]). Let $1 \le p \le 2$ and $s \ge 0$. Then a closed ball in $\mathbf{R}^n$ is a set of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$ and so is the complement of an open ball in $\mathbf{R}^n$.

7. Spectral synthesis revisited. As mentioned in Section 6.3, if $1 \le p \le 2$ and $0 \le s < 1$, then single points of $\mathbf{R}^n$ are sets of spectral synthesis for $M^{p,1}_s(\mathbf{R}^n)$. In this section we will prove this directly without using Theorem 6.3, and also for $p > 2$.

Theorem 7.1. *Let $1 \le p < \infty$, $0 \le s < 1$ and $x_0 \in \mathbf{R}^n$. Then $\{x_0\}$ is a set of spectral synthesis in $M^{p,1}_s(\mathbf{R}^n)$.*

7.1. Technical lemmas

Lemma 7.2. *For any $t_0 \in \mathbf{R}^n$ and $R > 0$, there exist $\psi^{(1)}, \psi^{(2)} \in C^\infty_c(\mathbf{R}^n)$ such that $\psi = \psi^{(1)} * \psi^{(2)}$ satisfies $\psi(x) = 1$ on $B_R(t_0)$ and* $\operatorname{supp}\psi \subset B_{5R}(t_0)$.

Proof. Let $g \in C^\infty_c(\mathbf{R}^n)$ be such that $g(x) \ge 0$ $(x \in \mathbf{R}^n)$, $g(x) = 1$ on $B_R(0)$ and $\operatorname{supp} g \subset B_{2R}(0)$. Define $\psi^{(1)}, \psi^{(2)} \in C^\infty_c(\mathbf{R}^n)$ by

$$\psi^{(1)}(x) = g\left(\frac{x}{2}\right) \quad \text{and} \quad \psi^{(2)}(x) = \frac{2^n}{\|g\|_{L^1}} g(2(x - t_0)),$$

respectively, and set $\psi = \psi^{(1)} * \psi^{(2)}$. Then ψ satisfies the desired conditions. Indeed,

$$\psi(x) = \int_{\mathbf{R}^n} \psi^{(1)}(x-y)\psi^{(2)}(y)\,dy = \frac{2^n}{\|g\|_{L^1}} \int_{\mathbf{R}^n} g\left(\frac{x-y}{2}\right) g(2(y-t_0))\,dy.$$

Moreover, if $x \in B_R(t_0)$ and $|2(y-t_0)| \le 2R$, then $\frac{|x-y|}{2} \le \frac{|x-t_0|}{2} + \frac{|t_0-y|}{2} \le R$, and thus $g(\frac{x-y}{2}) = 1$. Therefore,

$$\psi(x) = (\psi^{(1)} * \psi^{(2)})(x) = \frac{2^n}{\|g\|_1} \int_{\mathbf{R}^n} g(2(y-t_0))\,dy = 1$$

on $B_R(t_0)$. Furthermore, $\operatorname{supp}\psi^{(1)} \subset \{x \in \mathbf{R}^n \mid |x| \le 4R\}$ and $\operatorname{supp}\psi^{(2)} \subset \{x \in \mathbf{R}^n \mid |x-t_0| \le R\}$, and consequently $\operatorname{supp}\psi \subset \operatorname{supp}\psi^{(1)} + \operatorname{supp}\psi^{(2)} \subset \{x \in \mathbf{R}^n \mid |x-t_0| \le 5R\}$. ∎

LEMMA 7.3. *Let $\psi^{(1)}, \psi^{(2)} \in C_c^\infty(\mathbf{R}^n)$ be such that* $\operatorname{supp}\psi^{(j)} \subset B_R(0)$ *for some $R > 0$ $(j = 1, 2)$. Define $\psi = \psi^{(1)} * \psi^{(2)}$. Then for $0 \le s < 1$ and $\vartheta \in \mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} \langle\xi\rangle^s |\widehat{\psi}(\xi-\vartheta) - \widehat{\psi}(\xi)|\,d\xi \le C_\psi |\vartheta|^s \Big(\max_{|t|\le R} |e^{i\vartheta t} - 1|\Big)^{1-s}.$$

Proof. We first note that since $\widehat{\psi} = \widehat{\psi^{(1)}} \cdot \widehat{\psi^{(2)}}$, we have

$$\begin{aligned}&\widehat{\psi}(\xi-\vartheta) - \widehat{\psi}(\xi)\\ &\quad= (\widehat{\psi^{(1)}}(\xi-\vartheta) - \widehat{\psi^{(1)}}(\xi))\widehat{\psi^{(2)}}(\xi-\vartheta) + \widehat{\psi^{(1)}}(\xi)(\widehat{\psi^{(2)}}(\xi-\vartheta) - \widehat{\psi^{(2)}}(\xi)).\end{aligned}$$

Then it follows from the Cauchy–Schwarz inequality and the Plancherel theorem that

$$\begin{aligned}&\int_{\mathbf{R}^n} \langle\xi\rangle^s |\widehat{\psi}(\xi-\vartheta) - \widehat{\psi}(\xi)|\,d\xi\\ &\quad\le \Big(\int_{\mathbf{R}^n} |\widehat{\psi^{(1)}}(\xi-\vartheta) - \widehat{\psi^{(1)}}(\xi)|^2\,d\xi\Big)^{1/2} \Big(\int_{\mathbf{R}^n} \langle\xi\rangle^{2s} |\widehat{\psi^{(2)}}(\xi-\vartheta)|^2\,d\xi\Big)^{1/2}\\ &\qquad+ \Big(\int_{\mathbf{R}^n} |\widehat{\psi^{(2)}}(\xi-\vartheta) - \widehat{\psi^{(2)}}(\xi)|^2\,d\xi\Big)^{1/2} \Big(\int_{\mathbf{R}^n} \langle\xi\rangle^{2s} |\widehat{\psi^{(1)}}(\xi-\vartheta)|^2\,d\xi\Big)^{1/2}\\ &\quad= \|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x}-1)\psi^{(1)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)} \langle\vartheta\rangle^s \|\langle\xi\rangle^s \widehat{\psi^{(2)}}(\xi)\|_{L^2(\mathbf{R}^n_\xi)}\\ &\qquad+ \|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x}-1)\psi^{(2)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)} \langle\vartheta\rangle^s \|\langle\xi\rangle^s \widehat{\psi^{(1)}}(\xi)\|_{L^2(\mathbf{R}^n_\xi)}.\end{aligned}$$

We note that $|e^{i\vartheta x} - 1| \le \min\{2, |\vartheta x|\}$, and for $j = 1, 2$, it follows from the Plancherel theorem that

$$\|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x}-1)\psi^{(j)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)} = (2\pi)^{n/2} \|(e^{i\vartheta x}-1)\psi^{(j)}(x)\|_{L^2(\mathbf{R}^n_x)}.$$

Moreover, since $\operatorname{supp}\psi^{(j)} \subset B_R(0)$ $(j=1,2)$, we obtain

$$
\begin{aligned}
&\|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x}-1)\psi^{(j)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)}(1+|\vartheta|^s)\\
&\quad\lesssim \Big(\max_{|x|\le R}|e^{i\vartheta x}-1|\Big)^{1-s}\Big(\int_{\mathbf{R}^n}(|e^{i\vartheta x}-1|^s|\psi^{(j)}(x)|)^2\,dx\Big)^{1/2}(1+|\vartheta|^s)\\
&\quad\le |\vartheta|^s\Big(\max_{|x|\le R}|e^{i\vartheta x}-1|\Big)^{1-s}\Big(\int_{\mathbf{R}^n}(|x|^s|\psi^{(j)}(x)|)^2\,dx\Big)^{1/2}\\
&\qquad+2^s|\vartheta|^s\Big(\max_{|x|\le R}|e^{i\vartheta x}-1|\Big)^{1-s}\Big(\int_{\mathbf{R}^n}|\psi^{(j)}(x)|^2\,dx\Big)^{1/2},
\end{aligned}
$$

which yields the desired inequality. ∎

Now we prepare a lemma which corresponds to a weighted version of Bhimani–Ratnakumar's [3, Proposition 3.14].

Lemma 7.4. *Let $0 \le s < 1$, $f \in M^{1,1}_s(\mathbf{R}^n)$, $x_0 \in \mathbf{R}^n$ and $\varepsilon > 0$. Then there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that*

(i) $\|(f-f(x_0))\phi\|_{M^{1,1}_s} < \varepsilon$,

(ii) $\phi(x) = 1$ *in some neighborhood of* x_0.

Proof. Take $\psi = \psi^{(1)} * \psi^{(2)} \in C_c^\infty(\mathbf{R}^n)$ as in Lemma 7.2 with $t_0 = 0$ and $R = 1$, i.e., $\psi(x) = 1$ on $B_1(0)$ and $\operatorname{supp}\psi \subset B_5(0)$. Define $\psi_\lambda(x) = \psi(\lambda x)$ and $h^\lambda(x) = (f(x) - f(x_0))\psi_\lambda(x - x_0)$ for $\lambda > 0$. We note that if $\lambda > 5$ and $x \in \operatorname{supp} h^\lambda$, then $\psi(x - x_0) = 1$. Thus $h^\lambda(x) = h^\lambda(x)\psi(x - x_0)$.

Without loss of generality, we may assume $x_0 = 0$ (see Remark 6.2). Let $g_0(t) = e^{-|t|^2/2}$ $(t \in \mathbf{R}^n)$ and $\lambda > 5$. By (2.1), we have

$$
\begin{aligned}
\|h^\lambda\|_{M^{1,1}_s} &= \big\|\|\langle\xi\rangle^s V_{g_0}h^\lambda(x,\xi)\|_{L^1(\mathbf{R}^n_x)}\big\|_{L^1(\mathbf{R}^n_\xi)}\\
&= (2\pi)^{-n}\big\|\|\langle\xi\rangle^s V_{\widehat{g_0}}\widehat{h^\lambda}(\xi,-x)\|_{L^1(\mathbf{R}^n_x)}\big\|_{L^1(\mathbf{R}^n_\xi)}.
\end{aligned}
$$

Since $\widehat{g_0} = (2\pi)^{n/2}g_0$, $g_0^* = g_0$ and $h^\lambda(x) = h^\lambda(x)\psi(x)$, by (2.1) we obtain

$$
\begin{aligned}
V_{\widehat{g_0}}\widehat{h^\lambda}(\xi,-x) &= (2\pi)^{-n/2}V_{g_0}(\widehat{h^\lambda} * \widehat{\psi})(\xi,-x)\\
&= (2\pi)^{-n/2}e^{ix\xi}(\widehat{h^\lambda} * \widehat{\psi} * M_{-x}g_0)(\xi).
\end{aligned}
$$

Thus by the Fubini theorem and the Minkowski inequality for integrals,

$$
\begin{aligned}
\|h^\lambda\|_{M^{1,1}_s} &= (2\pi)^{-n}\big\|\|\langle\xi\rangle^s V_{\widehat{g_0}}\widehat{h^\lambda}(\xi,-x)\|_{L^1(\mathbf{R}^n_x)}\big\|_{L^1(\mathbf{R}^n_\xi)}\\
&\approx \big\|\|\langle\xi\rangle^s(\widehat{h^\lambda} * \widehat{\psi} * M_{-x}g_0)(\xi)\|_{L^1(\mathbf{R}^n_\xi)}\big\|_{L^1(\mathbf{R}^n_x)}\\
&\le \big\|\|\langle\xi\rangle^s\widehat{h^\lambda}(\xi)\|_{L^1(\mathbf{R}^n_\xi)}\|\langle\xi\rangle^s(\widehat{\psi} * M_{-x}g_0)(\xi)\|_{L^1(\mathbf{R}^n_\xi)}\big\|_{L^1(\mathbf{R}^n_x)}\\
&= \|\langle\cdot\rangle^s\widehat{h^\lambda}\|_{L^1}\|\psi\|_{M^{1,1}_s}.
\end{aligned}
$$

Since $\widehat{h^\lambda}(\zeta) = (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{f})(\zeta) - f(0)\widehat{\psi_\lambda}(\zeta)$ and $\widehat{\psi_\lambda}(\zeta) = \frac{1}{\lambda^n}\widehat{\psi}(\zeta/\lambda)$, we have

$$\widehat{h^\lambda}(\zeta) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{f}(\eta)\widehat{\psi_\lambda}(\zeta-\eta)\,d\eta - \frac{1}{(2\pi)^n}\Big(\int_{\mathbf{R}^n} \widehat{f}(\eta)\,d\eta\Big)\widehat{\psi_\lambda}(\zeta)$$
$$= \frac{1}{(2\pi\lambda)^n}\int_{\mathbf{R}^n} \widehat{f}(\eta)\Big(\widehat{\psi}\Big(\frac{\zeta-\eta}{\lambda}\Big) - \widehat{\psi}\Big(\frac{\zeta}{\lambda}\Big)\Big)d\eta$$

and thus we obtain by Lemma 7.3, choosing $\vartheta = \xi/\lambda$ and $\lambda > 5$,

$$\|\langle\cdot\rangle^s\widehat{h^\lambda}\|_{L^1(\mathbf{R}^n)} \lesssim \int_{\mathbf{R}^n}\Big(\frac{1}{\lambda^n}\int_{\mathbf{R}^n}\langle\zeta\rangle^s\Big|\widehat{\psi}\Big(\frac{\zeta-\eta}{\lambda}\Big) - \widehat{\psi}\Big(\frac{\zeta}{\lambda}\Big)\Big|\,d\zeta\Big)|\widehat{f}(\eta)|\,d\eta$$
$$\lesssim \lambda^s\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}\langle\xi\rangle^s\Big|\widehat{\psi}\Big(\xi-\frac{\eta}{\lambda}\Big) - \widehat{\psi}(\xi)\Big|\,d\xi\Big)|\widehat{f}(\eta)|\,d\eta$$
$$\lesssim \int_{\mathbf{R}^n}|\eta|^s\Big(\max_{|t|\le 4}|e^{i\frac{\eta}{\lambda}t}-1|\Big)^{1-s}|\widehat{f}(\eta)|\,d\eta.$$

We note that $M_s^{1,1}(\mathbf{R}^n) \hookrightarrow \mathcal{F}L_s^1(\mathbf{R}^n)$, and also $(\max_{|t|\le 4}|e^{i\frac{\eta}{\lambda}t}-1|)^{1-s} \le 2$ and $(\max_{|t|\le 4}|e^{i\frac{\eta}{\lambda}t}-1|)^{1-s} \to 0$ as $\lambda\to\infty$. Therefore the Lebesgue convergence theorem yields $\|\langle\cdot\rangle^s\widehat{h^\lambda}\|_{L^1(\mathbf{R}^n)} \to 0$ $(\lambda\to\infty)$. Therefore, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $\|h^{\lambda_0}\|_{M_s^{1,1}} < \varepsilon$. Hence, by putting $\phi(x) = \psi_{\lambda_0}(x)$, we arrive at the desired result. ∎

Next we prepare a lemma, which corresponds to a weighted version of Bhimani's [2, Proposition 4.7].

Lemma 7.5. *Let* $1 \le p \le \infty$, $0 \le s < 1$, $f \in M_s^{p,1}(\mathbf{R}^n)$, $x_0 \in \mathbf{R}^n$ *and* $\varepsilon > 0$. *Then there exists* $\tau \in C_c^\infty(\mathbf{R}^n)$ *such that*

(i) $\|(f - f(x_0))\tau\|_{M_s^{p,1}} < \varepsilon$,
(ii) $\tau(x) = 1$ *in some neighborhood of* x_0.

Proof. Let $\psi \in C_c^\infty(\mathbf{R}^n)$ be such that $\psi(x) = 1$ on some neighborhood of x_0, and set $h(x) = (f(x) - f(x_0))\psi(x)$. We note that $h(x_0) = 0$. Since $h \in M_s^{1,1}(\mathbf{R}^n)$, Lemma 7.4 implies that there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|(h - h(x_0))\phi\|_{M_s^{1,1}} < \varepsilon$ and $\phi = 1$ on some neighborhood of x_0. Now we define $\tau = \psi\phi \in C_c^\infty(\mathbf{R}^n)$. Then $\tau = 1$ on some neighborhood of x_0, and from $M_s^{1,1}(\mathbf{R}^n) \hookrightarrow M_s^{p,1}(\mathbf{R}^n)$ we have

$$\|(f - f(x_0))\tau\|_{M_s^{p,1}} = \|(h - h(x_0))\phi\|_{M_s^{p,1}} \lesssim \|(h - h(x_0))\phi\|_{M_s^{1,1}}.$$

Hence we obtain the desired result. ∎

Remark 7.6. Recall that $M_s^{p,1}(\mathbf{R}^n)$ is a Banach algebra. Then it follows from Lemma 7.5 that $M_s^{p,1}(\mathbf{R}^n)$ satisfies the condition of Wiener–Ditkin (see [16], [31, Ch. 2, §4.3], [32, Definition 2.4.7]), i.e., for every point $x_0 \in \mathbf{R}^n$, any function $f \in M_s^{p,1}(\mathbf{R}^n)$ vanishing at x_0 and any neighborhood $\mathcal{U}_0$ of 0 in $M_s^{p,1}(\mathbf{R}^n)$, there exists $\tau \in M_s^{p,1}(\mathbf{R}^n)$ such that (i) τ is constant 1 near x_0, and (ii) $f \cdot \tau \in \mathcal{U}_0$.

7.2. The proof of Theorem 7.1. We first observe that $J_0(\{x_0\}) \subset I(\{x_0\})$. Since $J(\{x_0\})$ is the smallest closed ideal containing $J_0(\{x_0\})$, it follows from Lemma 4.3 that $J(\{x_0\}) \subset I(\{x_0\})$. We next prove $I(\{x_0\}) \subset J(\{x_0\})$. Since $I(\{x_0\})$ is a closed ideal with $J_0(\{x_0\}) \subset I(\{x_0\})$, it suffices to prove that $I(\{x_0\}) \subset \overline{J_0(\{x_0\})}$. Let $f \in I(\{x_0\})$ and $\varepsilon > 0$. We note that $f(x_0) = 0$. By Lemma 7.5, there exists $\tau \in C_c^\infty(\mathbf{R}^n)$ such that $\|\tau f\|_{M_s^{p,1}} < \varepsilon$ and $\tau(x) = 1$ on some neighborhood of x_0. Now we set $g = (1-\tau)f$. Then $g = f - \tau f \in M_s^{p,1}(\mathbf{R}^n)$, $g(x) = 0$ on some neighborhood of x_0 and $\|f-g\|_{M_s^{p,1}} = \|\tau f\|_{M_s^{p,1}} < \varepsilon$. Hence we obtain the desired result.

8. Inclusion relation between $M^{p,1}$ and $\mathcal{F}A_p$. In this section, we consider the inclusion relation between the modulation space $M^{p,1}(\mathbf{R})$ and the Fourier Segal algebra $\mathcal{F}A_p(\mathbf{R})$. Here, $\mathcal{F}A_p(\mathbf{R})$ is the space defined by the norm

$$\|f\|_{\mathcal{F}A_p} = \|f\|_{L^p} + \|\widehat{f}\|_{L^1}.$$

We note that since $\mathcal{F}A_p(\mathbf{R})$ is the Fourier image of the Segal algebra $A_p(\mathbf{R})$ which is defined by the norm

$$\|f\|_{A_p} = \|\widehat{f}\|_{L^p} + \|f\|_{L^1}$$

(see [8, 31, 32, 40] for more details), $\mathcal{F}A_p(\mathbf{R})$ is an (abstract) Segal algebra in $A(\mathbf{R})(=\mathcal{F}L_0^1(\mathbf{R}))$ in the following sense.

(i) $\mathcal{F}A_p(\mathbf{R})$ is a commutative Banach algebra with pointwise multiplication and the norm $\|f\|_{\mathcal{F}A_p}$.
(ii) $\mathcal{F}A_p(\mathbf{R})$ is a dense subset of the Fourier algebra $A(\mathbf{R})$.
(iii) $\mathcal{F}A_p(\mathbf{R})$ is isometrically invariant under modulation operators:

$$\|M_\xi f\|_{\mathcal{F}A_p} = \|f\|_{\mathcal{F}A_p} \quad \forall \xi \in \mathbf{R}.$$

Moreover, $\mathcal{F}A_p(\mathbf{R})$ has similar properties to $A(\mathbf{R})$ (see [25, 26]). On the other hand, Lemmas 2.2, 2.7 and 2.9 imply that $M^{p,1}(\mathbf{R}) \hookrightarrow \mathcal{F}A_p(\mathbf{R})$ for $1 \le p \le 2$. Therefore, it is natural to ask whether $M^{p,1}(\mathbf{R})$ is a proper subset of $\mathcal{F}A_p(\mathbf{R})$.

THEOREM 8.1. *For $1 \le p \le 2$ we have the proper, dense inclusion $M^{p,1}(\mathbf{R}) \subsetneq \mathcal{F}A_p(\mathbf{R})$.*

We remark that Losert's [28, Theorem 2] implies that Theorem 8.1 holds for $p = 1$.

8.1. A technical lemma. To prove Theorem 8.1, we prepare the following lemma.

LEMMA 8.2. *Let K be a compact subset of $\mathbf{R}$. Then for any $\varepsilon > 0$, there exist a discrete measure $\mu \in M(\mathbf{R})$ and $m, N \in \mathbf{N}$ for which the following conditions hold:*

(i) $\|\mu\|_{M(\mathbf{R})} = 1$ *and* $\|\widehat{\mu}\|_{L^\infty} < \varepsilon$.
(ii) $\operatorname{supp}\mu$ *consists of* 2^m *different points and is expressed as*

$$\operatorname{supp}\mu = \Big\{x \in \mathbf{R} \;\Big|\; x = \sum_{j=1}^{m} \alpha_j N_j,\ \alpha_j = 0, 1,\ N_j = 2^{j-1}N\Big\}.$$

(iii) *The sets* $\{-x + K \mid x \in \operatorname{supp}\,\mu\}$ *are mutually disjoint.*

Proof. Our proof is based on the argument due to Kahane [20, pp. 34–36], which is also known as the Rudin–Shapiro method. Let $\varepsilon > 0$ and take $m \in \mathbf{N}$ such that $2^{\frac{1}{2}-\frac{m}{2}} < \varepsilon$. We choose $N \in \mathbf{N}$ such that the sets

$$\Big\{ -\sum_{j=1}^{m} \alpha_j N_j + K \;\Big|\; \alpha_j = 0, 1,\ N_j = 2^{j-1}N\Big\}$$

are mutually disjoint for different choices of $\alpha_j \in \{0, 1\}$, and then define $\mu_j, \nu_j \in M(\mathbf{R})$ $(j = 1, \ldots, m)$ by $\mu_0 = \nu_0 = \delta_0$ and

$$\mu_j = \mu_{j-1} + \nu_{j-1} * \delta_{N_j}, \quad \nu_j = \mu_{j-1} - \nu_{j-1} * \delta_{N_j} \quad (j = 1, \ldots, m).$$

We note that the set $\{\sum_{j=1}^{m} \alpha_j N_j \mid \alpha_j = 0, 1\}$ consists of 2^m different points, and thus $\|\mu_m\|_{M(\mathbf{R})} = 2^m$. Moreover, it follows from

$$(\mu_{j-1} \pm \mu_{j-1} * \delta_{N_j})^\wedge(\xi) = \widehat{\mu_{j-1}}(\xi) \pm \widehat{\nu_{j-1}}(\xi)e^{-i\xi N_j} \quad (j = 1, \ldots, m)$$

that

$$\begin{aligned}|\widehat{\mu_m}(\xi)|^2 + |\widehat{\nu_m}(\xi)|^2 &= 2(|\widehat{\mu_{m-1}}(\xi)|^2 + |\widehat{\nu_{m-1}}(\xi)|^2) \\ &= \cdots \\ &= 2^m(|\widehat{\mu_0}(\xi)|^2 + |\widehat{\nu_0}(\xi)|^2) = 2^{m+1}.\end{aligned}$$

This implies that

$$|\widehat{\mu_m}(\xi)| \le (|\widehat{\mu_m}(\xi)|^2 + |\widehat{\nu_m}(\xi)|^2)^{1/2} \le 2^{\frac{m+1}{2}}.$$

Now we set $\mu = 2^{-m}\mu_m$. Then we can easily see that $\|\mu\|_{M(\mathbf{R})} = 1$ and

$$\|\widehat{\mu}\|_{L^\infty} \le 2^{-m}2^{\frac{m+1}{2}} = 2^{\frac{1}{2}-\frac{m}{2}} < \varepsilon.$$

Hence, we obtain the desired result. ∎

REMARK 8.3. We note that μ can also be expressed as

$$\mu = 2^{-m} \sum_{\alpha_j=0,1} C_{(\alpha_1,\ldots,\alpha_{2m})}\delta_{\alpha_1 N_1+\cdots+\alpha_{2m}N_{2m}},$$

where each term $C_{(\alpha_1,\dots,\alpha_{2m})}$ is equal to 1 or -1. For the sake of simplicity, we simply write $\mu = \sum_{j=1}^{2^m} a_j \delta_{\ell(j)}$ with $\operatorname{supp}\mu = \{\ell(j) \in \mathbf{N} \mid j = 1,\dots,2^m\}$, where $a_j = 2^{-m}$ or $a_j = -2^{-m}$.

Remark 8.4. Let $1 \le p < 2$ and choose $r \in \mathbf{N}$ such that $2^{1/2-r(1/p-1/2)} < \varepsilon$. In the same way as above, we define $\nu \in M(\mathbf{R})$ by

$$\nu = 2^{-r/p}\mu_{m'} = \sum_{j=1}^{2^r} b_j \delta_{N(j)}.$$

Then $\sum_{j=1}^{2^r} |b_j|^p = 1$, $\|\widehat{\nu}\|_{L^\infty} < \varepsilon$ and the sets $\{-N(j) + K \mid N(j) \in \operatorname{supp}\nu\}$ are mutually disjoint.

8.2. The proof of Theorem 8.1. (i) We first consider the case $1 \le p < 2$. As mentioned at the beginning of Section 8, $M^{p,1}(\mathbf{R}) \subset \mathcal{F}A_p(\mathbf{R})$. Let $\phi \in C_c^\infty(\mathbf{R}) \setminus \{0\}$ be such that $K = \operatorname{supp}\phi \subset [-1/10, 1/10]$. Then Lemma 8.2 implies that for any $\varepsilon > 0$, there exist $\mu \in M(\mathbf{R})$ and $m \in \mathbf{N}$ such that $\|\mu\|_{M(\mathbf{R})} = 1$, $\|\widehat{\mu}\|_{L^\infty} < \varepsilon$ and

$$\operatorname{supp}\mu = \{\ell(j) \in \mathbf{N} \mid j = 1,\dots,2^m\},$$

where the sets $\{-\ell(j) + K \mid \ell(j) \in \operatorname{supp}\mu\}$ are mutually disjoint. It follows from Remark 8.3 that μ can be represented as $\mu = \sum_{j=1}^{2^m} a_j\delta_{\ell(j)}$ with $\ell(j) \in \operatorname{supp}\mu$ and $a_j = 2^{-m}$ or $a_j = -2^{-m}$. Moreover, since $\widehat{\phi} \in L^p(\mathbf{R})$, we see that for any $\eta > 0$, there exists a compact subset $K_\eta = [-R, R] \subset \mathbf{R}$ for some $R > 0$ such that $\int_{\mathbf{R}\setminus K_\eta} |\widehat{\phi}(\xi)|^p \, d\xi < \eta$. Furthermore, by Remark 8.4 there exist $r, N \in \mathbf{N}$ and $\nu = \sum_{j=1}^{2^r} b_j\delta_{N(j)} \in M(\mathbf{R})$ such that $\sum_{j=1}^{2^r} |b_j|^p = 1$, $\|\widehat{\nu}\|_{L^\infty} < \varepsilon$ and

$$\operatorname{supp}\nu = \{N(j) \in \mathbf{N} \mid j = 1,\dots,2^r\},$$

where the sets $\{-N(j) + K_\eta \mid N(j) \in \operatorname{supp}\nu\}$ are mutually disjoint, and $b_j = 2^{-r/p}$ or $b_j = -2^{-r/p}$.

Now we define f by $\widehat{f} = \mu * (\widehat{\nu}\phi)$.

Step 1. Firstly, we estimate $\|f\|_{M^{p,1}}$. Let $\varphi \in \mathcal{S}(\mathbf{R})$ be such that $\sum_{k\in\mathbf{Z}} \varphi(\xi - k) = 1$, $\operatorname{supp}\varphi \subset [-1, 1]$ and $\varphi(\xi) = 1$ on $[-1/10, 1/10]$. Since

$$\widehat{f}(\xi) = \Big(\Big(\sum_{j=1}^{2^m} a_j\delta_{\ell(j)}\Big) * (\widehat{\nu}\phi)\Big)(\xi) = \sum_{j=1}^{2^m} a_j(\widehat{\nu}\phi)(\xi - \ell(j)),$$

we have

$$\varphi(\xi - \ell(k))\widehat{f}(\xi) = a_k(\widehat{\nu}\phi)(\xi - \ell(k))$$

and

$$\varphi(D - k)f(x) = \frac{a_k}{2\pi}\int_{\mathbf{R}} (\widehat{\nu}\phi)(\xi - \ell(k))e^{ix\xi}\, d\xi = a_k e^{ix\ell(k)}\mathcal{F}^{-1}(\widehat{\nu}\phi)(x).$$

Therefore,

$$\|\varphi(D-k)f\|_{L^p} = |a_k|\,\|\mathcal{F}^{-1}(\widehat{\nu}\phi)\|_{L^p} = |a_k|\,\|\nu * (\mathcal{F}^{-1}\phi)\|_{L^p}.$$

On the other hand, since $\nu = \sum_{j=1}^{2^r} b_j \delta_{N(j)}$, we have

$$\|\nu * (\mathcal{F}^{-1}\phi)\|_{L^p} = \Big\| \sum_{j=1}^{2^r} b_j (\mathcal{F}^{-1}\phi)(\cdot - N(j)) \Big\|_{L^p}.$$

Moreover, since $\{-N(j) + K_\eta\}$ are mutually disjoint and

$$\Big| A + \sum_{j=1}^{2^r} B_j \Big|^p \geq \frac{1}{2^p}|A|^p - \Big| \sum_{j=1}^{2^r} B_j \Big|^p \geq \frac{1}{2^p}|A|^p - 2^{rp} \sum_{j=1}^{2^r} |B_j|^p,$$

we obtain

$$\begin{aligned}
&\Big| \sum_{j=1}^{2^r} b_j (\mathcal{F}^{-1}\phi)(x - N(j)) \Big|^p \\
&\quad = \Big| \sum_{j=1}^{2^r} b_j \big((\chi_{K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j)) + (\chi_{\mathbf{R}\setminus K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j) \big) \Big|^p \\
&\quad \geq \frac{1}{2^p} \sum_{j=1}^{2^r} |b_j|^p |(\chi_{K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j))|^p \\
&\qquad - 2^{rp} \sum_{j=1}^{2^r} |b_j|^p |(\chi_{\mathbf{R}\setminus K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j))|^p.
\end{aligned}$$

Since $\sum_{j=1}^{2^r} |b_j|^p = 1$, we have

$$\begin{aligned}
&\|\nu * (\mathcal{F}^{-1}\phi)\|_{L^p}^p \\
&\qquad \geq \sum_{j=1}^{2^r} |b_j|^p \Big(\frac{1}{2^p} \int_{\mathbf{R}} |(\chi_{K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j))|^p \, dx \\
&\qquad\qquad - 2^{rp} \int_{\mathbf{R}} |(\chi_{\mathbf{R}\setminus K_\eta} \cdot (\mathcal{F}^{-1}\phi))(x - N(j))|^p \, dx \Big) \\
&\qquad = \sum_{j=1}^{2^r} |b_j|^p \Big(\frac{1}{2^p} \int_{K_\eta} |\mathcal{F}^{-1}\phi(x)|^p \, dx - 2^{rp} \int_{\mathbf{R}\setminus K_\eta} |\mathcal{F}^{-1}\phi(x)|^p \, dx \Big) \\
&\qquad = \frac{1}{2^p} \|\mathcal{F}^{-1}\phi\|_{L^p}^p - (1 + 2^{rp})\eta.
\end{aligned}$$

Putting $\eta = \frac{1}{2^{p+1}(1+2^{rp})} \|\mathcal{F}^{-1}\phi\|_{L^p}^p$, we obtain

$$\|\nu * (\mathcal{F}^{-1}\phi)\|_{L^p} \geq \frac{1}{2^{1+1/p}} \|\mathcal{F}^{-1}\phi\|_{L^p}.$$

Hence from $\|\mu\|_{M(\mathbf{R})} = \sum_{k=1}^{2^m} |a_k| = 1$ we have

$$\|f\|_{M^{p,1}} = \sum_{k\in\mathbf{Z}} \|\phi(D-k)f\|_{L^p} \ge \frac{1}{2^{1+1/p}}\Big(\sum_{k=1}^{2^m} |a_k|\Big)\|\mathcal{F}^{-1}\phi\|_{L^1} = \frac{1}{2^{1+1/p}}\|\mathcal{F}^{-1}\phi\|_{L^1}.$$

STEP 2. Secondly, we show that $\|\widehat{f}\|_{L^1} \le \varepsilon\|\phi\|_{L^1}$. Since $\widehat{f} = \mu * (\widehat{\nu}\phi)$, $\|\widehat{\nu}\|_{L^\infty} < \varepsilon$ and $\mu = \sum_{j=1}^{2^m} a_j\delta_{\ell(j)}$ with $\|\mu\|_{M(\mathbf{R})} = \sum_{j=1}^{2^m} |a_j| = 1$, we find that

$$\|\widehat{f}\|_{L^1} \le \sum_{j=1}^{2^m} |a_j|\,\|(\widehat{\nu}\phi)(\cdot - \ell(j))\|_{L^1} \le \|\widehat{\nu}\|_{L^\infty}\|\phi\|_{L^1}\sum_{j=1}^{2^m} |a_j| < \varepsilon\|\phi\|_{L^1}.$$

STEP 3. Thirdly, we show that $\|f\|_{L^p} \le \varepsilon\|\mathcal{F}^{-1}\phi\|_{L^p}$. We note that

$$\begin{aligned} f(x) &= \mathcal{F}^{-1}(\mu * (\widehat{\nu}\phi))(x) = \mathcal{F}^{-1}\Big(\sum_{j=1}^{2^m} a_j(\widehat{\nu}\phi)(\cdot - \ell(j))\Big)(x) \\ &= \frac{1}{2\pi}\sum_{j=1}^{2^m} a_j\Big(\int_{\mathbf{R}} \widehat{\nu}(\xi - \ell(j))\phi(\xi - \ell(j))e^{ix\xi}\,d\xi\Big) \\ &= \sum_{j=1}^{2^m} a_j e^{ix\ell(j)}\mathcal{F}^{-1}(\widehat{\nu}\phi)(x) = (\widehat{\nu}(-x))((\mu * (\mathcal{F}^{-1}\phi))(x)). \end{aligned}$$

Therefore,

$$\|f\|_{L^p} \le \|\widehat{\nu}\|_{L^\infty}\|\mu * (\mathcal{F}^{-1}\phi)\|_{L^p} \le \|\widehat{\nu}\|_{L^\infty}\|\nu\|_{M(\mathbf{R})}\|\mathcal{F}^{-1}\phi\|_{L^p} \le \varepsilon\|\mathcal{F}^{-1}\phi\|_{L^p}.$$

STEP 4. We note that if $M^{p,1}(\mathbf{R}) = \mathcal{F}A_p(\mathbf{R})$ set-theoretically, then the closed graph theorem implies that the norms on both spaces are equivalent. Therefore

$$\begin{aligned} \frac{1}{2^{1+1/p}}\|\mathcal{F}^{-1}\phi\|_{L^1} \le \|f\|_{M^{p,1}} &\lesssim \|f\|_{\mathcal{F}A_p(\mathbf{R})} \\ &= \|f\|_{L^p} + \|\widehat{f}\|_{L^1} \le \varepsilon(\|\phi\|_{L^p} + \|\mathcal{F}^{-1}\phi\|_{L^1}). \end{aligned}$$

Since $\phi \ne 0$, this is impossible. Hence we obtain the desired result.

(ii) Finally, we consider the case $p = 2$. Set $I_k = [k - 1/(\log k)^2,\ k + 1/(\log k)^2]$ ($k \in \mathbf{N}$) and define f by $\widehat{f}(\xi) = \sum_{k=e^{10}}^{\infty} \chi_{I_k}(\xi)/k$, where χ_E denotes the characteristic function of a set $E \subset \mathbf{R}$. Moreover, let $\varphi \in \mathcal{S}(\mathbf{R})$ be such that $\sum_{k\in\mathbf{Z}} \varphi(\xi - k) = 1$ ($\xi \in \mathbf{R}$) and $\varphi(\xi) = 1$ on $[-1/10, 1/10]$. Then

$$\|\widehat{f}\|_{L^1} \le \sum_{k\in\mathbf{Z}} \|\varphi(\cdot - k)\widehat{f}\|_{L^1} \le \sum_{k=e^{10}}^{\infty} \frac{2}{k(\log_e k)^2} < \infty$$

and

$$\|f\|_{L^2} \approx \|\widehat{f}\|_{L^2} = \Big(\sum_{k=e^{10}}^{\infty} \frac{|I_k|}{k^2} \Big)^{1/2} = \Big(\sum_{k=e^{10}}^{\infty} \frac{2}{k^2(\log_e k)^2} \Big)^{1/2} < \infty.$$

Therefore $f \in \mathcal{F}A_2(\mathbf{R})$. On the other hand, it follows from the Plancherel theorem that

$$\|f\|_{M^{2,1}} \approx \sum_{k \in \mathbf{Z}} \|\varphi(\cdot - k)\widehat{f}\|_{L^2} = \sum_{k=e^{10}}^{\infty} \frac{\|\chi_{I_k}\|_{L^2}}{k} = \sum_{k=e^{10}}^{\infty} \frac{2}{k(\log_e k)} = \infty.$$

Thus $f \notin M^{2,1}(\mathbf{R})$. Hence $f \in \mathcal{F}A_2(\mathbf{R}) \setminus M^{2,1}(\mathbf{R})$.

Acknowledgements. This work was supported by JSPS KAKENHI Grant Numbers 22K03328, 22K03331.

References

[1] Á. Bényi and K. A. Okoudjou, *Modulation Spaces with Applications to Pseudodifferential Operators and Nonlinear Schrödinger Equations*, Appl. Numer. Harmonic Anal., Birkhäuser, New York, 2020.

[2] D. G. Bhimani, *Composition operators on Wiener amalgam spaces*, Nagoya Math. J. 240 (2020), 257–274.

[3] D. G. Bhimani and P. K. Ratnakumar, *Functions operating on modulation spaces and nonlinear dispersive equations*, J. Funct. Anal. 270 (2016), 621–648.

[4] L. H. Brandenburg, *On identifying the maximal ideals in Banach algebras*, J. Math. Anal. Appl. 50 (1975), 489–510.

[5] J. T. Burnham, *Closed ideals in subalgebras of Banach algebras. I*, Proc. Amer. Math. Soc. 32 (1972), 551–555.

[6] E. Cordero and K. A. Okoudjou, *Dilation properties for weighted modulation spaces*, J. Funct. Spaces Appl. 2012, art. 145491, 29 pp.

[7] E. Cordero and L. Rodino, *Time-Frequency Analysis of Operators*, De Gruyter Stud. Math. 75, De Gruyter, Berlin, 2020.

[8] H. G. Feichtinger, *On a new Segal algebra*, Monatsh. Math. 92 (1981), 269–289.

[9] H. G. Feichtinger, *Modulation spaces on locally compact abelian groups*, in: Wavelets and their Applications (Chennai, 2003), Allied Publ., New Delhi, 2003, 99–140.

[10] H. G. Feichtinger, *Banach spaces of distributions of Wiener's type and interpolation*, in: Functional Analysis and Approximation (Oberwolfach, 1980), Internat. Ser. Numer. Math. 69, Birkhäuser Boston, Basel, 1981, 153–165.

[11] H. G. Feichtinger, *A new family of functional spaces on the Euclidean n-space*, in: Proc. Conf. Theory of Approximation of Functions (Kiev, 1983), Teor. Priblizh. Funktsiĭ 1983, 493–497.

[12] H. G. Feichtinger, *Banach convolution algebras of Wiener's type*, in: Functions, Series, Operators (Budapest, 1980), Vol. I, Colloq. Math. Soc. János Bolyai 35, North-Holland, 1983, 509–524.

[13] H. G. Feichtinger, *Generalized amalgams, with applications to Fourier transform*, Canad. J. Math. 42 (1990), 395–409.

[14] H. G. Feichtinger, M. Kobayashi and E. Sato, *Further study of modulation spaces as Banach algebras*, Ann. Univ. Sci. Budapest. Sect. Comput. 56 (2024), 151–166.

[15] H. G. Feichtinger and T. Strohmer, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmonic Anal., Birkhäuser Boston, Boston, MA, 1998.

[16] J. E. Gilbert, *On a strong form of spectral synthesis*, Ark. Mat. 7 (1969) , 571–575.

[17] P. Gröbner, *Banachräume glatter Funktionen und Zerlegungsmethoden*, Ph.D. thesis, Univ. Wien, 1992.

[18] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Appl. Numer. Harmonic Anal., Birkhäuser Boston, Boston, MA, 2001.

[19] W. Guo, D. Fan, H. Wu and G. Zhao, *Sharp weighted convolution inequalities and some applications*, Studia Math. 241 (2018), 201–239.

[20] J.-P. Kahane, *Séries de Fourier Absolument Convergentes*, Ergeb. Math. Grenzgeb. 50, Springer, Berlin, 1970.

[21] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge Math. Library, Cambridge Univ. Press, 2004.

[22] M. Kobayashi and E. Sato, *Operating functions on modulation and Wiener amalgam spaces*, Nagoya Math. J. 230 (2018), 72–82.

[23] M. Kobayashi and E. Sato, *Operating functions on* $A_s^q(\mathbf{T})$, J. Fourier Anal. Appl. 28 (2022), art. 42, 28 pp.

[24] M. Kobayashi and E. Sato, *A note on operating functions of modulation spaces*, J. Pseudo-Differ. Oper. Appl. 13 (2022), art. 61, 17 pp.

[25] H.-C. Lai, *On some properties of* $A^p(G)$*-algebras*, Proc. Japan Acad. 45 (1969), 572–576.

[26] R. Larsen, *The algebras of functions with Fourier transforms in* L^p*: a survey*, Nieuw Arch. Wisk. (3) 22 (1974), 195–240.

[27] P. Lévy, *Sur la convergence absolue des séries de Fourier*, Compos. Math. 1 (1935), 1–14.

[28] V. Losert, *A characterization of the minimal strongly character invariant Segal algebra*, Ann. Inst. Fourier (Grenoble) 30 (1980), no. 3, 129–139.

[29] Y. Lu, *Sharp embedding between Besov–Triebel–Sobolev spaces and modulation spaces*, arXiv:2211.05336, 2022.

[30] K. A. Okoudjou, *A Beurling–Helson type theorem for modulation spaces.* J. Funct. Spaces Appl. 7 (2009), 33–41.

[31] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, Oxford, 1968.

[32] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups*, 2nd ed., London Math Soc. Monogr. New Ser. 22, Clarendon Press, 2000.

[33] W. Rudin, *Fourier Analysis on Groups*, Interscience Tracts Pure Appl. Math. 12, Wiley, New York, 1962.

[34] J. D. Stegeman, *Wiener–Ditkin sets for certain Beurling algebras*, Monatsh. Math. 82 (1976), 337–340.

[35] N. Teofanov and J. Toft, *An excursion to multiplications and convolutions on modulation spaces*, in: Operator and Norm Inequalities and Related Topics, Trends Math., Birkhäuser, 2022, 601–637.

[36] J. Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I*, J. Funct. Anal. 207 (2004), 399–429.

[37] J. Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II*, Ann. Global Anal. Geom. 26 (2004), 73–106.

[38] B. Wang, Z. Huo, C. Hao and Z. Guo, *Harmonic Analysis Method for Nonlinear Evolution Equations. I*, World Sci., Hackensack, NJ, 2011.

[39] N. Wiener, *Tauberian theorems*, Ann. of Math. (2) 33 (1932), 1–100.

[40] L. Y. H. Yap, *Nonfactorization of functions in Banach subspaces of* $L^1(G)$, Proc. Amer. Math. Soc. 51 (1975), 356–358.

Hans G. Feichtinger
Faculty of Mathematics
University of Vienna
A-1090 Wien, Austria
and
Acoustic Research Institute
Austrian Academy of Sciences
E-mail: hans.feichtinger@univie.ac.at

Masaharu Kobayashi
Department of Mathematics
Hokkaido University
Sapporo, Hokkaido 060-0810, Japan
E-mail: m-kobayashi@math.sci.hokudai.ac.jp

Enji Sato
Faculty of Science
Yamagata University
Yamagata-City, Yamagata 990-8560, Japan
E-mail: esato@sci.kj.yamagata-u.ac.jp