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SEMILOCAL CONVERGENCE ANALYSIS FOR A THREE-STEP SCHEME IN BANACH SPACES WITH A NEW TYPE MAJORANT: USING AVERAGE LIPSCHITZ CONDITIONS

Abstract. We analyze the semilocal convergence (S.C.) of the Newton-Traub scheme (NTS) used for finding solutions of nonlinear problems in Banach spaces. The analysis is based on the assumption that a generalized Lipschitz condition is satisfied by the first derivative of the relevant operator. The analysis establishes the fifth-order convergence of the NTS under an additional condition. Furthermore, we consider two special cases. The findings contribute to the theoretical understanding of NTS in Banach spaces and have practical applications, for example to integral equations.

1. Introduction. Consider an operator T that maps a nonempty open convex subset $\mathbb{D}$ of a Banach space U to another Banach space V. The aim is to obtain an approximation of a locally unique solution ξ^* to the nonlinear equation

$$T(g) = 0. \tag{1.1}$$

The numerical analysis of solutions of (1.1) is closely tied to variations of the Newton method, such as

$$g_{n+1} = g_n - [T'(g_n)]^{-1}T(g_n), \quad n \geq 0. \tag{1.2}$$

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Despite its relatively slow convergence, the Newton method is often favored and utilized. To gain further insight into this method, the interested readers can refer to Ortega's survey [15] as well as the literature cited by Rall [16] and Kantorovich [9].

In a significant number of contributions, researchers have analyzed the local convergence of iterative methods under Lipschitz, Hölder, or w-continuity conditions. However, some nonlinear problems fail to meet any of these conditions. To address this limitation, Wang [20] introduced the concept of generalized Lipschitz condition for analyzing the local convergence of Newton's method. Saxena et al. [17] realized that the earlier definition cannot be applied as is for multi-step Newton-type methods and therefore they presented a modified definition of generalized Lipschitz conditions.

Wang [21] introduced a Lipschitz condition with $\varkappa$-average, which is useful in investigating the Kantorovich-like convergence theorem. This condition has been used to study the semilocal convergence of various iterative methods, including the Gauss–Newton method as demonstrated by Xu and Li [22]. Further extensions of this concept have been discussed in [4, 5, 7].

We shall study the semilocal convergence of use classical Newton–Traub scheme (NTS) [3]

$$
\begin{aligned}
h_n &= g_n - [T'(g_n)]^{-1}T(g_n), \\
z_n &= h_n - [T'(g_n)]^{-1}T(h_n), \\
g_{n+1} &= z_n - [T'(h_n)]^{-1}T(z_n), \qquad n \geq 0,
\end{aligned}
\tag{1.3}
$$

under the $\varkappa$-average condition. The local convergence of the above iterative methods under generalized Lipschitz conditions has been thoroughly analyzed by Saxena et al. [18]. The majorizing function technique has found extensive use in analyzing the convergence of various Newton-type methods, including those for solving nonlinear operator equations [6, 5, 21], convex composite optimization using the Gauss–Newton method [11] and multiobjective optimization via the extended Newton method [19]. This analytical tool facilitates the establishment of more precise convergence criteria and convergence radii estimates for iterative methods.

Previous studies have explored semilocal convergence of iterative methods under different assumptions. Nguen and Zabrejko [23] used the majorizing function technique for the Newton method, while in 1995 Appell [1] established error estimates and semilocal convergence under the Lipschitz condition. Argyros [2] extended the analysis to include a generalization of the second Fréchet derivative of T. Ruiz and Argyros [14] relaxed the conditions and presented a new convergence analysis under Lipschitz and Lipschitz-like criteria for the first Fréchet derivative of T. George [3] studied the local convergence of multiple Newton-like methods, while Ling [12] presented a new semilocal convergence analysis of the Newton method which is two-step using a generalized Lipschitz condition for the first derivative of the operator, resulting in Q-cubic convergence.

This paper aims to demonstrate a general semilocal convergence result for the Newton–Traub scheme (1.3) by considering the first derivative of the nonlinear operator T under a generalized Lipschitz condition proposed by Wang [21]. This convergence analysis introduces several novel aspects, including clarifying the relationships between the majorizing function $\mathcal{H}$ and the nonlinear operator T and demonstrating that the Newton–Traub scheme exhibits fifth-order convergence under a slightly stronger condition. Several special cases, including Kantorovich-type and Y-type convergence results, are derived. The convergence result is supported by a numerical experiment and the use of the $\varkappa$-average Lipschitz condition is highlighted as a key advantage.

The paper is structured as follows: The generalized/average Lipschitz condition is introduced in Section 2. After some preliminary notions and properties of majorizing functions and majorizing sequences are reviewed in Section 3, the semilocal convergence analysis of the NTS is proven under the $\varkappa$-average Lipschitz condition in Section 4. An application to a nonlinear integral equation is given in Section 5. The final remarks are presented in Section 6.

2. Preliminaries. Making the paper as self-contained as possible, we present some essential concepts and notations taken from [12, 21]. The section concludes with the concepts of generalized/average Lipschitz condition.

Definition 2.1. Suppose $g_0 \in \mathbb{D}$ with $[T'(g_0)]^{-1}$ nonsingular, and let $\epsilon > 0$ with $M(g_0, \epsilon) \subseteq \mathbb{D}$ where $M(g_0, \epsilon)$ is the open ball with radius ϵ and center g_0. We suppose that $\varkappa(\cdot)$ is a positive nondecreasing function on $[0, \varrho)$. Then the *$\varkappa$-average Lipschitz condition* is satisfied by T' if on $M(g_0, \epsilon)$, for any $g, h \in M(g_0, \epsilon)$ with $\|g - g_0\| + \|h - g\| < \epsilon$,

$$\|[T'(g_0)]^{-1}(T'(h) - T'(g))\| \leq \int_{\|g-g_0\|}^{\|g-g_0\|+\|h-g\|} \varkappa(u)\, du. \tag{2.1}$$

In [21], Wang introduced this generalized Lipschitz condition, also known as the center Lipschitz condition in the inscribed sphere with $\varkappa$-average. Li et al. [10] examined the convergence behavior of the Gauss–Newton method for singular systems of equations, and Li and Ng [11] explored the same for convex composite optimization. They presented various adapted versions to study the convergence behavior. Evidently, the classical Lipschitz condition with Lipschitz constant $\varkappa(\epsilon)$ is implied by the $\varkappa$-average Lipschitz condition on $M(g_0, \epsilon)$.

3. Newton–Traub scheme (1.3) applied to the majorizing function. We suppose that $\varrho > 0$ satisfies the relation

$$\frac{\int_0^{\varrho} \varkappa(u)(\varrho - u)\,du}{\varrho} = 1. \tag{3.1}$$

The majorizing function $\mathcal{H} : [0, \varrho] \to \mathbb{R}$ is defined as

$$\mathcal{H}(k) = \beta - k + \int_0^k \varkappa(u)(k-u)\,du, \quad k \in [0, \varrho], \tag{3.2}$$

where β is a fixed constant. In the early 2000s, an important work of Wang [21] determined the semilocal convergence of the Newton method (1.2). It can be seen that this function is similar to the one used by Ferreira [6]. The intriguing reason for using this function is that it may yield a more precise convergence criterion and the error estimate for the three-step classical approach. We evidently have

$$\mathcal{H}'(k) = -1 + \int_0^k \varkappa(u)\,du, \quad k \in [0, \varrho),$$

with

$$\mathcal{H}''(k) = \varkappa(k) > 0 \quad \text{for a.e. } k \in [0, \varrho).$$

Thus

$$\int_j^k \varkappa(u)\,du = \mathcal{H}'(k) - \mathcal{H}'(j) \quad \text{for any } j, k \in [0, \varrho) \text{ with } j < k.$$

The connection between the majorizing function and $\varkappa$ presented here will be utilized often in the convergence analysis of the Newton–Traub scheme (1.3). Suppose ρ_0 satisfies

$$\int_0^{\rho_0} \varkappa(u)\,du = 1. \tag{3.3}$$

As a result, $\mathcal{H}(k)$ is strictly convex, $\mathcal{H}'(k)$ is increasing, convex and $-1 \leq \mathcal{H}'(k) < 0$ for any $k \in [0, \rho_0)$. In this section, we will begin by discussing some error estimates for the majorizing sequences v_i, s_i and k_i given by (3.7) below. Additionally, we will explore the relationship between the majorizing function $\mathcal{H}(k)$, as defined by (3.2), and the nonlinear operator T. We will then move to the convergence analysis of the NTS (1.3) under the $\varkappa$-average Lipschitz condition.

3.1. Intermediate results. The following auxiliary conclusion on scalar valued functions is recalled from any elementary convex analysis literature (see [8, Theorem 4.1.1 and Remark 4.1.2, p. 21]). These findings are significant and will be incorporated into our analysis.

LEMMA 3.1. *Assume $G : (0, \varrho) \to \mathbb{R}$ is a continuously differentiable and convex function, where $\varrho > 0$, and let $0 \le \phi \le 1$. Then*

(i) $(1-\phi)G'(\phi k) \le \dfrac{G(k) - G(\phi k)}{k} \le (1-\phi)G'(k), \quad \forall k \in (0, \varrho).$

(ii) $\dfrac{G(l) - G(\phi l)}{l} \le \dfrac{G(m) - G(\phi m)}{m}, \quad \forall l, m \in (0, \varrho),\ l < m.$

Furthermore, if G is strictly convex, then the inequalities above are strict.

Also, define

$$B := \int_0^{\rho_0} \varkappa(u)u\,du, \tag{3.4}$$

where ρ_0 is defined by (3.3). The preceding lemma is adapted from [21, Lemma 1.2] and provides some fundamental characteristics for the majorizing function $\mathcal{H}$ given by (3.2).

LEMMA 3.2 ([21]). *If $0 < \beta < B$, then $\mathcal{H}$ is decreasing on $[0, \rho_0]$ and increasing on $[\rho_0, \varrho]$ and*

$$\mathcal{H}(\beta) > 0, \quad \mathcal{H}(\rho_0) = \beta - B < 0, \quad \mathcal{H}(\varrho) = \beta > 0. \tag{3.5}$$

In addition, $\mathcal{H}$ has a unique zero in each of the two intervals, represented by ι^* and ι^{**}. They satisfy

$$\beta < \iota^* < \frac{\rho_0}{B}\beta < \rho_0 < \iota^{**} < \varrho. \tag{3.6}$$

Set the initial point for the sequence generated by iterating (3.7) to be $k_0 = 0$. Let s_i, v_i and k_i indicate the corresponding sequences generated by the Newton–Traub scheme for the majorizing function $\mathcal{H}$ provided in [12]:

$$\begin{cases} s_i = k_i - \frac{\mathcal{H}(k_i)}{\mathcal{H}'(k_i)}, \\ v_i = s_i - \frac{\mathcal{H}(s_i)}{\mathcal{H}'(k_i)}, \\ k_{i+1} = v_i - \frac{\mathcal{H}(v_i)}{\mathcal{H}'(s_i)}. \end{cases} \qquad i = 0, 1, \ldots. \tag{3.7}$$

NOTE. Suppose that $0 < \beta \le B$. Using Lemmas 3.1 and 3.2, as well as the usual analytical methods (see, for example, [13]), it is easy to demonstrate that the sequences s_i, v_i and k_i formed by (3.7) satisfy

$$0 \le k_i < s_i < v_i < k_{i+1} < \iota^*, \quad \forall i \ge 0, \tag{3.8}$$

and increasingly converge to the same point ι^*, which is the unique zero of $\mathcal{H}$ on $[0, \rho_0]$, and ρ_0 is provided by (3.3). Furthermore, we obtain

$$\iota^* - k_{i+1} \le \frac{1}{4}\frac{\mathcal{H}''(\iota^*)^4}{\mathcal{H}'(\iota^*)^3 H'(s_i)}(\iota^* - k_i)^5, \quad i \ge 0,$$

or

$$\iota^* - k_{i+1} \le \frac{1}{4}\frac{\mathcal{H}''(\iota^*)^4}{\mathcal{H}'(\iota^*)^4}(\iota^* - k_i)^5, \qquad i \ge 0. \tag{3.9}$$

In particular, if $1 + \iota^*\mathcal{H}''(\iota^*)/\mathcal{H}'(\iota^*) \ge 0$, we have

$$v_i - s_i \ge (\iota^* - s_i) + \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)(\iota^* - s_i), \quad i \ge 0. \tag{3.10}$$

The convergence features of the sequences v_i, s_i and k_i discussed above will be applied in the convergence analysis for the Newton–Traub scheme (1.3).

Let $g_0 \in \mathbb{D}$ be an initial guess such that the inverse $[T'(g_0)]^{-1}$ exists and let $M(g_0, \rho_0) \subset \mathbb{D}$, where ρ_0 satisfies (3.3). Let

$$\beta := \|[T'(g_0)]^{-1}T(g_0)\|. \tag{3.11}$$

Recall that (3.2) defines the majorizing function $\mathcal{H}$, (3.4) defines B, and ι^* and ι^{**} are the unique zeros of $\mathcal{H}$ on $[0, \rho_0]$ and $[\rho_0, \varrho]$, respectively, where ρ_0 and ϱ satisfy (3.3) and (3.1). Note that when $0 < \beta \le B$, the sequences v_i, s_i and k_i given by (3.7) converge to ι^*.

The succeeding lemmas, which establish explicit links between the majorizing function $\mathcal{H}$ and the nonlinear function T, will be crucial in the semilocal convergence analysis of the NTS (1.3).

LEMMA 3.3. *Assume that* $\|g-g_0\| \le k < \iota^*$. *Whenever the first derivative of* T *in* $M(\xi^*, k)$ *meets the* $\varkappa$*-average Lipschitz condition* (2.1), *then* $T'(g)$ *is nonsingular and*

$$\|[T'(g)]^{-1}T'(g_0)\| \le -\frac{1}{\mathcal{H}'(\|g - g_0\|)} \le -\frac{1}{\mathcal{H}'(k)}. \tag{3.12}$$

T *is, in addition, nonsingular in* $M(g_0, \iota^*)$.

Proof. Consider $g \in \overline{M(g_0, k)}$, $0 \le k < \iota^*$. The $\varkappa$-average Lipschitz condition (2.1) yields

$$\|[T'(g_0)]^{-1}T'(g) - I\| \le \int_0^{\|g-g_0\|} \varkappa(u)\, du = \mathcal{H}'(\|g - g_0\|) - \mathcal{H}'(0),$$

where I is the identity operator. Since $\mathcal{H}'(0) = 1$ and $\mathcal{H}$ strictly increases in $(0, \iota^*)$, we get

$$\|[T'(g_0)]^{-1}T'(g) - I\| \le \mathcal{H}'(g) + 1 < 1,$$

because $-1 < \mathcal{H}'(g) < 0$ for any $g \in (0, \iota^*)$. As a result, the Banach lemma can be used to conclude that $[T'(g_0)]^{-1}T'(g)$ is nonsingular and (3.12) holds. ∎

LEMMA 3.4. *Let* v_i, s_i *and* k_i *be generated by the scheme* (3.7). *Assume* T' *in* $M(g_0, \iota^*)$ *meets the* $\varkappa$*-average Lipschitz condition* (2.1). *If* $0 < \beta \le B$, *the sequences g_i, h_i and z_i generated using the three-step NTS* (1.3) *with the initial guess g_0 are well defined and contained in $M(g_0,k)$. Furthermore, for all $i=0,1,2,\dots$ we have*

(i) $[T'(g_i)]^{-1}$ *exists with* $\|[T'(g_i)]^{-1}T'(g_0)\| \le -1/\mathcal{H}'(\|g_i-g_0\|) \le -1/\mathcal{H}'(k)$,

(ii) $\|[T'(g_0)]^{-1}T(g_i)\| \le \mathcal{H}(g_i)$,

(iii) $\|h_i-g_i\| \le s_i-k_i$,

(iv) $\|z_i-h_i\| \le (v_i-s_i)\Big(\dfrac{\|h_i-g_i\|}{s_i-k_i}\Big)^2 \le v_i-s_i$,

(v) $\|z_i-g_i\| \le v_i-k_i$,

(vi) $\|g_{i+1}-z_i\| \le (k_{i+1}-v_i)\dfrac{\|z_i-h_i\|}{v_i-s_i}\,\dfrac{\|h_i-g_i\|+\tau\|z_i-h_i\|}{s_i-k_i+\tau(v_i-s_i)} \le k_{i+1}-v_i$,

(vii) $\|g_{i+1}-g_i\| \le k_{i+1}-k_i$.

Proof. We use induction. Obviously, the case $i=0$ holds true for (i)–(iv). As a result, $h_0\in M(g_0,\iota^*)$ because $\|h_0-g_0\| \le s_0-k_0=s_0<\iota^*$ and $\|z_0-h_0\| \le v_0-s_0$. As for (iv)–(vii), by (1.3), we can write

$$\begin{aligned}T(h_0) &= T(h_0)-T(h_0)-T'(h_0)(h_0-g_0)\\ &= \int_0^1 [T'(g_0+\tau(h_0-g_0))-T'(g_0)](h_0-g_0)\,d\tau.\end{aligned}$$

The $\varkappa$-average Lipschitz condition (2.1) is then used to deduce that

$$\begin{aligned}\|[T'(g_0)]^{-1}T(h_0)\| &\le \int_0^1 \|[T'(g_0)]^{-1}[T'(g_0+\tau(h_0-g_0))-T'(g_0)]\|\,\|h_0-g_0\|\,d\tau\\ &\le \int_0^1\Big(\int_0^{\tau\|h_0-g_0\|}\varkappa(u)\,du\Big)\|h_0-g_0\|\,d\tau\\ &= \int_0^1 [\mathcal{H}'(\tau\|h_0-g_0\|)-\mathcal{H}'(0)]\|h_0-g_0\|\,d\tau.\end{aligned}$$

As $\mathcal{H}'$ is strictly convex in $[0,\rho_0)$ and in addition $\|h_0-g_0\| \le s_0-k_0$ by (iii), Lemma 3.1 states that

$$\begin{aligned}\mathcal{H}'(\tau\|h_0-g_0\|)-\mathcal{H}'(0) &= \frac{\mathcal{H}'(\tau\|h_0-g_0\|)-\mathcal{H}'(0)}{\|h_0-g_0\|}\|h_0-g_0\|\\ &\le \frac{\mathcal{H}'(\tau(s_0-k_0))-\mathcal{H}'(0)}{s_0-k_0}\|h_0-g_0\|.\end{aligned}$$

By combining the above inequality and scheme (3.7), one obtains

$$\|[T'(g_0)]^{-1}T(h_0)\| \leq \int_0^1 [\mathcal{H}'(\tau s_0) - \mathcal{H}'(0)]s_0\, d\tau \left(\frac{\|h_0 - g_0\|}{s_0 - k_0}\right)^2$$
$$= \mathcal{H}(s_0)\left(\frac{\|h_0 - g_0\|}{s_0 - k_0}\right)^2 = (k_1 - s_0)\left(\frac{\|h_0 - g_0\|}{s_0 - k_0}\right)^2.$$

As a result,

$$\|z_0 - h_0\| = \|[T'(g_0)]^{-1}T(h_0)\| \leq (v_0 - s_0)\left(\frac{\|h_0 - g_0\|}{s_0 - k_0}\right)^2.$$

Through these two results, for (v) we have $\|z_0 - g_0\| \leq \|z_0 - h_0 + h_0 - g_0\| \leq v_0 - s_0 + s_0 - k_0 \leq v_0 - k_0$. Similarly for the next relation, we can see that

$$T(z_0) = T(z_0) - T(h_0) - T'(g_0)(z_0 - h_0)$$
$$= \int_0^1 [T'(h_0 + \tau(z_0 - h_0)) - T'(g_0)](z_0 - h_0)\, d\tau.$$

With the help of the $\varkappa$-average Lipschitz condition (2.1), we get

$$\|[T'(h_0)]^{-1}T(z_0)\|$$
$$\leq \int_0^1 \|[T'(h_0)]^{-1}[T'(h_0 + \tau(z_0 - h_0)) - T'(g_0)]\|\, \|z_0 - h_0\|\, d\tau$$
$$\leq \|[T'(h_0)]^{-1}T'(g_0)\| \int_0^1 \Big(\int_0^{\|h_0 - g_0\| + \tau\|z_0 - h_0\|} \varkappa(u)\, du\Big)\|z_0 - h_0\|\, d\tau$$
$$= \frac{-1}{\mathcal{H}'(\|h_0 - g_0\|)} \int_0^1 [\mathcal{H}'(\|h_0 - g_0\| + \tau\|z_0 - h_0\|) - \mathcal{H}'(0)]\|z_0 - h_0\|\, d\tau.$$

As $\mathcal{H}'$ is strictly convex in $[0, \rho_0]$ and $\|z_0 - h_0\| \leq v_0 - s_0$ by (iv), Lemma 3.1 states that

$$\mathcal{H}'(\|h_0 - g_0\| + \tau\|z_0 - h_0\|) - \mathcal{H}'(0)$$
$$= \frac{\mathcal{H}'(\|h_0 - g_0\| + \tau\|z_0 - h_0\|) - \mathcal{H}'(0)}{\|h_0 - g_0\| + \tau\|z_0 - h_0\|}\|h_0 - g_0\| + \tau\|z_0 - h_0\|$$
$$\leq \frac{\mathcal{H}'((s_0 - k_0) + \tau(v_0 - s_0)) - \mathcal{H}'(0)}{(s_0 - k_0) + \tau(v_0 - s_0)}\|h_0 - g_0\| + \tau\|z_0 - h_0\|.$$

By combining the above inequality and (3.7), one obtains

$$\|[T'(h_0)]^{-1}T(z_0)\|$$
$$\leq \frac{-1}{\mathcal{H}'(s_0)} \int_0^1 \frac{\mathcal{H}'((s_0 - k_0) + \tau(v_0 - s_0)) - \mathcal{H}'(0)}{(s_0 - k_0) + \tau(v_0 - s_0)}\|z_0 - h_0\|(\|h_0 - g_0\| + \tau\|z_0 - h_0\|)\, d\tau$$

$$= \frac{-\mathcal{H}(v_0)}{\mathcal{H}(s_0)} \frac{\|z_0 - h_0\|}{v_0 - s_0} \frac{\|h_0 - g_0 + \tau\|z_0 - h_0\|}{s_0 - k_0 + \tau(v_0 - s_0)}$$
$$= (k_1 - v_0) \frac{\|z_0 - h_0\|}{v_0 - s_0} \frac{\|h_0 - g_0 + \tau\|z_0 - h_0\|}{s_0 - k_0 + \tau(v_0 - s_0)}.$$

As a result,

$$\begin{aligned}\|g_1 - z_0\| &= \|[T'(h_0)]^{-1} T(z_0)\| \\ &\leq (k_1 - v_0) \frac{\|z_0 - h_0\|}{v_0 - s_0} \frac{\|h_0 - g_align0 + \tau\|z_0 - h_0\|}{s_0 - k_0 + \tau(v_0 - s_0)}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\|g_1 - g_0\| &\leq \|g_1 - z_0\| + \|z_0 - h_0\| + \|h_0 - g_0\| \\ &\leq (k_1 - v_0) + (v_0 - s_0) + (s_0 - k_0) = k_1 - k_0.\end{aligned}$$

That is, (v)–(vii) hold for $i = 0$, implying that $g_1 \in M(g_0, \iota^*)$.

We now suppose that $g_i, h_i, z_i \in M(g_0, \iota^*)$, $\|g_1 - g_0\| \leq k_i$, and (i)–(vii) are valid for some $i \geq 0$. Then, using the inductive hypothesis (iii), we get $\|h_i - g_0\| \leq \|h_i - g_i\| + \|g_i - g_0\| \leq s_i$. Furthermore, we employ the inductive hypothesis (vii) and (3.8) to produce

$$\|g_{i+1} - g_0\| \leq \sum_{k=0}^{i} \|g_{i+1} - g_i\| \leq \sum_{k=0}^{i} (k_{i+1} - k_i) = k_{i+1} < \iota^*.$$

This means that $g_{i+1} \in M(g_0, \iota^*)$. Together with (3.12), this implies that (i) holds for the case $i + 1$. By (1.3), the following identity can be obtained for (ii):

$$\begin{aligned}T(g_{i+1}) &= T(g_{i+1}) - T(z_i) - T'(h_i)(g_{i+1} - z_i) \\ &= \int_0^1 [T'(h_i + \tau(g_{i+1} - h_i)) - T'(g_i)](g_{i+1} - h_i)\, d\tau.\end{aligned}$$

According to the $\varkappa$-average Lipschitz condition (2.1),

$$\begin{aligned}&\|[T'(g_0)]^{-1} T(g_{i+1})\| \\ &\quad \leq \int_0^1 \|[T'(g_0)]^{-1} [T'(z_i + \tau(g_{i+1} - z_i)) - T'(h_i)]\| \, \|g_{i+1} - z_i\|\, d\tau \\ &\quad \leq \int_0^1 \Big(\int_{\|h_i - g_0\|}^{\|h_i - g_0\| + \|z_i - h_i + \tau(g_{i+1} - z_i)\|} \varkappa(u)\, du \Big) \|g_{i+1} - z_i\|\, d\tau.\end{aligned}$$

As $\mathcal{H}'$ is convex and increasing in $[0, \rho_0]$, one can deduce from Lemma 3.1

and (3.8) that

$$\begin{aligned}\int\limits_{\|h_i-g_0\|}^{\|h_i-g_0\|+\|z_i-h_i+\tau(g_{i+1}-z_i)\|} & \varkappa(u)\,du \\ &= \mathcal{H}'(\|h_i-g_0\|+\|z_i-h_i+\tau(g_{i+1}-z_i)\|)-\mathcal{H}'(\|h_i-g_0\|) \\ &\le \mathcal{H}'(\|h_i-g_0\|+\|z_i-h_i\|+\tau\|g_{i+1}-z_i\|)-\mathcal{H}'(\|h_i-g_0\|) \\ &\le \frac{\mathcal{H}'(v_i-\tau(k_{i+1}-v_i))-\mathcal{H}'(s_i)}{v_i-s_i+\tau(k_{i+1}-v_i)}(\|z_i-h_i\|+\tau\|g_{i+1}-z_i\|) \\ &\le \mathcal{H}'(v_i+\tau(k_{i+1}-v_i))-\mathcal{H}'(s_i).\end{aligned}$$

This enables us to obtain

$$\begin{aligned}(3.13)\qquad \|[T'(g_0)]^{-1}T(g_{i+1})\| \\ &\le \int\limits_0^1 [\mathcal{H}'(v_i+\tau(k_{i+1}-v_i))-\mathcal{H}'(s_i)]\|g_{i+1}-z_i\|\,d\tau \\ &= \mathcal{H}(k_{i+1})-\mathcal{H}(v_i)-\mathcal{H}'(s_i)(k_{i+1}-v_i)\frac{\|g_{i+1}-z_i\|}{k_{i+1}-v_i} \\ &= \mathcal{H}(k_{i+1}),\end{aligned}$$

which demonstrates that (ii) holds true for $i+1$. By combining (3.12) and (3.13), we get

$$\begin{aligned}(3.14)\qquad \|h_{i+1}-g_{i+1}\| &= \|[T'(g_{i+1})]^{-1}T(g_{i+1})\| \\ &\le \|[T'(g_{i+1})]^{-1}T'(g_0)\|\,\|[T'(g_0)]^{-1}T(g_{i+1})\| \\ &\le -\frac{\mathcal{H}(k_{i+1})}{\mathcal{H}'(k_{i+1})} = s_{i+1}-k_{i+1}.\end{aligned}$$

This implies that (iii) is true for $i+1$. Thus we reach the conclusion that $\|h_{i+1}-g_0\| \le \|h_{i+1}-g_{i+1}\|+\|g_{i+1}-g_0\| \le s_{i+1} < \iota^*$ and hence $h_{i+1} \in M(g_0,\iota^*)$. Regarding (iv), it is also worth noticing that

$$\begin{aligned}T(h_{i+1}) &= T(h_{i+1})-T(g_{i+1})-T'(g_{i+1})(h_{i+1}-g_{i+1}) \\ &= \int\limits_0^1 [T'(g_{i+1}+\tau(h_{i+1}-g_{i+1}))-T'(g_{i+1})](h_{i+1}-g_{i+1})\,d\tau.\end{aligned}$$

According to the $\varkappa$-average Lipschitz condition (2.1),

$$\begin{aligned}&\|[T'(g_{i+1})]^{-1}T(h_{i+1})\| \\ &\quad\le \frac{-1}{\mathcal{H}'(k_{i+1})}\int\limits_0^1 \Big(\int\limits_{\|g_{i+1}-g_0\|}^{\|g_{i+1}-g_0\|+\|\tau(h_{i+1}-g_{i+1})\|} \varkappa(u)\,du\Big)\|h_{i+1}-g_{i+1}\|\,d\tau.\end{aligned}$$

As $\mathcal{H}'$ is convex and increasing in $[0,\rho_0]$, one can deduce from Lemma 3.1 and (3.8) that

$$\begin{aligned}\int\limits_{\|g_{i+1}-g_0\|}^{\|g_{i+1}-g_0\|+\tau\|h_{i+1}-g_{i+1}\|} &\varkappa(u)\,du\\ &= \mathcal{H}'(\|g_{i+1}-g_0\|+\tau\|h_{i+1}-g_{i+1}\|)-\mathcal{H}'(\|g_{i+1}-g_0\|)\\ &\le \frac{\mathcal{H}'(k_{i+1}+\tau(s_{i+1}-k_{i+1}))-\mathcal{H}'(k_{i+1})}{s_{i+1}-k_{i+1}}\|h_{i+1}-g_{i+1}\|\\ &\le \mathcal{H}'(k_{i+1}+\tau(s_{i+1}-k_{i+1}))-\mathcal{H}'(k_{i+1}).\end{aligned}$$

This enables us to obtain

$$\begin{aligned}(3.15)\qquad &\|[T'(g_{i+1})]^{-1}T(h_{i+1})\|\\ &\le \frac{-1}{\mathcal{H}'(k_{i+1})}\int\limits_0^1[\mathcal{H}'(k_{i+1}+\tau(s_{i+1}-k_{i+1}))-\mathcal{H}'(k_{i+1})]\|h_{i+1}-g_{i+1}\|\,d\tau\\ &= \frac{-1}{\mathcal{H}'(k_{i+1})}[\mathcal{H}(s_{i+1})-\mathcal{H}(k_{i+1})-\mathcal{H}'(k_{i+1})(s_{i+1}-k_{i+1})]\frac{\|h_{i+1}-g_{i+1}\|^2}{(s_{i+1}-k_{i+1})^2}\\ &= \frac{-\mathcal{H}(s_{i+1})}{\mathcal{H}'(k_{i+1})}\,\frac{\|h_{i+1}-g_{i+1}\|^2}{(s_{i+1}-k_{i+1})^2},\end{aligned}$$

which leads to

$$(3.16)\qquad \begin{aligned}\|z_{i+1}-h_{i+1}\| &= \|[T'(g_{i+1})]^{-1}T(h_{i+1})\|\\ &\le \frac{-\mathcal{H}(s_{i+1})}{\mathcal{H}'(k_{i+1})}=v_{i+1}-s_{i+1}.\end{aligned}$$

This implies that (iv) is true for $i+1$. Through these two results, for (v) we have $\|z_{i+1}-g_{i+1}\|\le\|z_{i+1}-h_{i+1}+h_{i+1}-g_{i+1}\|\le v_{i+1}-s_{i+1}+s_{i+1}-k_{i+1}\le v_{i+1}-k_{i+1}$. Similarly for the next relation, we can see from (1.3) that

$$\begin{aligned}&g_{i+2}-z_{i+1}\\ &\quad= -[T'(h_{i+1})]^{-1}\int\limits_0^1[T'(h_{i+1}+\tau(z_{i+1}-h_{i+1}))-T'(g_{i+1})](z_{i+1}-h_{i+1})\,d\tau.\end{aligned}$$

With the help of the $\varkappa$-average Lipschitz condition (2.1),

$$\begin{aligned}&\|g_{i+2}-z_{i+1}\|\\ &\le \|[T'(h_{i+1})]^{-1}T'(g_0)\|\\ &\quad\times\int\limits_0^1\|[T'(g_0)]^{-1}[T'(h_{i+1}+\tau(z_{i+1}-h_{i+1}))-T'(g_{i+1})](z_{i+1}-h_{i+1})\|\,d\tau\end{aligned}$$

$$\leq \|[T'(h_0)]^{-1}T'(g_0)\| \times \int_0^1 \Big(\int_{\|g_{i+1}-g_0\|}^{\|g_{i+1}-g_0\|+\|h_{i+1}-g_{i+1}\|+\tau\|z_{i+1}-h_{i+1}\|} \varkappa(u)\,du \Big) \|z_{i+1}-h_{i+1}\|\,d\tau$$

$$= \frac{-1}{\mathcal{H}'(s_{i+1})} \int_0^1 [\mathcal{H}'(\|g_{i+1}-g_0\|+\|h_{i+1}-g_{i+1}\|+\tau\|z_{i+1}-h_{i+1}\|)-\mathcal{H}'(\|g_{i+1}-g_0\|)] \times \|z_{i+1}-h_{i+1}\|\,d\tau.$$

As $\mathcal{H}'$ is strictly convex in $[0,\rho_0]$ and $\|z_{i+1}-h_{i+1}\| \leq v_{i+1}-s_{i+1}$ by (iv), one obtains from Lemma 3.1

$$\begin{aligned} \|g_{i+2}-z_{i+1}\| &\leq \frac{-1}{\mathcal{H}'(s_{i+1})} \Big(\int_0^1 \frac{\mathcal{H}'(s_{i+1}+\tau(v_{i+1}-s_{i+1}))-\mathcal{H}'(k_{i+1})}{(s_{i+1}-k_{i+1})+\tau(v_{i+1}-s_{i+1})}\,d\tau \\ &\qquad \times \|z_{i+1}-h_{i+1}\|(\|h_{i+1}-g_{i+1}\|+\tau\|z_{i+1}-h_{i+1}\|)\Big) \\ &\leq \frac{-\mathcal{H}(v_{i+1})}{\mathcal{H}(s_{i+1})} \frac{\|z_{i+1}-h_{i+1}\|}{v_{i+1}-s_{i+1}} \frac{\|h_{i+1}-g_{i+1}+\tau\|z_{i+1}-h_{i+1}\|}{s_{i+1}-k_{i+1}+\tau(v_{i+1}-s_{i+1})} \\ &\leq (k_{i+2}-v_{i+1}) \frac{\|z_{i+1}-h_{i+1}\|}{v_{i+1}-s_{i+1}} \frac{\|h_{i+1}-g_{i+1}+\tau\|z_{i+1}-h_{i+1}\|}{s_{i+1}-k_{i+1}+\tau(v_{i+1}-s_{i+1})} \\ &\leq k_{i+2}-v_{i+1}, \end{aligned}$$

which proves (vi) for $i+1$. Furthermore, from (3.14) and (3.16) we derive

$$\begin{aligned} \|g_{i+2}-g_{i+1}\| &\leq \|g_{i+1}-z_{i+1}\|+\|z_{i+1}-h_{i+1}\|+\|h_{i+1}-g_{i+1}\| \\ &\leq (k_{i+2}-v_{i+1})+(v_{i+1}-s_{i+1})+(s_{i+1}-k_{i+1}) = k_{i+2}-k_{i+1}. \end{aligned}$$

As a result, all of the claims in the lemma are true by induction. ■

4. Main semilocal convergence result for (1.3). We are now prepared to demonstrate the semilocal convergence properties including convergence, uniqueness, and convergence rate of the Newton–Traub scheme (1.3) under the $\varkappa$-average Lipschitz condition (2.1). Furthermore, this main result yields two special cases: a Y-convergence result and, under the Lipschitz condition, a Kantorovich-type convergence result.

The following lemmas will play an important role for this purpose.

LEMMA 4.1. *Under the hypotheses of Lemma* 3.4, *the sequence* g_i *converges to a point* $\xi^* \in \overline{M(g_0,\iota^*)}$ *with* $T(\xi^*)=0$. *In addition,*

$$\|\xi^*-g_i\| \leq \iota^*-k_i, \quad i \geq 0, \tag{4.1}$$

$$\|\xi^* - h_i\| \le (\iota^* - s_i)\left(\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\right)^2, \qquad i \ge 0, \tag{4.2}$$

$$\|\xi^* - z_i\| \le (\iota^* - v_i)\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\,\frac{\|\xi^* - h_i\|}{\iota^* - s_i}, \qquad i \ge 0. \tag{4.3}$$

Proof. We use Lemma 3.4(vii) and (3.8) to prove that

$$\sum_{i=N}^{\infty} \|g_{i+1} - g_i\| \le \sum_{i=N}^{\infty} (k_{i+1} - k_i) = \iota^* - k_N < +\infty \quad \text{for any } N \in \mathbb{N}.$$

As a result, g_i is a Cauchy sequence in $M(g_0, \iota^*)$ and hence it converges to some $\xi^* \in \overline{M(g_0, \iota^*)}$. For each $i \ge 0$, the preceding inequality implies that $\|\xi^* - g_i\| \le \iota^* - k_i$.

We will demonstrate that $T(\xi^*) = 0$. Because of Lemma 3.3, we can see that $T(g_i)$ is bounded. Then, from Lemma 3.4, it follows that

$$\|T(g_i)\| \le \|T'(g_i)\|\,\|[T'(g_i)]^{-1}T(g_i)\| \le \|T'(g_i)\|(s_i - k_i).$$

Letting $i \to \infty$, and noting that s_i and k_i converge to the same point ι^* (see Note following (3.7)), we find that $\lim_{i\to\infty} T(g_i) = 0$. Because T is continuous in $\overline{M(g_0, \iota^*)}$, and $g_i \subset M(g_0, \iota^*)$ and g_i converges to ι^*, we have $\lim_{i\to\infty} T(g_i) = T(\xi^*)$, which confirms that $T(\xi^*) = 0$.

It remains to show the estimates (4.2) and (4.3). Due to Lemma 3.4, we have

$$\|h_i - g_0\| \le \|h_i - g_i\| + \|g_i - g_0\| \le s_i. \tag{4.4}$$

moreover, we can deduce the following identity:

$$\xi^* - h_i = -[T'(g_i)]^{-1}\int_0^1 [T'(g_i + \tau(\xi^* - g_i)) - T'(g_i)](\xi^* - g_i)\,d\tau.$$

Next, as $\mathcal{H}'$ is convex and increasing in $[0, \rho_0)$, one can combine (3.12), the $\varkappa$-average Lipschitz condition (2.1) and Lemma 3.1 to get

$$\begin{aligned}
\|\xi^* - h_i\| &\le \frac{-1}{\mathcal{H}'(k_i)}\int_0^1 \Big(\int_{\|g_i-g_0\|}^{\|g_i-g_0\|+\|\tau(\xi^*-g_i)\|} \varkappa(u)\,du\Big)\|\xi^* - g_i\|\,d\tau \\
&\le \frac{-1}{\mathcal{H}'(k_i)}\int_0^1 [\mathcal{H}'(\|g_i - g_0\| + \tau\|\xi^* - g_i\|) - \mathcal{H}'(\|g_i - g_0\|)]\|\xi^* - g_i\|\,d\tau \\
&\le \frac{-1}{\mathcal{H}'(k_i)}\int_0^1 \frac{\mathcal{H}'(k_i + \tau(\iota^* - k_i)) - \mathcal{H}'(k_i)}{\iota^* - k_i}\,d\tau\,\|\xi^* - g_i\|^2 \\
&\le (\iota^* - s_i)\frac{\|\xi^* - g_i\|^2}{(\iota^* - k_i)^2}.
\end{aligned}$$

As a result of Lemma 3.4, we deduce

$$\|z_i - g_0\| \leq \|z_i - h_i\| + \|h_i - g_0\| \leq v_i. \tag{4.5}$$

We can moreover draw the underlying identity:

$$\xi^* - z_i = -[T'(g_i)]^{-1} \int_0^1 [T'(h_i + \tau(\xi^* - h_i)) - T'(g_i)](\xi^* - h_i)\, d\tau.$$

As $\mathcal{H}'$ is convex and increasing in $[0, \rho_0)$, one may combine (3.12), Lemma 3.1 and the $\varkappa$-average Lipschitz condition (2.1) to obtain

$$\begin{aligned}
\|\xi^* - z_i\| &\leq \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 \Big(\int_{\|g_i - g_0\|}^{\|g_i-g_0\|+\|h_i-g_i+\tau(\xi^*-h_i)\|} \varkappa(u)\, du \Big) \|\xi^* - h_i\|\, d\tau \\
&= \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 \Big(\big[\mathcal{H}'\big(\|g_i - g_0\| + \|h_i - g_i + \tau(\xi^* - h_i)\|\big) \\
&\quad -\mathcal{H}'(\|g_i - g_0\|)\big] \|\xi^* - h_i\| \Big)\, d\tau \\
&\leq \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 \frac{\mathcal{H}'(s_i + \tau(\iota^* - s_i)) - \mathcal{H}'(k_i)}{\iota^* - k_i} \|\xi^* - g_i\|\, d\tau \|\xi^* - h_i\| \\
&\leq (\iota^* - v_i) \frac{\|\xi^* - g_i\|\, \|\xi^* - h_i\|}{(\iota^* - k_i)(\iota^* - s_i)},
\end{aligned}$$

as stated. ■

LEMMA 4.2. *Under the conditions of Lemma* 3.4 *and with the hypothesis that* $1 + \iota^* \mathcal{H}''(\iota^*)/\mathcal{H}'(\iota^*) > 0$, *we have*

$$\frac{\|z_i - h_i\|}{v_i - s_i} \leq \frac{1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)}{1 + \iota^* \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)} \, \frac{\|\xi^* - g_i\|^2}{(\iota^* - k_i)^2}. \tag{4.6}$$

Proof. We can deduce from (3.7) that

$$\iota^* - v_i = \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 [\mathcal{H}'(s_i + \tau(\iota^* - s_i)) - \mathcal{H}'(k_i)(\iota^* - s_i)]\, d\tau.$$

Considering $\mathcal{H}$'s convexity in $[0, \rho_0)$, Lemma 3.1 states that for any $\tau \in (0, 1]$,

$$\begin{aligned}
\mathcal{H}'(s_i + \tau(\iota^* - s_i\|) - \mathcal{H}'(k_i) &\leq \frac{\mathcal{H}'(\iota^*) - \mathcal{H}'(k_i)}{\iota^* - k_i}(s_i + \tau(\iota^* - s_i)) \\
&\leq \mathcal{H}''(\iota^*)(\iota^* - k_i).
\end{aligned}$$

Therefore, given the positivity of $1/\mathcal{H}'(k)$, one can deduce from Lemma 3.1

that

$$\iota^* - v_i \leq \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 \frac{\mathcal{H}'(\iota^*) - \mathcal{H}'(k_i)}{\iota^* - k_i} (\iota^* - k_i)(\iota^* - s_i)\, d\tau$$
$$= \frac{-\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)} (\iota^* - k_i)(\iota^* - s_i),$$

because $\mathcal{H}'$ is strictly increasing. Using this idea from the scheme (3.7) that

$$\iota^* - s_i = \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 [\mathcal{H}'(k_i + \tau(\iota^* - k_i)) - \mathcal{H}'(k_i)(\iota^* - k_i)]\, d\tau.$$

As $\mathcal{H}$ is convex in $[0, \rho_0)$, and $1/\mathcal{H}'(k) > 0$, one can deduce from Lemma 3.1 that

$$\iota^* - s_i \leq \frac{-1}{\mathcal{H}'(k_i)} \int_0^1 \frac{\mathcal{H}'(\iota^*) - \mathcal{H}'(k_i)}{\iota^* - k_i} (\iota^* - k_i)^2 \tau\, d\tau \leq \frac{-1}{2} \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)} (\iota^* - k_i)^2. \tag{4.7}$$

Because $\|z_i - h_i\| \leq \|\xi^* - z_i\| + \|\xi^* - h_i\|$, it follows from (4.2) and (4.3) that

$$\|z_i - h_i\| \leq (\iota^* - v_i) \frac{\|\xi^* - g_i\|}{\iota^* - k_i} \frac{\|\xi^* - h_i\|}{\iota^* - s_i} + (\iota^* - s_i) \left(\frac{\|\xi^* - g_i\|}{\iota^* - k_i} \right)^2.$$

Finally, by (4.7), we may go even further as

$$\|z_i - h_i\| \leq \left(1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)} (\iota^* - k_i) \right) \frac{-1}{2} \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)} \|\xi^* - g_i\|^2.$$

As a result of (3.10), we can derive

$$\frac{\|z_i - h_i\|}{v_i - s_i} \leq \frac{\left(1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)\right)\left(\frac{-1}{2} \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}\right)}{v_i - s_i} \|\xi^* - g_i\|^2$$
$$\leq \frac{\left(1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)\right)\left(\frac{-1}{2} \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}\right)}{\left(1 + \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)\right)\left(\frac{-1}{2} \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}\right)} \frac{\|\xi^* - g_i\|^2}{(\iota^* - k_i)^2}. \ \blacksquare$$

By utilizing Lemmas 4.1 and 4.2, we will prove

THEOREM 4.3. *Consider a nonlinear operator $T : \mathbb{D} \subset U \to V$ that is continuously Fréchet differentiable and defined in an open and convex subset $\mathbb{D}$. Assume there is $g_0 \in \mathbb{D}$ for which $[T(g_0)]^{-1}$ exists and T fulfills the $\varkappa$-average Lipschitz condition* (2.1) *in the ball $M(g_0, \iota^*)$. Consider the sequence $\{g_i\}$ obtained by applying the Newton–Traub scheme* (1.3) *with initial value g_0. If $0 < \beta \leq B$, then the sequence $\{g_i\}$ is well defined and converges to a unique solution ξ^* of* (1.1) *in the ball $\overline{M(g_0, \rho)}$ with rate* 5, *where $\rho := \sup\{k \in (\iota^*, \varrho) : \mathcal{H}(k) \leq 0\}$. Moreover, this solution is unique in the* *larger ball* $\overline{M(g_0, \rho)}$, *where* $\iota^* \leq \rho < k^{**}$. *Additionally, if*

$$1 + \frac{\iota^* \mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)} > 0 \iff 1 - \frac{\iota^* \varkappa(\iota^*)}{1 - \int_0^{\iota^*} \varkappa(u)\, du} > 0, \tag{4.8}$$

then at least fifth order of convergence can be expected, and the following error bounds can be obtained:

$$\|\xi^* - g_{i+1}\| \leq \frac{1}{4} \mathcal{H}_*^4 \frac{1 - \iota^* \mathcal{H}_*}{1 + \iota^* \mathcal{H}_*} \|\xi^* - g_i\|^5, \quad i \geq 0, \tag{4.9}$$

where $\mathcal{H}_* := \mathcal{H}''(\iota^*)/\mathcal{H}'(\iota^*)$.

Proof. Using Lemma 3.4, we can confirm that the sequence g_i is well-defined. From Lemma 3.4(vii) and (3.8), it can be deduced that $\|g_i - g_0\| \leq k_i < \iota^*$ for $i \geq 0$ and so g_i is in the ball $M(g_0, \iota^*)$. Furthermore, Lemma 4.1 implies that g_i converges to a solution ξ^* of (1.1) in $M(g_0, \iota^*)$.

In the next step, we will confirm the fifth order convergence of the iterative method. To achieve this, we employ conventional analytical methods to obtain

$$\begin{aligned}
\xi^* - g_{i+1} &= \xi^* - z_i + [T'(h_i)]^{-1} T(z_i) \\
&= -[T'(h_i)]^{-1} [T(\xi^*) - T(z_i) - T'(z_i)(\xi^* - z_i) + (T'(z_i) - T'(h_i))(\xi^* - z_i)] \\
&= -[T'(h_i)]^{-1} \Big[\int_0^1 (T'(g_i^\tau) - T'(z_i))(\xi^* - z_i)\, d\tau + (T'(z_i) - T'(h_i))(\xi^* - z_i) \Big],
\end{aligned}$$

where $g_i^\tau := g_i + \tau(\xi^* - g_i)$. Using the $\varkappa$-average Lipschitz condition (2.1) and (3.12), we infer that

$$\begin{aligned}
\|\iota^* - g_{i+1}\| \leq \frac{-1}{\mathcal{H}'(s_i)} \Big[\int_0^1 \Big(\int_{\|z_i - g_0\|}^{\|z_i - g_0\| + \|z_i - z_i + \tau(\iota^* - z_i)\|} \varkappa(u)\, du \Big) \|\iota^* - z_i\|\, d\tau \\
+ \int_{\|h_i - g_0\|}^{\|h_i - g_0\| + \|z_i - h_i\|} \varkappa(u)\, du\, \|\iota^* - z_i\| \Big].
\end{aligned}$$

Since $\mathcal{H}'$ is convex and increasing in $[0, \rho_0)$, one may combine (4.1), (4.5), Lemma 3.1, Lemma 3.4 and the $\varkappa$-average Lipschitz condition (2.1) to get

$$\begin{aligned}
\|\xi^* - g_{i+1}\| \leq \frac{-1}{\mathcal{H}'(s_i)} \Big[\int_0^1 \frac{\mathcal{H}'(v_i + \tau(\iota^* - v_i)) - \mathcal{H}'(v_i)}{\iota^* - v_i} \|\xi^* - z_i\|^2\, d\tau \\
+ \frac{\mathcal{H}'(v_i) - \mathcal{H}'(s_i)}{v_i - s_i} \|\xi^* - z_i\|\, \|z_i - h_i\| \Big]
\end{aligned}$$

$$= \frac{-1}{\mathcal{H}'(s_i)}\Big[\big(\mathcal{H}(\iota^*) - \mathcal{H}(v_i) - \mathcal{H}'(v_i)(\iota^* - v_i)\big)\frac{\|\xi^* - z_i\|^2}{(\iota^* - v_i)^2} + (\mathcal{H}'(v_i) - \mathcal{H}'(s_i))(\iota^* - v_i)\frac{\|\xi^* - z_i\|}{(\iota^* - v_i)}\frac{\|z_i - h_i\|}{v_i - s_i}\Big].$$

Using Lemma 3.4 and (4.1), (4.5) once more, we hence derive

$$\|\xi^* - g_{i+1}\| \le (\iota^* - k_{i+1})\left[\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\right]^3. \tag{4.10}$$

Subsequently, we can deduce from (3.9) that

$$\frac{\|\xi^* - g_{i+1}\|}{\|\xi^* - g_i\|^3} \le \frac{\iota^* - k_{i+1}}{(\iota^* - k_i)^3} \le \frac{1}{4}\,\frac{\mathcal{H}''(\iota^*)^4}{\mathcal{H}'(\iota^*)^3 H'(s_i)}(\iota^* - k_i)^2. \tag{4.11}$$

Letting $i \to \infty$, since $k_i \to \iota^*$, we obtain

$$\lim_{i\to\infty} \frac{\|\xi^* - g_{i+1}\|}{\|\xi^* - g_i\|^3} = 0. \tag{4.12}$$

Moreover, if condition (4.8) is satisfied as well, we can use (4.1), (4.5), (4.6), and (3.9) to deduce from (4.9) that

(4.13)

$$\begin{aligned}\|\xi^* - g_{i+1}\| &\le \frac{-1}{\mathcal{H}'(s_i)}\big[(\mathcal{H}(\iota^*) - \mathcal{H}(v_i) - \mathcal{H}'(v_i)(\iota^* - v_i))\,d\tau \\ &\quad + (\mathcal{H}'(v_i) - \mathcal{H}'(s_i))(\iota^* - v_i)\big]\frac{1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)}{1 + \iota^*\frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)}\left(\frac{\|\xi^* - z_i\|}{(\iota^* - k_i)}\right)^5 \\ &= (\iota^* - k_{i+1})\frac{1 - \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)}{1 + \iota^*\frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}(\iota^* - k_i)}\left(\frac{\|\xi^* - z_i\|}{(\iota^* - k_i)}\right)^5 \\ &\le \frac{1}{4}\left(\frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}\right)^4 \frac{1 - \frac{k^*\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}}{1 + \frac{k^*\mathcal{H}''(\iota^*)}{\mathcal{H}'(\iota^*)}}\|\xi^* - g_i\|^5.\end{aligned}$$

As a result, the estimate (4.9) is demonstrated, indicating that the order of convergence of the iterations is 5.

Ultimately, we demonstrate that the solution is unique. Initially, we establish the uniqueness of the solution ξ^* for (1.1) in $\overline{M(g_0, \iota^*)}$. Suppose there is another solution ξ^{**} in $\overline{M(g_0, \iota^*)}$. Then $\|\xi^{**} - g_0\| \le \iota^*$. We will demonstrate by induction that

$$\|\xi^{**} - g_i\| \le \iota^* - k_i, \qquad i = 0, 1, 2, \ldots. \tag{4.14}$$

As $k_0 = 0$, the case $i = 0$ is evidently true. Assuming that the above estimate holds for a particular $i \ge 0$, we apply the same procedure to estimate

$\|\xi^{**} - h_i\|$ in (4.2) and $\|\xi^* - z_i\|$ in (4.3), resulting in

$$\|\xi^* - h_i\| \le (\iota^* - s_i)\left(\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\right)^2, \qquad i \ge 0,$$

and

$$\|\xi^* - z_i\| \le (\iota^* - v_i)\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\,\frac{\|\xi^* - h_i\|}{\iota^* - s_i}, \qquad i \ge 0.$$

Moreover, by utilizing the same method to estimate $\|\xi^* - z_i\|$ in (4.10), it can be shown that

$$\|\xi^* - g_{i+1}\| \le (\iota^* - k_{i+1})\left[\frac{\|\xi^* - g_i\|}{\iota^* - k_i}\right]^3.$$

Hence, by utilizing the inductive hypothesis (4.14), it can be shown that (4.14) holds for $i+1$. As $\{g_i\}$ converges to ξ^* and $\{k_i\}$ converges to ι^*, it follows from (4.14) that $\xi^{**} = \xi^*$. Consequently, ξ^* is the only root of (1.1) in $\overline{M(g_0,\iota^*)}$. It is still necessary to demonstrate that T has no roots in $M(g_0,\rho)\backslash\overline{M(g_0,\iota^*)}$. Assuming the contrary, suppose that there exists $\xi^{**} \in \mathbb{D} \subset X$ such that $\iota^* < \xi^{**} - g_0 < \rho$ and $T(\xi^{**}) = 0$. We have

$$T(\xi^{**}) = T(g_0) + T'(g_0)(\xi^{**} - g_0) + \int_0^1 [T'(g_0^\tau) - T'(g_0)](\xi^{**} - g_0)\,d\tau, \tag{4.15}$$

where $g_0^\tau := g_0 + \tau(\xi^{**} - g_0)$. Notice that

$$\begin{aligned}\|[T'(g_0)]^{-1}[T(g_0) + T'(g_0)(\xi^{**} - g_0)]\| &\ge \|\xi^{**} - g_0\| - \|T'(g_0)]^{-1}T(g_0)\| \\ &= \|\xi^{**} - g_0\| - \mathcal{H}(0).\end{aligned}$$

Furthermore, the $\varkappa$-average Lipschitz condition (2.1) yields

$$\begin{aligned}&\left\|[T'(g_0)]^{-1}\int_0^1 [T'(g_0^\tau) - T'(g_0)](\xi^{**} - g_0)\,d\tau\right\| \\ &\quad\le \int_0^1 \Big(\int_0^{\tau(\|\xi^{**}-g_0)\|} \varkappa(u)\,du\Big)\|\xi^{**} - g_0\|\,d\tau \\ &\quad= \int_0^1 [\mathcal{H}'(\tau\|\xi^{**} - g_0\|) - \mathcal{H}'(0)]\|\xi^{**} - g_0\| \\ &\quad= \mathcal{H}(\|\xi^{**} - g_0\|) - \mathcal{H}(0) - \mathcal{H}'(0)\|\xi^{**} - g_0\|.\end{aligned}$$

Since $T(\xi^{**}) = 0$ and $\mathcal{H}'(0) = -1$, it follows from (4.15) that

$$\mathcal{H}(\|\xi^{**} - g_0\|) - \mathcal{H}(0) - \mathcal{H}'(0)\|\xi^{**} - g_0\| \ge \|\xi^{**} - g_0\| - \mathcal{H}(0).$$

This is equivalent to $\mathcal{H}(\|\xi^{**} - g_0\|) \ge 0$. Based on Lemma 3.2, it can be deduced that $\mathcal{H}$ is strictly positive in $(\|\xi^{**} - g_0\|, \mathbb{R})$. Thus, $\rho < \|\xi^{**} - g_0\|$, which contradicts the initial assumptions. ∎

NOTE. The convergence criterion $0 < \beta \leq B$, as stated in Theorem 4.3, was in fact originally derived by Wang [21] to examine the Newton method's convergence (1.2). To achieve fifth order convergence, it is also necessary to meet condition (4.8).

The results admit extensions as follows. Let $\epsilon := \sup\{t \geq 0 : M(g_0, t) \subset \mathcal{D}\}$.

DEFINITION 4.4. The operator T' satisfies the *center $\varkappa_0$-average Lipschitz criterion* on the ball $M(g_0, \epsilon)$ if for each $x \in M(g_0, \epsilon)$,

$$\|[T'(g_0)]^{-1}(T'(x) - T'(g_0))\| \leq \int_0^{\|x-g_0\|} \varkappa_0(u)\, du \tag{4.16}$$

for some nondecreasing continuous and nonnegative function $\varkappa_0$ defined on $[0, \epsilon]$.

Suppose that the equation $\int_0^\epsilon \varkappa_0(u)\, du - 1 = 0$ has a smallest positive solution $\epsilon_0 \in (0, \epsilon]$. Since $M(g_0, \epsilon_0) \subset M(g_0, \epsilon)$, it follows that

$$\varkappa_0(u) \leq \varkappa(u) \quad \text{for each } u \in [0, \epsilon_0].$$

Moreover, $T'(x)$ is invertible for $x \in M(g_0, \epsilon_0)$ and

$$\|[T'(x)]^{-1}T'(g_0)\| \leq \frac{1}{1 - \int_0^{\epsilon_0} \varkappa_0(u)\, du}.$$

This estimate is more precise than

$$\|[T'(x)]^{-1}T'(g_0)\| \leq \frac{1}{1 - \int_0^{\epsilon} \varkappa(u)\, du}.$$

used in the previous sections.

DEFINITION 4.5. The operator T' satisfies *the restricted $\overline{\varkappa}$-average Lipschitz criterion* on the ball $M(g_0, \epsilon_0)$ if for x, y with $\|y - x\| + \|x - x_0\| \leq \epsilon_0$,

$$\|[T'(g_0)]^{-1}(T'(y) - T'(x))\| \leq \int_0^{\|x-g_0\|} \overline{\varkappa}(u)\, du,$$

where $\overline{\varkappa}$ is a continuous, nondecreasing and nonnegative function defined on $[0, \epsilon_0]$.

It follows from these definitions that

$$\overline{\varkappa}(u) \leq \varkappa(u) \quad \text{for each } u \in [0, \epsilon_0].$$

Hence, the tighter function $\overline{\varkappa}$ can replace $\varkappa$ in all the previous results. This way the sufficient convergence criteria are weaker and the error estimates $\|x_{i+1} - x_i\|, \|\xi^* - x_i\|$ at least as precise. Notice that $\varkappa_0 = \varkappa_0(M(g_0, \epsilon))$, $\varkappa = \varkappa(M(g_0, \epsilon))$ but $\overline{\varkappa} = \overline{\varkappa}(M(g_0, \epsilon))$. It is also worth noting that the functions $\varkappa_0$ and $\overline{\varkappa}$ are specializations of the original function $\varkappa$. Thus, no additional conditions are used to obtain these improvements.

It turns out that the uniqueness region can be made more precise.

THEOREM 4.6. *Suppose that there exists a solution $\xi^{**} \in M(g_0, \rho_1)$ of the equation $T(g) = 0$ for some $\rho_1 > 0$, condition* (4.16) *holds on the ball $M(g_0, \rho_1)$ and there exists $\rho_2 \geq \rho_1$ such that for $b_\tau = (1-\tau)\|\xi^* - g_0\| + \tau\|\xi^{**} - g_0\|$,*

$$\int_0^1 \int_0^{(1-\tau)\rho_1+\tau\rho_2} \varkappa_0(u)\, du\, d\tau < 1. \tag{4.17}$$

Define $D_1 = M(g_0, \epsilon_0) \cap \overline{M}(g_0, \rho_2)$. Then the equation $T(g) = 0$ is uniquely solvable in D_1.

Proof. Define the linear operator $S = \int_0^1 T'(\xi^* + \tau(\xi^{**} - \xi^*))\, d\tau$, where $\xi^{**} \in D_1$ with $T(\xi^{**}) = 0$. It follows from this definition and conditions (4.16) and (4.17) that

$$\|[T'(g_0)]^{-1}(S - T'(g_0))\| \leq \int_0^1 \int_0^{b_\tau} \varkappa_0(u)\, du\, d\tau < 1,$$

since $b_\tau < (1-\tau)\rho_1 + \tau\rho_2$. Thus, S is invertible and from

$$\xi^{**} - \xi^* = S^{-1}(T(\xi^{**}) - T(\xi^*)) = S^{-1}(0) - 0,$$

we deduce $\xi^{**} = \xi^*$. ∎

REMARK. (a) The limit point ι^* can be replaced by ρ in Theorem 4.3.

(b) If all the assumptions of Theorem 4.3 hold, let $\rho_1 = \iota^*$ and $\xi^{**} = \xi^*$ in Theorem 4.6.

Special cases. By utilizing Theorem 4.3, we will derive several corollaries by considering different positive functions $\varkappa$. To begin, if $\varkappa$ is a positive constant, the $\varkappa$-average Lipschitz condition (2.1) simplifies to the following Lipschitz condition.

COROLLARY 4.7. *Let $T : \mathbb{D} \subset U \to V$ be a nonlinear operator that is continuously Fréchet differentiable in an open convex subset $\mathbb{D}$. Suppose that $g_0 \in \mathbb{D}$ is such that $[T'(g_0)]^{-1}$ exists, and T fulfills the Lipschitz condition*

$$\|[T'(g_0)]^{-1}(T'(h) - T'(g))\| \leq \varkappa\|h - g\|, \quad g, h \in M(g_0, \rho_0), \tag{4.18}$$

where $\rho_0 = 1/\varkappa$. The majorizing function $\mathcal{H}$ defined by (3.2) *is now*

$$\mathcal{H}(k) = \beta - k + \frac{\varkappa}{2}k^2, \quad k \in [0, R].$$

The value of ϱ can be obtained using (3.1) *as $\varrho = 2/\varkappa$. The constant B defined in* (3.4) *is now $B = 1/(2\varkappa)$. Additionally, Lemma* 3.2 *implies that if*

$\varkappa\beta \leq 1/2$, then the roots of $\mathcal{H}$ in the intervals $(0, 1/\varkappa)$ and $(1/\varkappa, 2/\varkappa)$ are

$$\iota^* = \frac{1 - \sqrt{1 - 2\varkappa\beta}}{\varkappa} \quad \text{and} \quad \iota^{**} = \frac{1 + \sqrt{1 - 2\varkappa\beta}}{\varkappa}. \tag{4.19}$$

Let $\{g_i\}$ denote the iterates generated by the Newton–Traub scheme (1.3) *with initial guess g_0. Assuming $0 < \varkappa\beta \leq 1/2$, the iterates $\{g_i\}$ are defined and converge Q-cubically to a unique solution $\xi^* \in M(g_0, \iota^*)$ of* (1.1), *where $\iota^* < \rho < \iota^{**}$ and ι^* and ι^{**} are given in (4.19). Additionally, if $0 < \varkappa\beta \leq 3/8$, the convergence order is at least five, and the following error bound holds:*

$$\|\xi^* - g_{i+1}\| \leq \frac{1}{4} \frac{\varkappa^4}{(1 - 2\varkappa\beta)^2} \frac{1}{2\sqrt{1 - 2\varkappa\beta} - 1} \|\xi^* - g_i\|^5, \qquad i \geq 0. \tag{4.20}$$

Secondly, let us assume that $Y > 0$. We consider the positive function $\varkappa$ defined by

$$\varkappa(u) := \frac{2Y}{(1 - Yu)^3}, \qquad u \in [0, 1/Y). \tag{4.21}$$

COROLLARY 4.8. *Assume that $T : \mathbb{D} \subset U \to V$ is a nonlinear operator that is continuously Fréchet differentiable in an open convex subset $\mathbb{D}$, and there exists $g_0 \in \mathbb{D}$ such that $[T(g_0)]^{-1}$ exists and*

$$\begin{aligned} &\|[T'(g_0)]^{-1}(T'(h) - T'(g))\| \\ &\qquad \leq \frac{1}{(1 - Y\|g - g_0\| - \|h - g\|)^2} - \frac{1}{(1 - Y\|g - g_0\|)^2}. \end{aligned} \tag{4.22}$$

The majorizing function $\mathcal{H}$ defined by (3.2) *is now*

$$\mathcal{H}(k) = \beta - k + \frac{Yt^2}{1 - Yt}, \qquad k \in [0, 1/Y].$$

The value of ρ_0 can be obtained using (3.3) *as $\rho_0 = \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{Y}$. The constant B, defined in* (3.4), *is now $B = 0.1715728/Y$. If $a = \beta Y \leq 0.1715728$, then the roots of $\mathcal{H}$ are*

$$\iota^* = \frac{1 + a - \sqrt{(1 + a)^2 - 8a}}{4Y} \quad \text{and} \quad \iota^{**} = \frac{1 + a + \sqrt{(1 + a)^2 - 8a}}{4Y}.$$

The constant $\mathcal{H}^* := \frac{\mathcal{H}''(\iota^*)}{\mathcal{H}'(k^*)}$ given in Theorem 4.3 *now has a specific form:*

$$\mathcal{H}^* = -\frac{32Y}{\sqrt{(1 + a)^2 - 8a}(3 - a + \sqrt{(1 + a)^2 - (8a)^2})}.$$

The iterates $\{g_i\}$ are generated by the Newton–Traub scheme (1.3) *with initial guess g_0. Assuming $0 < a \leq 0.1715728$, the iterates $\{g_i\}$ are defined and converge Q-cubically to a unique solution $\xi^* \in M(g_0, \iota^*)$ of* (1.1), *where*

$\iota^* < \rho < \iota^{**}$. *Additionally, if*

$$0 < a \leq \frac{1}{6}\left(17 - \frac{49}{(937 - 48\sqrt{330})^{1/3}} - (937 - 48\sqrt{330})^{1/3}\right),$$

the convergence order is at least 5, *and the following error bound holds:*

$$\|\xi^* - g_{i+1}\| \leq \frac{l}{4}(\mathcal{H}^*)^4\|\xi^* - g_i\|^5, \quad i \geq 0, \tag{4.23}$$

where

$$l := -\frac{7 - a^3 + \sqrt{1-6a+a^2} + a^2(9+\sqrt{1-6a+a^2}) - 3a(5+2\sqrt{1-6a+a^2})}{1 + a^3 - 9\sqrt{1-6a+a^2} - a^2(9+\sqrt{1-6a+a^2}) + a(23+6\sqrt{1-6a+a^2})}.$$

5. Numerical example demonstrating the application. We give an application based on the semilocal convergence results established in the previous section.

Example 5.1. Let $X = C[0,1]$, the space of continuous functions on $[0,1]$ with the max norm

$$\|g\| = \max_{s\in[0,1]} |g(s)|.$$

Let $\phi = M(0,1)$ and suppose the operator T on ϕ is

$$T(g)(s) = g(s) - 2\lambda \int_0^1 \gamma(s,k) g(k)^3 dk. \tag{5.1}$$

Here, γ denotes the Green's function kernel defined on $[0,1] \times [0,1]$ by

$$\gamma(s,k) = \begin{cases} (1-s)k, & k \leq s, \\ s(1-k), & s \leq k. \end{cases}$$

Here, $s \in [0,1]$, λ is a real parameter to be chosen arbitrarily, and $g \in C[0,1]$ has to be determined. As a result, we have

$$T'(g)h(s) = h(s) - 6\lambda \int_0^1 \gamma(s,k) g(k)^2 h(k) dk, \quad h \in \phi. \tag{5.2}$$

Next, let $S = \max_{s\in[0,1]} \int_0^1 |\gamma(s,k)|\, dk$, which results in $S = 1/8$. In addition, choosing $g_0(k) = 0.25$ as the initial approximate solution, for any $g, h \in \phi$, we obtain

$$\beta = \|[T'(g_0)]^{-1} T(g_0)\| \leq \frac{0.0039063|\lambda|}{1 - 0.046875|\lambda|}. \tag{5.3}$$

By the definition of $\varkappa$-average, from Corollary 4.7, we get $\varkappa = \frac{3}{2}|\lambda|\frac{1}{1-0.046875|\lambda|}$. Since $\beta < B$, the convergence criterion is satisfied. Theorem 4.3 can be used to

Table 1. Domains of uniqueness and existence of solution for NTS

λ	Ball of convergence Existence, $\overline{M(g_0, \iota^*)}$	Uniqueness, $\overline{M(g_0, \iota^{**})}$
1	$\overline{M(0.25, 0.00409923)}$	$\overline{M(0.25, 1.26672)}$
0.5	$\overline{M(0.25, 0.00200157)}$	$\overline{M(0.25, 2.60217)}$
0.25	$\overline{M(0.25, 0.00098834)}$	$\overline{M(0.25, 5.26984)}$
0.125	$\overline{M(0.25, 0.00049118)}$	$\overline{M(0.25, 10.6037)}$
0.0625	$\overline{M(0.25, 0.000244864)}$	$\overline{M(0.25, 21.2706)}$

Table 2. Convergence criteria and the error bounds for NTS

λ	$\varkappa\beta < 3/8$	Error bound
1	0.006449	1.59456
0.5	0.00153602	0.0877798
0.25	0.000374952	0.00519424
0.125	0.0000926362	0.000316515
0.0625	0.0000230232	0.0000195426

assert that the NTS sequence (1.3) generated with the initial guess g_0 converges to a zero of T. Table 1 shows the domain of uniqueness and existence of solution for various values of λ, namely $\lambda = 0.0625, 0.125, 0.25, 0.5, 1$. Table 2 shows the criteria $\varkappa\beta < 3/8$ and the error bounds for similar values of λ. In comparison to the semilocal convergence of two-step method in [12], we can see from Table 2 that the new convergence criterion is stronger than the previous one.

6. Conclusion. In conclusion, this paper has contributed to the analysis of semilocal convergence for the multistep Newton–Traub scheme (1.3) in a Banach space by assuming that the $\varkappa$-average Lipschitz conditions (2.1) are satisfied by the first derivative T. The main result, Theorem 4.3, establishes the existence and uniqueness of a solution $\xi^* \in M(g_0, \rho)$ of (1.1), as well as the cubic convergence of the sequence $\{g_i\}$ when the convergence criterion $0 < \beta \leq B$ given by Wang [21] is fulfilled. Furthermore, we have shown that the sequence $\{g_i\}$ is fifth order convergent if the condition (4.8) is satisfied additionally. Finally, we have provided two special cases, including Kantorovich-type conditions [9] and Y-conditions, which extend the applicability of the results presented in this paper.

Overall, our findings provide insights into the behavior of the Newton–Traub scheme (1.3) in Banach spaces with potential for future advancements and suggest further research into their application in optimization and scientific computing. Empirical evidence obtained from numerical tests validates the effectiveness and applicability of our results.

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