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Persistence exponents via perturbation theory: Gaussian MA(1)-processes

by

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Abstract. For the moving average process $X_n = \rho\xi_{n-1} + \xi_n$, $n \in \mathbb{N}$, where $\rho \in \mathbb{R}$ and $(\xi_i)_{i\geq -1}$ is an i.i.d. sequence of standard normally distributed random variables, we study the persistence probabilities $\mathbb{P}(X_0 \geq 0, \dots, X_N \geq 0)$ for $N \to \infty$. We exploit the fact that the exponential decay rate λ_ρ of that quantity, called the persistence exponent, is given by the leading eigenvalue of a concrete integral operator. This makes it possible to study the problem with purely functional-analytic methods. In particular, using methods from perturbation theory, we show that the persistence exponent λ_ρ can be expressed as a power series in ρ. Finally, we consider the persistence problem for the Slepian process, transform it into the moving average setup, and show that our perturbation results are applicable.

1. Introduction

1.1. Persistence probabilities for moving average processes. Let $(\xi_i)_{i\geq -1}$ be a sequence of i.i.d. random variables and let $\rho \in \mathbb{R}$. A moving average process of order 1 (MA(1)-process) is given by

$$X_n := \rho\xi_{n-1} + \xi_n \quad \text{for } n \in \mathbb{N}.$$

Throughout the paper, we consider standard normally distributed random variables $(\xi_i)_i$, i.e. ξ_0 has the density $\phi(x) := (2\pi)^{-1/2}\exp(-x^2/2)$, $x \in \mathbb{R}$.

We are interested in the *persistence probabilities*

$$\mathbb{P}(X_0 \geq 0, \dots, X_N \geq 0) \quad \text{for } N \to \infty, \tag{1.1}$$

and in particular in the exponential rate of decay of this quantity, which is called the *persistence exponent*. Non-exit probabilities are a classic and fun-

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damental topic in probability with numerous applications in finance, insurance, queueing, and other subjects. The question is also studied intensively in theoretical physics; there, the rationale is that the persistence exponent is a measure of how fast the underlying physical system returns to equilibrium. We refer to the survey [8] and the monograph [22] for an overview of the relevance of the question to physical systems and to [6] for a survey of the mathematical literature.

The persistence problem for moving average processes was studied before in [19, 17, 5]. For MA(1)-processes, the persistence question can be rewritten as a non-exit problem for a two-dimensional Markov chain (see e.g. [5, Section 2.1]). It is well-known that non-exit probabilities of Markov chains have close relations to eigenvalues of operators (see [3, 5, 13]; also see [11, 9, 21, 27, 28] for the quasi-stationary approach). The purpose of this paper is to establish and to use the connection between the persistence exponent λ_ρ, i.e. the exponential decay rate of (1.1), and the leading eigenvalue of a suitable operator. Then, powerful tools from functional analysis can be applied. In particular, methods of perturbation theory in the spirit of [14] can be used to obtain a series representation of the eigenvalue in the parameter ρ. This ansatz was previously used in [4] for autoregressive processes.

1.2. The eigenvalue problem. In the recent work [5, Section 2.1] it is shown that for $\rho \neq -1$ the persistence probability (1.1) of the MA(1)-process decays exponentially fast and that the exponential decay rate, i.e. the persistence exponent, is the leading eigenvalue of the following explicit integral operator:

$$(1.2)\qquad S_\rho\colon \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathbb{R}), \qquad S_\rho f(x) := \int_{-\rho x}^{\infty} f(y)\phi(y)\,\mathrm{d}y,$$

where $\mathcal{B}(\mathbb{R})$ is the space of bounded measurable real-valued functions on $\mathbb{R}$. Bearing this connection in mind, the purpose of this paper is to study the largest solution $\lambda = \lambda_\rho$ of the eigenvalue equation

$$(1.3)\qquad \lambda f(x) = \int_{-\rho x}^{\infty} f(y)\phi(y)\,\mathrm{d}y, \qquad x \in \mathbb{R},\ f \in \mathcal{B}(\mathbb{R}).$$

The approach taken in this paper is to show that a modification of the integral operator S_ρ can be represented as a power series in ρ if we consider the operator on a suitable space of functions. The definition of this function space (see Section 2) is motivated by the following observation:

Assume that f is analytic, that is, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for $x \in \mathbb{R}$. Further, if we assume that $\lim_{x\to\infty}(f\phi)^{(n-1)}(x) = 0$ for all $n \geq 1$, we can write $(f\phi)^{(n-1)}(0) = (-1)\int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y$. Then

$$
\begin{aligned}
(1.4)\qquad S_\rho f(x) &= \int_{-\rho x}^{\infty} f(y)\phi(y)\,\mathrm{d}y \\
&= \int_0^\infty f(y)\phi(y)\,\mathrm{d}y + (-1)\int_0^{-\rho x} f(y)\phi(y)\,\mathrm{d}y \\
&= \int_0^\infty f(y)\phi(y)\,\mathrm{d}y + \sum_{n=1}^\infty \rho^n(-1)^{n+1}x^n\frac{(f\phi)^{(n-1)}(0)}{n!} \\
&= \int_0^\infty f(y)\phi(y)\,\mathrm{d}y + \sum_{n=1}^\infty \rho^n(-1)^n x^n\frac{1}{n!}\int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y \\
&= \sum_{n=0}^\infty \rho^n(-1)^n x^n\frac{1}{n!}\int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y.
\end{aligned}
$$

Hence, under the above conditions on f, the expression $S_\rho f(x)$ can be written as a power series in ρ.

Here and elsewhere, we use the following notation: In order to enhance readability, we will use $f^{(0)} := f$, while for $n \geq 1$, $f^{(n)}$ is the nth derivative. Further, for a complex-valued function f we use $\int f(y)\,\mathrm{d}y = \int \Re(f)(y)\,\mathrm{d}y + i\int \Im(f)(y)\,\mathrm{d}y$, and $\Re$ and $\Im$ denote the real and imaginary part, respectively.

With (1.4) in mind, we will consider a specific space of analytic functions such that we obtain a well-defined holomorphic operator (see Theorem 2.1). From this, by using perturbation techniques in the spirit of [14], we can conclude that the leading eigenvalue and the corresponding eigenfunction are holomorphic in ρ, too. This is the content of our main result, Theorem 2.3. In other words, the persistence exponent and the eigenfunction admit a power series representation in ρ. Additionally, we have iterative formulas for the coefficients of the power series representation of the persistence exponent and the eigenfunction; and we compute the first coefficients (see Theorem 2.4 and the subsequent remark).

1.3. Application to the Slepian process. As a further application, we look at the persistence problem for the so-called Slepian process. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and define the Slepian process by

$$(1.5)\qquad S_t := B_{t+1} - B_t, \qquad t \geq 0.$$

In his seminal paper [26], D. Slepian computed the distribution of the supremum of S on $[0,1]$ and found

$$(1.6)\qquad \mathbb{P}\Big(\sup_{u\in[0,1]} S_u \leq a\Big) = \Phi(a)^2 - \phi(a)(a\Phi(a) + \phi(a)),$$

where again ϕ is the standard normal density and Φ is the corresponding cumulative distribution function. The general formula for the distribution of $\sup_{u\in[0,t]} S_u$ for any $t \geq 0$ was then obtained by Shepp [25]. Shepp leaves it as an open question to study the asymptotics

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(\sup_{u\in[0,N]} S_u \leq a\Big),$$

because his formulas, which involve iterated integrals, are not well-suited for such computations. The existence of this limit was then obtained by Li & Shao [18], and numerical computations have been proposed recently by Noonan & Zhigljavsky [23]. The contribution of this paper is the observation that one can rewrite the persistence problem for the Slepian process as a persistence problem for an MA(1)-process. Further, we can show that our perturbation results are applicable.

The outline of this paper is as follows. In Section 2, our main results are stated. In particular, we show that the leading eigenvalue λ_ρ of (1.3) can be expanded into a power series in ρ and we discuss the coefficients of this power series. In Section 3, we consider the persistence problem for the Slepian process, transform it into the MA(1)-process setup, and show that our main results are applicable. Section 4 is devoted to the proofs of the main theorems. Finally, in Section 5 we deal with the radius of convergence of the series for λ_ρ.

We mention that a large portion of the results of this paper is part of the PhD thesis [15].

2. Main results. Let γ be the standard Gaussian measure on $\mathbb{R}$ and let h_n denote the nth Hermite polynomial given by

$$h_n(x) := (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2}.$$

For the facts on Hermite polynomials that we use in this paper, we refer the reader to [2, Chapter 6]. Further, we set $\widehat{h}_n(x) := (n!)^{-1/2} h_n(x)$. Here, the normalization is chosen so that $\|\widehat{h}_n\|_{L^2(\mathbb{R},\gamma)} = 1$.

Fix $0 < q < 1$ and let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ be a sequence with $\sum_{n=0}^\infty |a_n|^2 q^{-n} < \infty$. By [1, Theorem 2], $\sum_{n=0}^\infty |a_n \widehat{h}_n(x)|$ converges uniformly on compact subsets of $\mathbb{R}$. Hence, we can define an analytic function $f\colon \mathbb{R}\to\mathbb{C}$ via $f(x) := \sum_{n=0}^\infty a_n \widehat{h}_n(x)$. In particular, $\Re(f)$ and $\Im(f)$ are analytic. Let

$$\mathcal{H}_q := \Big\{ f\colon \mathbb{R}\to\mathbb{C} : f(x) = \sum_{n=0}^\infty a_n \widehat{h}_n(x) \text{ with } \sum_{n=0}^\infty |a_n|^2 q^{-n} < \infty \Big\}$$

and set

$$\langle f, g\rangle_{\mathcal{H}_q} := \sum_{n=0}^{\infty} a_n \overline{b_n} q^{-n} \qquad \text{for } f = \sum_{n=0}^{\infty} a_n \widehat{h}_n,\ g = \sum_{n=0}^{\infty} b_n \widehat{h}_n.$$

We note that $(\mathcal{H}_q, \langle\cdot,\cdot\rangle_{\mathcal{H}_q})$ is a Hilbert space of functions [1, Proposition 1]. In fact, we will see that it is a reproducing kernel Hilbert space and we will exploit this structure in the proofs. Note that we consider a complex Hilbert space instead of a real one, since we need a complex space to apply the powerful methods of perturbation theory in the spirit of [14].

We set

$$T_\rho\colon \mathcal{H}_q \to \mathcal{H}_q, \qquad T_\rho f(x) := \int_{-\rho x}^{\infty} f(y)\phi(y)\,\mathrm{d}y.$$

Note that this is the version of the operator S_ρ on the space $\mathcal{H}_q$ (see (1.2)), in the sense that S_ρ acts on bounded real-valued functions, while T_ρ acts on complex-valued functions in $\mathcal{H}_q$.

Our first main result states that T_ρ is a holomorphic operator.

THEOREM 2.1. *Fix $0 < q < 1$. Let $-\sqrt{\frac{1-q}{1+q^{-1}}} < \rho < \sqrt{\frac{1-q}{1+q^{-1}}}$ and define for $n \in \mathbb{N}$ the integral operator*

$$T^{(n)}\colon \mathcal{H}_q \to \mathcal{H}_q, \qquad T^{(n)} f(x) := (-1)^n x^n \frac{1}{n!} \int_0^{\infty} (f\phi)^{(n)}(y)\,dy.$$

The operator T_ρ is well-defined, bounded, compact and admits the representation

$$T_\rho = \sum_{n=0}^{\infty} \rho^n T^{(n)}. \tag{2.1}$$

REMARK 2.2. To optimize the radius of convergence of the power series for T_ρ, the best choice of $0 < q < 1$ for the Hilbert space $\mathcal{H}_q$ is $q^* := \sqrt{2} - 1$. Then T_ρ can be represented as a power series for $-(\sqrt{2}-1) < \rho < \sqrt{2} - 1$.

Let

$$r(T_\rho) := \sup\{|\lambda|\colon \lambda \in \Sigma(T_\rho)\}$$

be the spectral radius of T_ρ, where $\Sigma(T_\rho)$ denotes the spectrum of T_ρ.

Our second main result deals with the leading eigenvalue λ_ρ of (1.3) on $\mathcal{H}_q$, i.e. the persistence exponent for moving average processes, and states that λ_ρ can be expanded into a power series.

THEOREM 2.3. *For $-\sqrt{\frac{1-q}{1+q^{-1}}} < \rho < \sqrt{\frac{1-q}{1+q^{-1}}}$ we have*

$$\mathbb{P}(X_0 \geq 0, \ldots, X_N \geq 0) = \lambda_\rho^{N+o(N)},$$

where $\lambda_\rho := r(T_\rho) \in (0,1)$ *is the largest eigenvalue of* T_ρ*. The corresponding eigenfunction* f_ρ *is non-negative, i.e.* $f_\rho(x) \geq 0$ *for all* $x \in \mathbb{R}$*.*

Further, there are numbers $K_n \in \mathbb{R}$ *such that* λ_ρ *admits the representation*

$$\lambda_\rho = \sum_{n=0}^{\infty} \rho^n K_n, \tag{2.2}$$

for all $|\rho| < r_0$*, where* $r_0 > 0.332$*.*

As an application, we are going to see in Section 3 that one can transform the persistence problem for the Slepian process into a question about persistence of MA(1)-processes and that this case can be covered by Theorem 2.3; see Proposition 3.1.

As the third and last main result, we determine the coefficients K_n, $n \in \mathbb{N}$, of the power series of the persistence exponent λ_ρ in (2.2).

By Theorem 2.1 the operator T_ρ, and by Theorem 2.3 the eigenvalue λ_ρ can be expressed as a power series in ρ. Additionally, the corresponding eigenfunction f_ρ can be expressed as a power series in ρ (see [4, Theorem 4]). Let us write

$$T_\rho = \sum_{k=0}^{\infty} \rho^k T^{(k)}, \quad \lambda_\rho = \sum_{k=0}^{\infty} \rho^k K_k, \quad f_\rho = \sum_{m=0}^{\infty} \rho^m g_m.$$

Note that for $\rho = 0$ the MA(1)-process is a sequence of i.i.d. random variables, so that

$$K_0 = \lambda_0 = \mathbb{P}(\xi_0 \geq 0) = \frac{1}{2}.$$

THEOREM 2.4. *For all* $m \in \mathbb{N}$ *the function* g_m *is a polynomial of degree at most* m*. A full iterative description of the* g_m*,* $m \in \mathbb{N}$*, is given by the equations* $g_0 = \mathbb{1}$*,* $K_0 = 1/2$*,* $g_m(0) = 0$ *for* $m \geq 1$*, and*

$$g_m = \frac{1}{K_0}\Big(\sum_{j=1}^{m} T^{(j)} g_{m-j} - \sum_{j=1}^{m-1} T^{(0)} g_j \cdot g_{m-j}\Big), \qquad m \geq 1. \tag{2.3}$$

Further, the K_n*,* $n \geq 1$*, can be computed as*

$$K_n = T^{(0)} g_n. \tag{2.4}$$

REMARK 2.5. Using the iterative formulas from Theorem 2.4, the first coefficients can be computed:

$$K_0 = \frac{1}{2},$$

$$K_1 = \frac{1}{\pi},$$

$$K_2 = -\frac{2}{\pi^2},$$

$$
\begin{aligned}
K_3 &= -\frac{5}{6\pi} + \frac{8}{\pi^3},\\
K_4 &= \frac{13}{3\pi^2} - \frac{40}{\pi^4},\\
K_5 &= \frac{23}{40\pi} - \frac{28}{\pi^3} + \frac{224}{\pi^5},\\
K_6 &= -\frac{1069}{180\pi^2} + \frac{580}{3\pi^4} - \frac{1344}{\pi^6},\\
K_7 &= -\frac{37}{112\pi} + \frac{842}{15\pi^3} - \frac{4144}{3\pi^5} + \frac{8448}{\pi^7},\\
K_8 &= \frac{943}{168\pi^2} - \frac{1535}{3\pi^4} + \frac{10080}{\pi^6} - \frac{54912}{\pi^8}.
\end{aligned}
$$

It would be very interesting to obtain a closed-form expression for the coefficients K_n. We suspect that the last term of each K_n is given by $\tau_n := (-1)^{n-1}2^{n-1}\binom{2(n-1)}{n-1}\frac{1}{n}\pi^{-n}$. The second-to-last term of K_n seems to be of the form $-\tau_{n-2}\frac{8(n-3)+5}{6}$. Obtaining more terms seems complicated.

3. Slepian process. In this section, we consider the persistence problem for the Slepian process, show how this can be transformed into a persistence question for an MA(1)-process, and prove that the above main results can be applied.

Recall the Slepian process (S_t) defined in (1.5). Let us denote by

$$
F_N(a) := \mathbb{P}\Big(\sup_{u\in[0,N]} S_u \le a\Big)
$$

the *persistence probability* for the Slepian process, where $a \in \mathbb{R}$ and $N \in \mathbb{N}$.

PROPOSITION 3.1. *The persistence probability $F_N(a)$ may be written as*

$$
F_N(a) = \mathbb{P}(X_0 \le b, \dots, X_{N-1} \le b),
$$

where $(X_n)_{n\ge 1}$ is an MA(1)-process with standard normally distributed random variables and with parameter

$$
\widehat{\rho} = \frac{\sqrt{1-4\cos^2(2\pi F_2(\widehat{a}))}-1}{2\cos(2\pi F_2(\widehat{a}))} \simeq 0.3186,
$$

where $\widehat{a} = F_1^{-1}(1/2)$ and $b = \Phi^{-1}(F_1(a))\sqrt{1+\widehat{\rho}^2}$.

When $b = 0$, i.e. $a = \widehat{a}$, Theorem 2.3 shows that the desired exponential decay rate of the persistence probability of the Slepian process can be expressed as a power series since $\widehat{\rho} < 0.332$. For arbitrary $a \in \mathbb{R}$ a shifted version of $T_{\widehat{\rho}}$ has to be considered. Presumably, as in Theorems 2.1 and 2.3, a power series representation can be obtained.

Remark 3.2. The value of F_1 was originally computed in [26, (1.6)]. Similarly, following the computation of [23], the value of $F_2(\widehat{a})$ is given by

$$F_2(\widehat{a}) = \Phi(\widehat{a}) - \Phi^3(\widehat{a}) - \frac{1}{\sqrt{2}} \int_0^\infty \Phi(\widehat{a}-y)\varphi(\sqrt{2}y)\Big(\Phi(\sqrt{2}y) - \frac{1}{2}\Big)\,dy$$

$$+ \frac{\varphi^2(\widehat{a})}{2}\big(\Phi(\widehat{a})(\widehat{a}^2+1) + \widehat{a}\varphi(\widehat{a})\big) + \int_0^\infty \Phi^2(\widehat{a}-y)\varphi(y+\widehat{a})\,dy.$$

Proof of Proposition 3.1. We start by noting the decomposition

$$F_N(a) = \mathbb{P}(M_0 \le a, \dots, M_{N-1} \le a),$$

where the random variables $(M_i)_{i\ge 0}$ are defined by

$$M_i := \sup_{u\in[0,1]} S_{u+i}.$$

Using the stationarity and the independence of the increments of Brownian motion, it is clear that the random variables (M_i) are identically distributed with common distribution F_1, and for any i and j with $|i-j|\ge 2$, M_i and M_j are independent. Define next the centered Gaussian random variables

$$Z_i := \Phi^{-1}(F_1(M_i)), \qquad i \ge 0,$$

and observe that the sequence (Z_i) is stationary with covariance given by $\mathbb{E}[Z_i^2] = 1$, $\mathbb{E}[Z_iZ_j] = \mathbb{E}[Z_0Z_1] =: s$ for $|i-j| = 1$, and $\mathbb{E}[Z_iZ_j] = 0$ for $|i-j| \ge 2$. Setting $b_0 := \Phi^{-1}(F_1(a))$, we are thus led to compute

$$F_N(a) = \mathbb{P}(Z_0 \le b_0, Z_1 \le b_0, \dots, Z_{N-1} \le b_0), \tag{3.1}$$

where the density of the Gaussian vector $(Z_0,\dots,Z_{N-1})$ only depends on s. To compute the value of s, observe that taking $N=2$ and $b_0=0$, we obtain

$$\begin{aligned} F_2(\widehat{a}) &= \mathbb{P}(Z_0 \le 0, Z_1 \le 0) \\ &= \frac{\sqrt{1-s^2}}{2\pi}\int_{-\infty}^0\int_{-\infty}^0 \exp\Big(-\frac{1}{2}(x^2+y^2-2sxy)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{4} + \frac{1}{2\pi}\arctan\Big(\frac{s}{\sqrt{1-s^2}}\Big). \end{aligned}$$

Inverting this relation yields the value of s (note that $2\pi F_2(\widehat{a}) \in [\pi/2, \pi]$ so that we can use the last formula):

$$s = \frac{\tan(2\pi F_2(\widehat{a}) - \pi/2)}{\sqrt{1+\tan^2(2\pi F_2(\widehat{a}) - \pi/2)}} = -\cos(2\pi F_2(\widehat{a})).$$

As above, let (ξ_i) be i.i.d. standard normal random variables and set

$$\widehat{\rho} := \frac{1-\sqrt{1-4s^2}}{2s}, \quad \text{so that} \quad s = \frac{\widehat{\rho}}{1+\widehat{\rho}^2}.$$

Therefore, the sequence (Z_i) has the same distribution as the sequence

$$\frac{\widehat{\rho}\xi_{i-1}+\xi_i}{\sqrt{1+\widehat{\rho}^2}}, \qquad i\geq 0,$$

and going back to (3.1), we obtain

$$F_N(a)=\mathbb{P}\big(\widehat{\rho}\xi_{-1}+\xi_0\leq b_0\sqrt{1+\widehat{\rho}^2},\ldots,\widehat{\rho}\xi_{N-2}+\xi_{N-1}\leq b_0\sqrt{1+\widehat{\rho}^2}\big).$$

This is exactly the statement of Proposition 3.1 after setting $X_i:=\widehat{\rho}\xi_{i-1}+\xi_i$ for $i\in\mathbb{N}$. ∎

4. Proofs of the main theorems. The following lemma allows us to use the computation (1.4) and provides a helpful representation of the inner product and the norm on $\mathcal{H}_q$.

LEMMA 4.1. *Let $f,g\in\mathcal{H}_q$. Then*

(a) $\|f^{(k)}\|_{L^2(\mathbb{R},\gamma)}<\infty$ *for all* $k\in\mathbb{N}$,
(b) $\lim_{x\to\infty}(f\phi)^{(n-1)}(x)=0$ *for all* $n\geq 1$,
(c) $\langle f,g\rangle_{\mathcal{H}_q}=\sum_{k=0}^{\infty}\frac{(q^{-1}-1)^k}{k!}\langle f^{(k)},g^{(k)}\rangle_{L^2(\mathbb{R},\gamma)}$,
(d) $\|f\|_{\mathcal{H}_q}^2=\sum_{k=0}^{\infty}\frac{(q^{-1}-1)^k}{k!}\|f^{(k)}\|^2_{L^2(\mathbb{R},\gamma)}$.

Proof. (a) Let $f\in\mathcal{H}_q$, i.e. suppose that $f(x)=\sum_{n=0}^{\infty}a_n\widehat{h}_n(x)$ for $x\in\mathbb{R}$, with $\sum_{n=0}^{\infty}|a_n|^2q^{-n}<\infty$. Note that $\widehat{h}_n^{(k)}(x)=\sqrt{\frac{n!}{(n-k)!}}\,\widehat{h}_{n-k}(x)$ for $k\leq n$ (see e.g. [2, Section 6.1]). Thus, the derivatives of f are given by

$$f^{(k)}(x)=\sum_{n=0}^{\infty}a_n\widehat{h}_n^{(k)}(x)=\sum_{n=0}^{\infty}a_{n+k}\sqrt{\frac{(n+k)!}{n!}}\,\widehat{h}_n(x).$$

Recall that $(\widehat{h}_n)_n$ is an orthonormal basis for $L^2(\mathbb{R},\gamma)$. By Parseval's identity,

$$\|f^{(k)}\|_{L^2(\mathbb{R},\gamma)}=\sum_{n=0}^{\infty}\Big(|a_{n+k}|\sqrt{\frac{(n+k)!}{n!}}\Big)^2\leq\sum_{n=0}^{\infty}|a_{n+k}|^2(n+k)^k<\infty.$$

(b) Note that $(f\phi)^{(n-1)}=(\Re(f)\phi)^{(n-1)}+i(\Im(f)\phi)^{(n-1)}$. We have

$$\begin{aligned}\int_0^\infty|(\Re(f)\phi)^{(n-1)}(x)|\,\mathrm{d}x&=\int_0^\infty\Big|\sum_{k=0}^{n-1}\binom{n-1}{k}\Re(f)^{(k)}(x)\phi^{(n-1-k)}(x)\Big|\,\mathrm{d}x\\&=\int_0^\infty\Big|\sum_{k=0}^{n-1}\binom{n-1}{k}\Re(f)^{(k)}(x)h_{n-1-k}(x)\Big|\,\mathrm{d}\gamma(x)\\&\leq\sum_{k=0}^{n-1}\binom{n-1}{k}\int_{\mathbb{R}}|\Re(f)^{(k)}(x)h_{n-1-k}(x)|\,\mathrm{d}\gamma(x)\end{aligned}$$

$$\le \sum_{k=0}^{n-1} \binom{n-1}{k} \|\Re(f)^{(k)}\|_{L^2(\mathbb{R},\gamma)} \|h_{n-1-k}\|_{L^2(\mathbb{R},\gamma)} < \infty,$$

by using Hölder's inequality in the last but one step and statement (a) in the last step. Therefore, if $\lim_{x\to\infty}(\Re(f)\phi)^{(n-1)}(x)$ exists, then the limit must be zero. The limit exists since

$$(\Re(f)\phi)^{(n-1)}(x) = \int_0^x (\Re(f)\phi)^{(n)}(y)\,\mathrm{d}y + (\Re(f)\phi)^{(n-1)}(0) \xrightarrow{x\to\infty} \int_0^\infty (\Re(f)\phi)^{(n)}(y)\,\mathrm{d}y + (\Re(f)\phi)^{(n-1)}(0).$$

Similarly, $\lim_{x\to\infty}(\Im(f)\phi)^{(n-1)}(x) = 0$ follows. This proves the assertion.

(c) Let $f(x) = \sum_{n=0}^\infty a_n \widehat{h}_n(x)$, $g(x) = \sum_{n=0}^\infty b_n \widehat{h}_n(x)$ for $x \in \mathbb{R}$. As in the proof of statement (a) we have $f^{(k)}(x) = \sum_{n=k}^\infty a_n \sqrt{\frac{n!}{(n-k)!}}\,\widehat{h}_{n-k}(x)$ and $g^{(k)}(x) = \sum_{n=k}^\infty b_n \sqrt{\frac{n!}{(n-k)!}}\,\widehat{h}_{n-k}(x)$ for $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Using the fact that $(\widehat{h}_n)_n$ is an orthonormal basis for $L^2(\mathbb{R},\gamma)$, we compute

$$\begin{aligned}\sum_{k=0}^\infty \frac{(q^{-1}-1)^k}{k!} \langle f^{(k)}, g^{(k)}\rangle_{L^2(\mathbb{R},\gamma)} &= \sum_{k=0}^\infty \frac{(q^{-1}-1)^k}{k!} \sum_{n=k}^\infty \frac{n!}{(n-k)!} a_n \overline{b_n} \\ &= \sum_{n=0}^\infty a_n \overline{b_n} \sum_{k=0}^n \binom{n}{k} (q^{-1}-1)^k \\ &= \sum_{n=0}^\infty a_n \overline{b_n} (q^{-1}-1+1)^n \\ &= \langle f, g\rangle_{\mathcal{H}_q}.\end{aligned}$$

(d) follows directly from (c). ■

Combining Lemma 4.1(b) with the fact that functions in $\mathcal{H}_q$ are analytic, we find that, as in (1.4),

$$T_\rho f(x) = \sum_{n=0}^\infty \rho^n (-1)^n x^n \frac{1}{n!} \int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y$$

for all $f \in \mathcal{H}_q$ and $x \in \mathbb{R}$, i.e. (2.1) holds.

Proof of Theorem 2.1. We want to compute an upper bound for the operator norm of $T^{(n)}$ for $n \in \mathbb{N}$, which will simultaneously show that these operators are well-defined. Let $m_n(x) := x^n$ for $x \in \mathbb{R}$. Note that

$$\begin{aligned}\|T^{(n)} f\|_{\mathcal{H}_q} &= \left\| (-1)^n \frac{1}{n!} \int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y \cdot m_n \right\|_{\mathcal{H}_q} \\ &= \left| \frac{1}{n!} \int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y \right| \|m_n\|_{\mathcal{H}_q}.\end{aligned}$$

We recall the explicit inverse formula for the Hermite polynomials (see e.g. [24, Section 2]):

$$m_n = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{h_{n-2j}}{2^j j! (n-2j)!} = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^j j! \sqrt{(n-2j)!}} \widehat{h}_{n-2j}.$$

On the one hand, by using the inequality $(2j)! \le 2^{2j} j!^2$ for $j \in \mathbb{N}$, we obtain

$$\begin{aligned}\|m_n\|_{\mathcal{H}_q}^2 &= \sum_{j=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^j j! \sqrt{(n-2j)!}} \right)^2 q^{-(n-2j)} = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^{2j} j!^2 (n-2j)!} q^{-(n-2j)} \\ &\le n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{(2j)!(n-2j)!} q^{-(n-2j)} = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} q^{-(n-2j)} \\ &\le n! \sum_{j=0}^{n} \binom{n}{j} q^{-(n-j)} = n!(1+q^{-1})^n.\end{aligned}$$

Hence,

$$\|m_n\|_{\mathcal{H}_q} \le \sqrt{n!} \cdot (1+q^{-1})^{n/2}.$$

On the other hand, by using Hölder's inequality and Lemma 4.1(d), we find that

$$\begin{aligned}&\left| \frac{1}{n!} \int_0^\infty (f\phi)^{(n)}(y)\,\mathrm{d}y \right| \\ &\qquad \le \frac{1}{n!} \int_0^\infty \sum_{k=0}^{n} \binom{n}{k} |f^{(k)}(y) \phi^{(n-k)}(y)|\,\mathrm{d}y \\ &\qquad = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty |f^{(k)}(y) h_{n-k}(y) \phi(y)|\,\mathrm{d}y \\ &\qquad \le \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \int_{\mathbb{R}} |f^{(k)}(y) h_{n-k}(y)|\,\mathrm{d}\gamma(y) \\ &\qquad \le \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \|f^{(k)}\|_{L^2(\mathbb{R},\gamma)} \cdot \|h_{n-k}\|_{L^2(\mathbb{R},\gamma)}\end{aligned}$$

$$= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \|f^{(k)}\|_{L^2(\mathbb{R},\gamma)} \sqrt{(n-k)!}$$
$$= \sum_{k=0}^{n} \frac{(q^{-1}-1)^{k/2}}{\sqrt{k!}} \|f^{(k)}\|_{L^2(\mathbb{R},\gamma)} \cdot \frac{1}{\sqrt{k!(n-k)!}(q^{-1}-1)^{k/2}}$$
$$\le \Big(\sum_{k=0}^{n} \frac{(q^{-1}-1)^k}{k!} \|f^{(k)}\|^2_{L^2(\mathbb{R},\gamma)}\Big)^{1/2} \cdot \Big(\sum_{k=0}^{n} \frac{1}{k!(n-k)!(q^{-1}-1)^k}\Big)^{1/2}$$
$$\le \|f\|_{\mathcal{H}_q} \Big(\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(q^{-1}-1)^k}\Big)^{1/2}$$
$$= \|f\|_{\mathcal{H}_q} \frac{1}{\sqrt{n!}} \Big(\frac{1}{1-q}\Big)^{n/2}.$$

Putting these computations together, we obtain

$$\|T^{(n)}\| \le \Big(\frac{1}{1-q}\Big)^{n/2} \cdot (1+q^{-1})^{n/2} = \Big(\frac{1+q^{-1}}{1-q}\Big)^{n/2}. \tag{4.1}$$

Thus, for $|\rho| < \sqrt{\frac{1-q}{1+q^{-1}}}$,

$$\sum_{n=0}^{\infty} \|\rho^n T^{(n)}\| < \infty.$$

The set of all bounded linear operators on $\mathcal{H}_q$ is a Banach space and thus $T_\rho = \sum_{n=0}^{\infty} \rho^n T^{(n)}$ is a bounded linear operator on $\mathcal{H}_q$.

For the compactness of T_ρ, note that $T^{(n)}$ is a rank 1 operator for all $n \in \mathbb{N}$, i.e. the range of $T^{(n)}$ is one-dimensional. As a finite-rank operator, $T^{(n)}$ is compact. The subset of all compact operators in the Banach space of bounded linear operators on $\mathcal{H}_q$ is itself a Banach space (see e.g. [14, III. Theorem 4.7]). Hence, $T_\rho = \sum_{n=0}^{\infty} \rho^n T^{(n)}$ is compact. ∎

Proof of Theorem 2.3. Let $-\sqrt{\frac{1-q}{1+q^{-1}}} < \rho < \sqrt{\frac{1-q}{1+q^{-1}}}$. We begin by relating the eigenvalue problem for T_ρ to the persistence problem for the MA(1)-process. First, note that $S_\rho^N(\mathbb{1}) = T_\rho^N(\mathbb{1})$ for all $N \in \mathbb{N}$. By [5, Section 2.1] we can rewrite the persistence probability as follows:

$$\mathbb{P}(X_0 \ge 0, \dots, X_N \ge 0) = \int_S T_\rho^N(\mathbb{1})(x_2)\, \mathrm{d}(\gamma \otimes \gamma)(x_1, x_2)$$

with $S := \{(x_1, x_2) \in \mathbb{R}^2 \colon \rho x_1 + x_2 \ge 0\}$. Let $r(T_\rho)$ be the spectral radius of T_ρ. We need to show that

$$\int_S T_\rho^N(\mathbb{1})(x_2)\, \mathrm{d}(\gamma \otimes \gamma)(x_1, x_2) = r(T_\rho)^{N+o(N)}.$$

A priori we cannot exclude that $r(T_\rho) = 0$. For the upper bound note that

$$\|f\|_{L^1(\mathbb{R},\gamma)} \le \|f\|_{L^2(\mathbb{R},\gamma)} \le \|f\|_{\mathcal{H}_q}$$

for all $f \in \mathcal{H}_q$ due to Lemma 4.1(d). Using this, we obtain

$$\int_S T_\rho^N(\mathbb{1})(x_2)\,\mathrm{d}(\gamma\otimes\gamma)(x_1,x_2) \le \|T_\rho^N(\mathbb{1})\|_{L^1(\mathbb{R},\gamma)} \le \|T_\rho^N(\mathbb{1})\|_{\mathcal{H}_q}$$
$$\le \|T_\rho^N\|\cdot\|\mathbb{1}\|_{\mathcal{H}_q} = r(T_\rho)^{N+o(N)},$$

where the last step is due to Gelfand's formula, i.e. $r(T_\rho) = \lim_{N\to\infty}\|T_\rho^N\|^{1/N}$. Now, we turn to the lower bound. We need to consider two cases. If $r(T_\rho) = 0$, then clearly

$$\int_S T_\rho^N(\mathbb{1})(x_2)\,\mathrm{d}(\gamma\otimes\gamma)(x_1,x_2) \ge r(T_\rho)^N.$$

If $r(T_\rho) > 0$, we show the lower bound by using the Krein–Rutman theorem (see [16], [12, Theorem 19.2]). For this purpose, let us define the cone

$$C := \{f \in \mathcal{H}_q \colon f(x) \ge 0 \text{ for all } x \in \mathbb{R}\}.$$

From [1, Proposition 1] it follows that $\mathcal{H}_q$ is a reproducing kernel Hilbert space with reproducing kernel

$$K_q(x,y) := \sum_{n=0}^{\infty} q^n \widehat{h}_n(x)\widehat{h}_n(y) = \frac{1}{\sqrt{1-q^2}} e^{-\frac{q^2x^2+q^2y^2-2qxy}{2(1-q^2)}},$$

where the last equality is due to Mehler's formula [20]. We have

$$K_q^y(\cdot) := K_q(\,\cdot\,,y) \in C$$

for all $y \in \mathbb{R}$. Since span $\{K_q^y \colon y \in \mathbb{R}\}$ is dense in $\mathcal{H}_q$ (see e.g. [7]), the closure of $C + (-C)$ is equal to $\mathcal{H}_q$. Further, $T_\rho(C) \subseteq C$. Therefore, the Krein–Rutman theorem can be applied and yields the existence of an eigenfunction $g \in C$ with eigenvalue $r(T_\rho)$. Note that any eigenfunction of T_ρ is bounded since

$$|T_\rho f(x)| \le \int_{-\rho x}^{\infty} |f(y)\phi(y)|\,\mathrm{d}y \le \|f\|_{L^1(\mathbb{R},\gamma)} \le \|f\|_{\mathcal{H}_q},$$

for all $f \in \mathcal{H}_q$ and $x \in \mathbb{R}$. Hence, $\|g\|_\infty < \infty$. We obtain

$$\int_S T_\rho^N(\mathbb{1})(x_2)\,\mathrm{d}(\gamma\otimes\gamma)(x_1,x_2) \ge \int_S T_\rho^N\Big(\frac{g}{\|g\|_\infty}\Big)(x_2)\,\mathrm{d}(\gamma\otimes\gamma)(x_1,x_2)$$
$$= r(T_\rho)^N \int_S \frac{g(x_2)}{\|g\|_\infty}\,\mathrm{d}(\gamma\otimes\gamma)(x_1,x_2)$$
$$= r(T_\rho)^{N+o(N)}.$$

Thus,

$$\mathbb{P}(X_0 \geq 0, \ldots, X_N \geq 0) = r(T_\rho)^{N+o(N)}.$$

Now, note that [5, Proposition 2.3] implies $r(T_\rho) > 0$. Hence, $\lambda_\rho := r(T_\rho) > 0$ is the largest eigenvalue of T_ρ by the Krein–Rutman theorem. Further, $\mathbb{P}(X_0 \geq 0, \ldots, X_N \geq 0) \leq \mathbb{P}(\min_{0\leq n\leq \lfloor N/2\rfloor} X_{2n} \geq 0)$. Note that the random variables $\{X_{2n} \colon 0 \leq n \leq \lfloor N/2\rfloor\}$ are independent. Hence,

$$\mathbb{P}(X_0 \geq 0, \ldots, X_N \geq 0) \leq \mathbb{P}(X_0 \geq 0)^{\lfloor N/2\rfloor+1},$$

which implies $\lambda_\rho < 1$.

It remains to show that λ_ρ admits a representation as a power series. We will see that this follows by [4, Theorem 4], which is based on the classical work of [14]. From the eigenvalue equation

$$\lambda f(x) = T_0 f(x) = \mathbb{1} \int_0^\infty f(y)\phi(y)\,\mathrm{d}y \quad \text{for all } x \in \mathbb{R}, \tag{4.2}$$

it follows that the largest eigenvalue of T_0 is given by $\lambda_0 = 1/2$. To obtain the analyticity in ρ of the eigenvalue at 0 by methods of perturbation theory, it is necessary to show that the algebraic multiplicity of λ_0 is 1.

For this purpose, let P_{λ_0} be the spectral projection of λ_0. The algebraic multiplicity is defined to be the dimension of $P_{\lambda_0}L^2(\mathbb{R},\gamma)$. Due to the compactness of T_0, we get $P_{\lambda_0}L^2(\mathbb{R},\gamma) = \ker(\lambda_0 - T_0)^v$, where $v \in \mathbb{N}$ is the smallest natural number such that $\ker(\lambda_0 - T_0)^v = \ker(\lambda_0 - T_0)^{v+1}$ (see e.g. [10]). From the eigenvalue equation (4.2), we see that $\ker(\lambda_0 - T_0)^1$ is equal to the constant functions and is therefore one-dimensional.

If we prove that $\ker(\lambda_0 - T_0)^1 = \ker(\lambda_0 - T_0)^2$, it follows that the algebraic multiplicity of λ_0 is 1. Let $g \in \ker(\lambda_0 - T_0)^2$. Then

$$\begin{aligned} 0 &= (\lambda_0 - T_0)^2(g) = (\lambda_0 - T_0)(\lambda_0 g - T_0(g)) \\ &= \lambda_0^2 g - 2\lambda_0 T_0(g) + T_0(T_0(g)). \end{aligned}$$

Since $T_0(g)$ is constant, the above equation implies that g is constant, i.e. $g \in \ker(\lambda_0 - T_0)^1$. By [4, Theorem 4], λ_ρ can be represented as a power series for $|\rho| < r_0$ for some $r_0 > 0$. The question of the radius of convergence is tricky and will be tackled in Section 5. In particular, the only thing that is still to be proved is the bound for the radius of convergence, which can be found by combining Corollary 5.8 with Lemma 5.9. ∎

Remark 4.2. We remark that the operator T_ρ is not normal. If it were, further results from perturbation theory would be applicable. In particular, a concrete bound for the radius of convergence would follow from [4, Corollary 6] together with the bound (4.1).

Proof of Theorem 2.4. The eigenvalue equation $T_\rho(f_\rho) = \lambda_\rho f_\rho$ reads

$$\sum_{k=0}^{\infty} \rho^k T^{(k)}\Big(\sum_{m=0}^{\infty} \rho^m g_m\Big) = \sum_{k=0}^{\infty} \rho^k K_k \cdot \sum_{m=0}^{\infty} \rho^m g_m.$$

Sorting this in powers of ρ (which is allowed within the radius of convergence) gives

$$\sum_{n=0}^{\infty} \rho^n \sum_{k=0}^{n} T^{(k)} g_{n-k} = \sum_{n=0}^{\infty} \rho^n \sum_{k=0}^{n} K_k g_{n-k}.$$

Since this holds for any $\rho \in (-\rho_0, \rho_0)$ with $\rho_0 > 0$, we must have

$$\sum_{k=0}^{n} T^{(k)} g_{n-k} = \sum_{k=0}^{n} K_k g_{n-k} \quad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

For $n = 0$, this is

$$T^{(0)} g_0 = K_0 g_0.$$

Since the left-hand side is a constant, by the definition of $T^{(0)} = T_0$, we know that g_0 must be constant, so without loss of generality (via multiplication of the eigenfunction by a scalar) take

$$g_0 = \mathbb{1}.$$

Fix $n \geq 1$. We now analyse the iterative structure of (4.3):

$$\sum_{k=1}^{n} T^{(k)} g_{n-k} + T^{(0)} g_n = K_n g_0 + \sum_{k=1}^{n-1} K_k g_{n-k} + K_0 g_n. \tag{4.4}$$

Observe that, by the definition of the operators $T^{(k)}$, the first term on the left is a polynomial. Further, the second term on the left and the first term on the right are constants. The second term on the right involves only the g_ℓ, $\ell < n$. Therefore, inductively we know that g_n is a polynomial of degree at most n.

Let us denote the coefficients of the polynomials g_ℓ by g_ℓ^i, i.e. $g_\ell(x) =: \sum_{i=0}^{\ell} g_\ell^i x^i$. Then equating the coefficients of x^0 in (4.4) gives

$$0 + T^{(0)} g_n = K_n g_0 + \sum_{k=1}^{n-1} K_k g_{n-k}^0 + K_0 g_n^0.$$

Further, by the definition of $T^{(0)}$ and the representation of g_n, we have

$$T^{(0)} g_n = \sum_{i=0}^{n} g_n^i \int_0^\infty y^i \phi(y)\, \mathrm{d}y.$$

Combining the last two equations and using $\int_0^\infty y^0 \phi(y)\, \mathrm{d}y = K_0$, we see that the terms involving g_n^0 cancel, giving

$$\sum_{i=1}^{n} g_n^i \int_0^\infty y^i \phi(y)\,\mathrm{d}y = K_n g_0 + \sum_{k=1}^{n-1} K_k g_{n-k}^0. \tag{4.5}$$

None of the other polynomial terms in (4.4) uses g_n^0 either. Therefore, g_n^0 is in fact arbitrary and can be chosen to be zero. This however simplifies (4.4) in the sense that

$$g_n = \frac{1}{K_0}\Big(\sum_{k=1}^{n} T^{(k)} g_{n-k} - \sum_{k=1}^{n-1} K_k g_{n-k}\Big), \tag{4.6}$$

because these are the parts of (4.4) involving the coefficients of x^i, $i > 0$, only. Note that (4.6) computes g_n with the help of g_ℓ and K_ℓ for $\ell < n$ only.

Similarly, now (4.5) simplifies (using $g_\ell^0 = 0$ and $g_0 = \mathbb{1}$) to (2.4). Putting this back into (4.6) shows (2.3). ■

5. Radius of convergence. This section is devoted to the study of the radius of convergence of the series in Theorem 2.3. In particular, we will finish the proof of that theorem by showing that the radius of convergence is at least 0.332.

For this purpose, we first give an alternative description of the leading eigenvalue of the eigenvalue equation (1.3), which might be interesting in its own right; see (5.11). We also show that the two descriptions must have the same radius of convergence. Using the alternative description, we can prove a lower bound for the radius of convergence; see Corollary 5.8.

We start with a reformulation of the eigenvalue equation (1.3) that gets rid of the eigenfunction.

LEMMA 5.1. *Let $\rho \in [0,1]$. The leading eigenvalue $\lambda = \lambda_\rho$ of the eigenvalue equation* (1.3) *is the largest positive root of the equation*

$$\lambda = \sum_{k=0}^{\infty} \frac{\kappa_k(\rho)}{\lambda^k}, \tag{5.1}$$

where $\kappa_0 \equiv 1/2$, and for $k \geq 1$,

(5.2)

$$\begin{aligned}\kappa_k(\rho) &:= \frac{1}{(2\pi)^{\frac{k+1}{2}}} \int_0^\infty \int_{-\rho s_0}^0 \cdots \int_{-\rho s_{k-2}}^0 \int_{-\rho s_{k-1}}^0 \exp\Big(-\frac{1}{2}\sum_{i=0}^{k} s_i^2\Big)\,\mathrm{d}s_k \ldots \mathrm{d}s_0 \\ &= \frac{\rho^{\frac{k(k+1)}{2}}}{(2\pi)^{\frac{k+1}{2}}} \int_0^\infty \int_{-s_0}^0 \cdots \int_{-s_{k-2}}^0 \int_{-s_{k-1}}^0 \exp\Big(-\frac{1}{2}\sum_{i=1}^{k} (\rho^i s_i)^2\Big) e^{-\frac{s_0^2}{2}}\,\mathrm{d}s_k \ldots \mathrm{d}s_0.\end{aligned}$$

Before we give the proof of Lemma 5.1, we note a useful technical bound for the coefficients (κ_k).

LEMMA 5.2. *For the (κ_k) defined in* (5.2), *we have, for all $k \ge 0$,*

(5.3)
$$|\kappa_k(\rho)| = \frac{\rho^{\frac{k(k+1)}{2}}}{(2\pi)^{\frac{k+1}{2}}} \int_0^\infty \int_0^{s_0} \ldots \int_0^{s_{k-2}} \int_0^{s_{k-1}} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}(\rho^i s_i)^2\right) e^{-\frac{s_0^2}{2}} \,\mathrm{d}s_k \ldots \mathrm{d}s_0$$
$$\le \frac{\rho^{\frac{k(k+1)}{2}}}{2^{k+1}\pi^{\frac{k}{2}}} \frac{1}{\Gamma\big(\frac{k}{2}+1\big)}.$$

Proof. Bounding all the exponentials in (5.3) by 1, except the last one, we obtain

$$|\kappa_k(\rho)| \le \frac{\rho^{\frac{k(k+1)}{2}}}{(2\pi)^{\frac{k+1}{2}} k!} \int_0^\infty s^k e^{-\frac{s^2}{2}} \,\mathrm{d}s = \frac{\rho^{\frac{k(k+1)}{2}}}{2\pi^{\frac{k+1}{2}} k!} \Gamma\left(\frac{k+1}{2}\right) = \frac{\rho^{\frac{k(k+1)}{2}}}{2^{k+1}\pi^{\frac{k}{2}}} \frac{1}{\Gamma\big(\frac{k}{2}+1\big)},$$

where we have used the Legendre duplication formula

$$\Gamma(2t) = 2^{2t-1}\Gamma(t)\Gamma(t+1/2)/\sqrt{\pi}$$

for $t = (k+1)/2$ in the last step. ∎

REMARK 5.3. A result similar to Lemma 5.2 (and subsequently a result similar to Lemma 5.1) can be proved in the same way as long as the density ϕ has a superexponential decay.

REMARK 5.4. We cannot expect explicit values for the coefficients $\kappa_k(\rho)$. In fact, they are persistence probabilities themselves:

$$\kappa_k(\rho) = \mathbb{P}(0 \le \xi_0 \le \rho\xi_1 \le \cdots \le \rho^k \xi_k),$$

where the (ξ_i) are independent random variables with density ϕ.

Proof of Lemma 5.1. The only thing to do is to transform the eigenvalue equation:

$$\lambda f(x) = \int_{-\rho x}^{\infty} f(s)\phi(s)\,\mathrm{d}s = \int_0^\infty f(s)\phi(s)\,\mathrm{d}s + \int_{-\rho x}^{0} f(s)\phi(s)\,\mathrm{d}s, \qquad x \in \mathbb{R}. \tag{5.4}$$

We may assume $f(0) := 1$. Inserting this into (5.4) gives

$$\lambda = \int_0^\infty f(s)\phi(s)\,\mathrm{d}s. \tag{5.5}$$

Replacing the corresponding term in (5.4) gives

$$f(x) = 1 + \lambda^{-1} \int_{-\rho x}^{0} f(s)\phi(s)\,\mathrm{d}s. \tag{5.6}$$

Inserting (5.6) into (5.5) we obtain

$$\begin{aligned}\lambda &= \int_0^\infty \Big[1+\lambda^{-1}\int_{-\rho s_0}^0 f(s_1)\phi(s_1)\,\mathrm{d}s_1\Big]\phi(s_0)\,\mathrm{d}s_0\\ &= \int_0^\infty \phi(s_0)\,\mathrm{d}s_0 + \lambda^{-1}\int_0^\infty\int_{-\rho s_0}^0 f(s_1)\phi(s_1)\,\mathrm{d}s_1\,\phi(s_0)\,\mathrm{d}s_0\\ &= \frac{1}{2}+\lambda^{-1}\int_0^\infty\int_{-\rho s_0}^0 f(s_1)\phi(s_1)\,\mathrm{d}s_1\,\phi(s_0)\,\mathrm{d}s_0.\end{aligned}$$

Iterating this procedure gives

$$\begin{aligned}\lambda = {} & \sum_{k=0}^{N-1}\lambda^{-k}\kappa_k(\rho)\\ & + \lambda^{-N}\int_0^\infty\int_{-\rho s_0}^0\cdots\int_{-\rho s_{N-1}}^0 f(s_N)\phi(s_N)\,\mathrm{d}s_N\ldots\phi(s_1)\,\mathrm{d}s_1\phi(s_0)\,\mathrm{d}s_0.\end{aligned}$$

We now let $N\to\infty$ and show that the remainder term vanishes. For this purpose, we use that the eigenfunction f is bounded and we employ the bound from Lemma 5.2:

$$\begin{aligned}\Big|\lambda^{-N}\int_0^\infty\int_{-\rho s_0}^0\cdots\int_{-\rho s_{N-1}}^0 f(s_N)\phi(s_N)\,\mathrm{d}s_N\ldots\phi(s_1)\,\mathrm{d}s_1\phi(s_0)\,\mathrm{d}s_0\Big|\\ \le \|f\|_\infty\cdot\frac{\mathrm{const}^N}{\Gamma(N/2+1)}\to 0.\end{aligned}$$ ∎

We are going to rewrite the eigenvalue equation in the following way. Define

$$\Psi_\rho(\xi) := \sum_{k=1}^\infty 2^{k+1}\kappa_k(\rho)(1+\xi)^{-k}.$$

Note that finding solutions of (5.1) for $\lambda\ge 1/2$ is equivalent to finding solutions of the following equation for $\xi\ge 0$:

$$\xi=\Psi_\rho(\xi),\qquad \lambda=\frac{1}{2}(1+\xi). \tag{5.7}$$

That means we are looking for the largest root $\xi\ge 0$ of (5.7). Next, we define

$$\Gamma_\rho(\xi) := \frac{\xi}{\Psi_\rho(\xi)},\qquad \xi\ge 0.$$

Let us state some important properties of the function Γ_ρ that will allow us to invert this function.

Lemma 5.5. *We have $\Gamma_\rho(0)=0$ and $\lim_{\xi\to\infty}\Gamma_\rho(\xi)=\infty$. Further, for $\rho\in[0,1]$, the function $\xi\mapsto\Gamma_\rho(\xi)$ is strictly increasing.*

Proof. The first two properties follow immediately from Lemma 5.2. To see the last property, note that the derivative of Γ_ρ has the same sign as

$$\begin{aligned}
(5.8)\quad & \Psi_\rho(\xi) - \xi\Psi'_\rho(\xi) \\
&= \sum_{k=1}^{\infty} 2^{k+1}\kappa_k(\rho)(1+\xi)^{-k} + \sum_{k=1}^{\infty} 2^{k+1}\kappa_k(\rho)k\xi(1+\xi)^{-k-1} \\
&= 4\kappa_1(\rho)\frac{1+2\xi}{(1+\xi)^2} + \sum_{k=2}^{\infty} 2^{k+1}\kappa_k(\rho)\frac{1+(k+1)\xi}{(1+\xi)^{k+1}} \\
&\geq 4\kappa_1(\rho)\frac{1+2\xi}{(1+\xi)^2} - \sum_{k=2}^{\infty} \frac{\rho^{\frac{k(k+1)}{2}}}{\pi^{k/2}}\,\frac{1}{\Gamma\big(\frac{k}{2}+1\big)}\,\frac{1+(k+1)\xi}{(1+\xi)^{k+1}} \\
&= 4\,\frac{1+2\xi}{(1+\xi)^2}\left[\frac{1}{2\pi}\arctan(\rho) - \sum_{k=2}^{\infty}\frac{\rho^{\frac{k(k+1)}{2}}}{\pi^{k/2}}\,\frac{1}{\Gamma\big(\frac{k}{2}+1\big)}\,\frac{1+(k+1)\xi}{(1+\xi)^{k-1}(1+2\xi)}\right],
\end{aligned}$$

where differentiating under the sum in the first step is allowed by Lemma 5.2 and we note that

$$(5.9)\qquad \kappa_1(\rho) = \frac{1}{2\pi}\int_0^\infty\int_{-\rho z}^0 e^{-x^2/2}e^{-z^2/2}\,\mathrm{d}x\,\mathrm{d}z = \frac{1}{2\pi}\arctan(\rho) > 0.$$

To see the last equality, differentiate with respect to ρ. Now note that, for $k \geq 2$, the function

$$\xi \mapsto \frac{1+(k+1)\xi}{(1+\xi)^{k-1}(1+2\xi)}$$

is strictly decreasing, which can be seen by differentiation. Therefore, we can replace it in (5.8) by its value at $\xi = 0$ and thus get the estimate

$$\begin{aligned}
\Psi_\rho(\xi) - \xi\Psi'_\rho(\xi) &\geq 4\,\frac{1+2\xi}{(1+\xi)^2}\left[\frac{1}{2\pi}\arctan(\rho) - \sum_{k=2}^{\infty}\frac{\rho^{\frac{k(k+1)}{2}}}{\pi^{k/2}}\,\frac{1}{\Gamma\big(\frac{k}{2}+1\big)}\right] \\
&\geq 4\,\frac{1+2\xi}{(1+\xi)^2}\left[\frac{1}{2\pi}\arctan(\rho) - \sum_{k=2}^{\infty}\frac{\rho^{k+1}}{\pi^{k/2}}\right] \\
&= 4\,\frac{1+2\xi}{(1+\xi)^2}\left[\frac{1}{2\pi}\arctan(\rho) - \frac{\rho^3}{\sqrt{\pi}(\sqrt{\pi}-\rho)}\right],
\end{aligned}$$

which is strictly positive for all $\rho \in (0,1]$ and all $\xi \geq 0$. ∎

COROLLARY 5.6. *The largest solution of* (5.1) *admits the following representation*

$$(5.10)\qquad \lambda = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\partial^{n-1}}{\partial\xi^{n-1}}[\Psi_\rho(\xi)^n]\right)\bigg|_{\xi=0}.$$

Proof. We use the Lagrange–Bürmann formula, a variant of the Lagrange inversion formula. Knowing that $\Gamma_\rho(0) = 0$, $\Gamma_\rho(\infty) = \infty$, and that it is strictly increasing, we can solve the equation $\Gamma_\rho(\xi) = 1$, i.e. compute the inverse of Γ_ρ at 1. The result obtained from the Lagrange–Bürmann formula is precisely the statement of the corollary. Note that (5.1) admits several solutions in λ, while here we show that there is only one solution in $\xi \geq 0$, i.e. one solution in λ with $\lambda \geq 1/2$. ■

COROLLARY 5.7. *The largest solution of* (5.1) *admits the following representation:*

$$\lambda = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{n-1}}{n!} \times \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \Big(\prod_{j=1}^{n} 2^{k_j} \kappa_{k_j}(\rho)\Big) \frac{\Gamma(k_1+\cdots+k_n+n-1)}{\Gamma(k_1+\cdots+k_n)}. \tag{5.11}$$

Proof. To compute the successive derivatives in (5.10), we write Ψ_ρ as a Laplace transform

$$\Psi_\rho(\xi) = \sum_{k=1}^{\infty} 2^{k+1}\kappa_k(\rho)\frac{1}{\Gamma(k)} \int_0^\infty e^{-\xi x} e^{-x} x^{k-1}\,\mathrm{d}x = \int_0^\infty e^{-\xi x}\psi_\rho(x)\,\mathrm{d}x,$$

where we can exchange the sum and integral, due to Lemma 5.2, and we set

$$\psi_\rho(x) := e^{-x}\sum_{k=1}^{\infty} \frac{2^{k+1}}{\Gamma(k)}\kappa_k(\rho)x^{k-1}.$$

As a consequence,

$$\begin{aligned}\Big(\frac{\partial^{n-1}}{\partial\xi^{n-1}}(\Psi_\rho(\xi))^n\Big)\Big|_{\xi=0} &= \Big(\int_0^\infty (-1)^{n-1}x^{n-1}e^{-\xi x}\psi_\rho^{*(n)}(x)\,\mathrm{d}x\Big)\Big|_{\xi=0}\\ &= (-1)^{n-1}\int_0^\infty x^{n-1}\psi_\rho^{*(n)}(x)\,\mathrm{d}x,\end{aligned}$$

where $*$ denotes the usual convolution product. For the first term, we have

$$\psi_\rho^{*(1)}(x) = \psi_\rho(x) = e^{-x}\sum_{k_1=1}^{\infty}\frac{2^{k_1+1}}{\Gamma(k_1)}\kappa_{k_1}(\rho)x^{k_1-1}.$$

For the second term, we get

$$\begin{aligned}\psi_\rho^{*(2)}(x) &= e^{-x}\int_0^x \sum_{k_1=1}^{\infty}\frac{2^{k_1+1}}{\Gamma(k_1)}\kappa_{k_1}(\rho)(x-y)^{k_1-1}\sum_{k_2=1}^{\infty}\frac{2^{k_2+1}}{\Gamma(k_2)}\kappa_{k_2}(\rho)y^{k_2-1}\,\mathrm{d}y\\ &= e^{-x}\sum_{k_1=1}^{\infty}\frac{2^{k_1+1}}{\Gamma(k_1)}\kappa_{k_1}(\rho)\sum_{k_2=1}^{\infty}\frac{2^{k_2+1}}{\Gamma(k_2)}\kappa_{k_2}(\rho)x^{k_1+k_2-1}\int_0^1 (1-y)^{k_1-1}y^{k_2-1}\,\mathrm{d}y\\ &= e^{-x}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} 2^{k_1+k_2+2}\kappa_{k_1}(\rho)\kappa_{k_2}(\rho)\frac{x^{k_1+k_2-1}}{\Gamma(k_1+k_2)},\end{aligned}$$

where exchanging the sums and the integral is permitted by Lemma 5.2. By induction, we obtain

$$\psi^{*(n)}(x) = e^{-x} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} 2^{k_1+\cdots+k_n+n} \Big(\prod_{j=1}^{n} \kappa_{k_j}(\rho)\Big) \cdot \frac{x^{k_1+\cdots+k_n-1}}{\Gamma(k_1+\cdots+k_n)}$$

and so

$$\int_0^\infty x^{n-1}\psi_\rho^{*(n)}(x)\,\mathrm{d}x = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \Big(\prod_{j=1}^{n} 2^{k_j+1}\kappa_{k_j}(\rho)\Big) \cdot \frac{\Gamma(k_1+\cdots+k_n+n-1)}{\Gamma(k_1+\cdots+k_n)}. \quad \blacksquare$$

Together with the bound in Lemma 5.2, it is now possible to obtain a bound for the radius of convergence – despite the fact that we cannot determine the $\kappa_n(\rho)$ explicitly.

Corollary 5.8. *The representation in* (5.11) *converges absolutely for* $|\rho| < 0.332$.

Proof. Observe that the coefficient $\kappa_k(\rho)$ admits a series expansion of the form

$$\kappa_k(\rho) =: \frac{\rho^{\frac{k(k+1)}{2}}}{(2\pi)^{\frac{k+1}{2}}} \sum_{n=0}^{\infty} \psi_n^{(k)} \rho^n.$$

By taking absolute values everywhere, we thus obtain the following bound for the absolute value of the term in (5.11):

$$\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \Bigg(\prod_{j=1}^{n} \frac{\rho^{\frac{k_j(k_j+1)}{2}} 2^{k_j}}{(2\pi)^{\frac{k_j+1}{2}}} \sum_{p=0}^{\infty} |\psi_p^{(k_j)}|\,|\rho|^p\Bigg) \times \frac{\Gamma(k_1+\cdots+k_n+n-1)}{\Gamma(k_1+\cdots+k_n)}. \tag{5.12}$$

For the remainder of this proof, we consider $\rho > 0$ only to avoid the absolute value signs. To compute the multiple sum, we first bound the ratio of Gamma functions. Note that, by the binomial theorem, for $x \in \mathbb{N}$,

$$\frac{\Gamma(x+n-1)}{\Gamma(x)} = \frac{(x+n-2)!}{(x-1)!} = (n-1)!\,\frac{(x+n-2)!}{(n-1)!(x-1)!} = (n-1)!\binom{x+n-2}{n-1} \le (n-1)!\,2^{x+n-2}.$$

As a consequence, using the Fubini–Tonelli theorem, we get the upper bound for the term in (5.12):

(5.13)

$$\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{n!}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\bigg(\prod_{j=1}^{n}\frac{\rho^{\frac{k_j(k_j+1)}{2}}4^{k_j}}{(2\pi)^{\frac{k_j+1}{2}}}\sum_{p=0}^{\infty}|\psi_p^{(k_j)}|\rho^p\bigg)(n-1)!\,2^{n-2}$$
$$=\frac{1}{2}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{4^n}{n}\bigg(\sum_{k=1}^{\infty}\frac{\rho^{\frac{k(k+1)}{2}}4^k}{(2\pi)^{\frac{k+1}{2}}}\sum_{p=0}^{\infty}|\psi_p^{(k)}|\rho^p\bigg)^n.$$

We thus arrive at the following sufficient condition for the sum in (5.13) to converge:

$$\sum_{k=1}^{\infty}\frac{\rho^{\frac{k(k+1)}{2}}4^k}{(2\pi)^{\frac{k+1}{2}}}\sum_{p=0}^{\infty}|\psi_p^{(k)}|\rho^p<\frac{1}{4}. \tag{5.14}$$

We now study $\sum_{p=0}^{\infty}|\psi_p^{(k)}|\rho^p$. For $k=1$, we have, due to (5.9),

$$\kappa_1(\rho)=\frac{\arctan(\rho)}{2\pi}=\frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\rho^{2n+1}}{2n+1};$$

hence, the definition $\kappa_1(\rho)=\frac{\rho}{2\pi}\sum_{p=0}^{\infty}\psi_p^{(1)}\rho^p$ yields

$$\sum_{n=0}^{\infty}|\psi_n^{(1)}|\rho^n=\sum_{n=0}^{\infty}\frac{\rho^{2n}}{2n+1}=\frac{1}{2\rho}\log\Big(\frac{1+\rho}{1-\rho}\Big).$$

Turning to the terms for $k\geq 2$ in (5.14), similarly to the proof of Lemma 5.2, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}|\psi_n^{(k)}|\rho^n &\leq \int_0^{\infty}\int_0^{s_0}\cdots\int_0^{s_{k-1}}\int_0^{s_k}\exp\Big(\frac{1}{2}\sum_{i=1}^{k}(\rho^i s_i)^2\Big)e^{-\frac{s_0^2}{2}}\,\mathrm{d}s_k\ldots\mathrm{d}s_0\\ &\leq \int_0^{\infty}\frac{s^k}{k!}\exp\Big(\frac{1}{2}\sum_{i=1}^{k}(\rho^i s)^2\Big)e^{-\frac{s^2}{2}}\,\mathrm{d}s\\ &\leq \int_0^{\infty}\frac{s^k}{k!}\exp\Big(-\frac{s^2}{2}\frac{1-2\rho^2}{1-\rho^2}\Big)\,\mathrm{d}s\\ &=\frac{1}{2\cdot k!}\Gamma\Big(\frac{k+1}{2}\Big)2^{\frac{k+1}{2}}\Big(\frac{1-\rho^2}{1-2\rho^2}\Big)^{\frac{k+1}{2}}.\end{aligned} \tag{5.15}$$

Therefore, (5.14) will be implied by

$$\frac{1}{\pi}\log\Big(\frac{1+\rho}{1-\rho}\Big)+\sum_{k=2}^{\infty}\frac{\rho^{\frac{k(k+1)}{2}}4^k}{\pi^{\frac{k+1}{2}}}\frac{1}{2\cdot k!}\Gamma\Big(\frac{k+1}{2}\Big)\Big(\frac{1-\rho^2}{1-2\rho^2}\Big)^{\frac{k+1}{2}}<\frac{1}{4}.$$

One can check numerically that this is true at least for all $\rho<0.332$. ∎

Note that the bound in the last corollary can be improved by evaluating more terms in the sum (5.14) explicitly instead of using the estimate (5.15).

LEMMA 5.9. *The representation in* (5.11) *holds if and only if the representation* $\lambda_\rho = \sum_{i=0}^{\infty} K_i \rho^i$ *holds, where the* (K_i) *are as in Theorem* 2.3.

Proof. First observe that the $\kappa_k(\rho)$ can be written as a series in ρ. Now, within the radius of convergence, one may rearrange all sums in (5.11). Therefore, the representation in (5.11) may be rewritten as a series in ρ. Since the representation $\lambda_\rho = \sum_{i=0}^{\infty} K_i \rho^i$ is analytic in a neighborhood of 0, the coefficients of the two series have to agree. Therefore, their radii of convergence have to be identical. ∎

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