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p-Integrality of canonical coordinates

by

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Abstract. Let L be a differential operator with coefficients in $\mathbb{Q}(z)$ of order $n \geq 2$ with maximal unipotent monodromy at zero. We are interested in determining when the canonical coordinate of L belongs to $\mathbb{Z}_p[[z]]$. For this purpose, motivated by a recent conjecture due to P. Candelas, X. de la Ossa and D. van Straten (2021), we study the situation when L has a strong Frobenius structure $\Phi = (\phi_{i,j})_{1\leq i,j\leq n} \in M_n(\mathbb{Z}_p[[z]])$ such that $\phi_{1,1}(0) = 1$. We then give a necessary and sufficient condition for the canonical coordinate of L to belong to $\mathbb{Z}_p[[z]]$ when L has such a strong Frobenius structure.

1. Introduction. In this paper we study the p-integrality of the *canonical coordinate* of a differential operator with *maximal unipotent monodromy* at zero. We recall that

$$L = \delta^n + a_{n-1}(z)\delta^{n-1} + \cdots + a_1(z)\delta + a_0(z) \in \mathbb{Q}(z)[\delta], \qquad \delta = z\frac{d}{dz},$$

has maximal unipotent monodromy at zero (*MUM type*) if, for all $i \in \{0, \ldots, n-1\}$, $a_i(z) \in \mathbb{Q}(z) \cap z\mathbb{Q}[[z]]$. By the Frobenius method, we know that if L is of MUM type and $n \geq 2$ then there are unique power series $\mathfrak{f}(z) \in 1 + z\mathbb{Q}[[z]]$ and $\mathfrak{g}(z) \in z\mathbb{Q}[[z]]$ such that

$$y_0 = \mathfrak{f}(z) \quad \text{and} \quad y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$$

are solutions of L.

The canonical coordinate of L is the power series

$$q(z) := \exp(y_1/y_0) = z\exp(\mathfrak{g}(z)/\mathfrak{f}(z)) = z\bigg(1 + \sum_{j\geq 1} \frac{1}{j!}(\mathfrak{g}(z)/\mathfrak{f}(z))^n\bigg) \in \mathbb{Q}[[z]].$$

This power series is also often called the *q-coordinate* of L. We are interested in determining the *p-integrality* of $y_0(z)$ and $q(z)$, that is, we want to know when $y_0(z)$ and $q(z)$ belong to $\mathbb{Z}_p[[z]]$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

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Our main source of motivation comes from a recent conjecture formulated by P. Candelas, X. de la Ossa and D. van Straten [6, Section 4.1] about *Calabi–Yau differential operators*. A typical example of a Calabi–Yau differential operator is

$$\mathcal{H} = \delta^4 - 5z(5\delta+1)(5\delta+2)(5\delta+3)(5\delta+4).$$

This differential operator appears in the work of Candelas et al. [5], where they study a mirror family for quintic threefolds in $\mathbb{P}^4$. The differential operator $\mathcal{H}$ satisfies some *algebraic properties*, MUM being one of them, and also satisfies some *arithmetic properties*, namely, F_0 and $\exp(F_1/F_0)$ belong to $\mathbb{Z}[[z]]$, where F_0 and F_1 are solutions of $\mathcal{H}$ given by

$$F_0 = 1 + \sum_{n\geq 1} \frac{(5n)!}{(n!)^5} z^n \quad \text{and} \quad F_1 = F_0 \log z + G(z),$$

with

$$G(z) = \sum_{n\geq 1} \frac{(5n)!}{(n!)^5}(5H_{5n} - 5H_n)z^n \quad \text{and} \quad H_n = \sum_{i=1}^{n} 1/i.$$

It is clear that $F_0 \in \mathbb{Z}[[z]]$, and it was proven by Lian and Yau [22] that $\exp(F_1/F_0) \in \mathbb{Z}[[z]]$. The second fact is very surprising as $G(z)$ is not integral because it has unbounded denominators. In addition, it seems that for many differential operators L of MUM type, y_0 and $\exp(y_1/y_0)$ are N-integral ([1]). These operators are usally called *Calabi–Yau operators*. We refer the reader to [4, Definition 6.5] for a precise definition.

So, a natural question is to determine when a differential operator is of Calabi–Yau type. Almkvist et al. [1] have gathered more than 400 differential operators of order 4 which are good candidates to be Calabi–Yau operators. For many differential operators appearing in this list they also give the analytic solution y_0 and it turns out for numerous cases that y_0 belongs to $\mathbb{Z}[[z]]$. Moreover, from Krattenthaler and Rivoal [21] and Delaygue [12], we also know that the canonical coordinate of some differential operators of this list belongs to $\mathbb{Z}[[z]]$.

It is also expected that every differential operator appearing in [1] can be obtained as a *Picard–Fuchs operator* associated with families of *Calabi–Yau threefolds* in a one-parameter family. Following [19, Theorem 22.2.1] or [2, p. 111], if L is a Picard–Fuchs operator then L is equipped with a matrix $\Phi_p = (\phi_{i,j})_{1\leq i,j\leq n}$ with coefficients in E_p, called a *strong Frobenius structure* ([2]). Recently, P. Candelas, X. de la Ossa and D. van Straten [6,

([1]) A power series $f(z) \in \mathbb{Q}[[z]]$ is said to be *N-integral* if there exists a non-zero $N \in \mathbb{Q}$ such that $f(Nz) \in \mathbb{Z}[[z]]$.

([2]) The field E_p is the field of analytic elements. In Section 2 we give the definition of E_p and we also give the definition of strong Frobenius structure there.

Section 4.1] formulated a conjecture about the strong Frobenius structure for differential operators associated with a family of Calabi–Yau threefolds in a one-parameter family.

CONJECTURE 1.1 ([6]). *Let $\mathcal{L} \in \mathbb{Q}(z)[\delta]$ be a differential operator associated with a family of Calabi–Yau threefolds in a one-parameter family. Then, for almost every prime number p, there exists a strong Frobenius structure $\Phi_p = (\phi_{i,j})_{1\leq i,j\leq 4}$ for $\mathcal{L}$ such that $\Phi_p \in M_4(\mathbb{Z}_p[[z]])$ and $\phi_{1,1}(0) = 1$, $\phi_{1,2}(0) = 0 = \phi_{1,3}(0)$, and $\phi_{1,4}(0) = p^3\lambda\zeta_p(3)$, where $\zeta_p(3)$ denotes the p-adic analog of $\zeta(3)$, and λ is a rational number independent of p.*

As already mentioned, Conjecture 1.1 is expected to be true for the differential operators appearing in [1] and it is also expected that such differential operators are of Calabi–Yau type. So, an interesting question is to determine if Conjecture 1.1 implies the N-integrality of $y_0(z)$ and $q(z)$. The initial motivation of this paper is to answer a weaker question, namely, whether Conjecture 1.1 implies the p-integrality of $y_0(z)$ and $q(z)$. On the one hand, we show that the existence of a strong Frobenius structure $\Phi_p \in M_4(\mathbb{Z}_p[[z]])$ implies $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$. On the other hand, under the assumption $\phi_{1,1}(0) = 1$, we give a necessary and sufficient condition for $q(z) \in \mathbb{Z}_p[[z]]$. This condition relies on certain properties of a *p-integral Frobenius structure* for a differential operator of order 2.

We point out that, thanks to Proposition 5.2, if $\mathcal{L} \in \mathbb{Q}(z)[\delta]$ is an irreducible MUM Picard–Fuchs operator of order 4 then, for almost every prime p, there exists a strong Frobenius structure $\Phi_p = (\phi_{i,j}(z))_{1\leq i,j\leq 4} \in M_4(\mathbb{Z}_p[[z]])$ for $\mathcal{L}$ such that $\|\Phi_p\| = 1$. Since a strong Frobenius structure is unique up to a constant (see [17]), Conjecture 1.1 says that we should have $\phi_{1,1}(0) = 1$. Furthermore, following Dwork [14, Lemma 6.2, (6.16)], we know that, for almost every prime p, there is a strong Frobenius structure $\Gamma_p = (\gamma_{i,j}(z))_{1\leq i,j\leq 4} \in M_4(E_p)$ for $\mathcal{L}$ such that $\gamma_{1,1}(0)^2 = 1$. Since a strong Frobenius structure is unique up to a constant, Conjecture 1.1 claims that we should have $\Phi_p = \frac{1}{\gamma_{1,1}(0)}\Gamma_p$.

1.1. Main result. Recall that for a differential operator

$$L = \delta^n + a_{n-1}(z)\delta^{n-1} + \cdots + a_1(z)\delta + a_0(z)$$

in $\mathbb{Q}(z)[\delta]$ the *companion matrix* of L is

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0(z) & -a_1(z) & -a_2(z) & \ldots & -a_{n-2}(z) & -a_{n-1}(z) \end{pmatrix}.$$

Definition 1.2 (p-integral Frobenius structure). Let L be a monic differential operator of order n in $\mathbb{Q}(z)[\delta]$ and let $A(z)$ be the companion matrix of L. A *p-integral Frobenius structure* for L is a matrix $\Phi \in M_n(\mathbb{Z}_p[[z]])$ such that $\det(\Phi) \neq 0$ and

$$\delta\Phi = A(z)\Phi - p\Phi A(z^p).$$

This definition is inspired by the definition of a *strong Frobenius structure* introduced by Dwork [15]. The reader can find this definition in Section 2. The relation between both definitions is given by Proposition 5.2, where it is shown that if $L \in \mathbb{Q}(z)[\delta]$ is MUM, irreducible and equipped with a Frobenius structure for almost every prime p then L has a p-integral Frobenius structure for almost every prime p.

In order to state our main result, we need to introduce some notations. With every MUM differential operator $L \in \mathbb{Q}(z)[\delta]$ of order $n \geq 2$ there is associated a MUM type differential operator $L^{(2)} \in \mathbb{Q}_p[[z]][\delta]$ of order 2 which is uniquely determined by the fact that $y_0 = \mathfrak{f}(z)$ and $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ are its solutions at zero. In other words, $L^{(2)} = (\delta - t_2)(\delta - t_1)$, where

$$t_1 = \frac{\delta\mathfrak{f}(z)}{\mathfrak{f}(z)}, \quad t_2 = \frac{\delta\mathfrak{h}(z)}{\mathfrak{h}(z)} \quad \text{with } \mathfrak{h}(z) = \mathfrak{f}(z) + \delta\mathfrak{g}(z) - t_1\mathfrak{g}(z).$$

Finally, we consider the $\mathbb{Q}_p$-linear operator $\Lambda_p : \mathbb{Q}_p[[z]] \to \mathbb{Q}_p[[z]]$ given by $\Lambda_p(\sum_{n\geq 0} a(n)z^n) = \sum_{n\geq 0} a(np)z^n$. This $\mathbb{Q}_p$-linear operator is often called the *Cartier operator*.

We are now ready to state our main result.

Theorem 1.3. *Let L be a differential operator with coefficients in $\mathbb{Q}(z)$ of order $n \geq 2$ and of MUM type. Suppose that $L \in \mathbb{Z}_p[[z]][\delta]$ and that $\Phi = (\phi_{i,j}(z))_{1\leq i,j\leq n}$ is a p-integral Frobenius structure for L. Then $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$. Moreover, if $|\phi_{1,1}(0)| = 1$ then*

(1) *the differential operator $L^{(2)}$ belongs to $\mathbb{Z}_p[[z]][\delta]$ and has a p-integral Frobenius structure $\Psi = (\psi_{i,j}(z))_{1\leq i,j\leq 2}$ such that $\Psi(0) = \mathrm{diag}(1, p)$,*
(2) *the following statements are equivalent:*
   - (a) $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$,
   - (b) $y_0(z) = \psi_{1,1}(z)y_0(z^p) \bmod p$,
   - (c) $\Lambda_p(\psi_{1,1}(z))y_0(z) = \Lambda_p(y_0(z)) \bmod p$.

Let us make a few comments.

• The condition $L \in \mathbb{Z}_p[[z]][\delta]$ is satisfied for almost every prime p because L has coefficients in $\mathbb{Q}(z)\cap\mathbb{Q}[[z]]$. Moreover, as a consequence of Theorem 1.3, we prove in Corollary 5.1 that if L is an irreducible Picard–Fuchs equation then $y_0(z)$ belongs to $1 + z\mathbb{Z}_p[[z]]$ for almost every prime p.

• An explicit expression for Ψ is given in the proof of Theorem 1.3. This expression depends on the power series $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$.

• According to Theorem 6.7, the condition $|\phi_{1,1}(0)|=1$ implies $\exp(y_1/y_0)^p \in \mathbb{Z}_p[[z]]$.

• Thanks to Lucas' Theorem, the analytic solution $y_0(z)$ at zero of many differential operators appearing in [1] satisfies $\Lambda_p(y_0(z)) = y_0(z) \bmod p$ for almost every prime p. We recall that it is hoped that the differential operators appearing in [1] are Calabi–Yau operators. In particular, it is expected that $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$ for almost every prime p. Thus, in view of Theorem 1.3, in order to prove $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$, it is sufficient to show that $\Lambda_p(\psi_{1,1}(z)) = 1 \bmod p$.

• We conjecture that if under the assumptions of Theorem 1.3 we assume additionally that $\Lambda_p(y_0(z)) = y_0(z) \bmod p$ then

$$\Psi \bmod p = \begin{pmatrix} \mathfrak{f}_p(z) \bmod p & 0 \\ \delta(\mathfrak{f}_p(z)) \bmod p & 0 \end{pmatrix},$$

where $\mathfrak{f}_p(z)$ is the p-truncation of $\mathfrak{f}(z)$ (3). Note that if the conjecture holds then Theorem 1.3 implies $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$ and that $y_0(z)$ is p-Lucas (4).

1.2. Strategy of the proof. Let us briefly explain the strategy of the proof of Theorem 1.3. The p-integrality of $y_0(z)$ is a direct consequence of Proposition 3.3 and Theorem 3.4. Proposition 3.3 is proven in Section 3 and Theorem 3.4 is proven in Section 5. In Section 4 we use the theory of p-adic differential equations in order to prove some results that are crucial in the proof of Theorem 3.4. Items (1) and (2) of Theorem 1.3 are proven in Section 7. For this purpose, we prove in Section 6 some results on p-integrality of certain power series with coefficients in $\mathbb{Q}_p$.

1.3. Previous results and comparison. It has been shown, in numerous cases, that the canonical coordinate belongs to $\mathbb{Z}[[z]]$. For example, from the works of Lian and Yau [22, Theorem 1.2], Zudilin [26, Theorem 3], Krattenthaler and Rivoal [20, Theorem 1] and Delaygue [12], we know that the canonical coordinate for a large class of MUM hypergeometric operators belongs to $\mathbb{Z}[[z]]$. Furthermore, Delaygue, Rivoal and Roques [13] gave a characterization of the hypergeometric operators whose canonical coordinate belongs to $\mathbb{Z}[[z]]$. The works of Krattenthaler and Rivoal [21] and Delaygue [12] provide examples of canonical coordinate in $\mathbb{Z}[[z]]$ for some non-hypergeometric operators. The technique used by Krattenthaler and Rivoal and by Delaygue is based on a multi-variate version of Dwork's formal congruences [16, Theorem 1.1]. In the work of Vologodsky [25] there is an approach using p-adic cohomology to prove the integrality of the canonical

(3) The *p-truncation* of a power series $\sum_{n\geq 0} c_n z^n$ is the polynomial $\sum_{n=0}^{p-1} c_n z^n$.

(4) A power series $t(z) \in \mathbb{Z}_p[[z]]$ is *p-Lucas* if $t(z) = t_p(z)t(z^p) \bmod p$, where $t_p(z)$ is the p-truncation of $t(z)$.

coordinate of Picard–Fuchs operators coming from families of Calabi–Yau operators. We do not know the real status of [25] since it has not been published in any journal yet. In contrast, the present work offers a more elementary approach which is based on the theory of p-adic differential equations. The idea of using p-adic tools for proving the integrality of the canonical coordinate goes back to Stienstra [23]. Finally, let us mention a result due to Beukers and Vlasenko [3]. Let $L \in \mathbb{Z}_p[[z]][\delta]$ be a differential operator of order n. Those authors show that if there exists $\mathcal{A} = \sum_{i=1}^{n} A_i(z)\delta^{i-1} \in \mathbb{Z}_p[[z]][\delta]$ with $A_1(0) = 1$ such that, for every solution y of L, the composition $\mathcal{A}(y(z^p))$ is a solution of L, then $y_0, \exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$. Nevertheless, according to Remark 7.1 below, Theorem 1.3 implies that $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$ if and only if there exists $\mathcal{B} = B_1(z) + B_2(z)\delta \in \mathbb{Z}_p[[z]][\delta]$ with $B_1(0) = 1$ and $B_2(0) = 0$ such that, for every solution y of $L^{(2)}$, the composition $\mathcal{B}(y(z^p))$ is a solution of $L^{(2)}$.

2. Frobenius structure and MUM operators. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$. It is well-known that the p-adic norm of $\mathbb{Q}_p$ extends uniquely to $\overline{\mathbb{Q}_p}$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ with respect to the p-adic norm. The field $\mathbb{C}_p(z)$ is equipped with the Gauss norm which is defined as follows:

$$\left|\frac{\sum_{i=0}^{n} a_i z^i}{\sum_{j=0}^{m} b_j z^j}\right|_{\mathcal{G}} = \frac{\sup\{|a_i|\}_{1\leq i\leq n}}{\sup\{|b_j|\}_{1\leq j\leq m}}.$$

The field of *analytic elements*, denoted E_p, is the completion of $\mathbb{C}_p(z)$ with respect to the Gauss norm. The field E_p has a derivation $\delta = z\frac{d}{dz}$. It is easily seen that $E_p \subset \mathcal{W}_p$, where

$$\mathcal{W}_p = \Big\{\sum_{n\in\mathbb{Z}} a_n z^n : a_n \in \mathbb{C}_p,\ \lim_{n\to-\infty} |a_n| = 0, \text{ and } \sup_{n\in\mathbb{Z}} |a_n| < \infty\Big\}.$$

The ring $\mathcal{W}_p$ is usually called the *Amice ring*. This ring is also equipped with the Gauss norm

$$\Big|\sum_{n\in\mathbb{Z}} a_n z^n\Big|_{\mathcal{G}} = \sup\{|a_n|\}_{n\in\mathbb{Z}}.$$

We let $\mathcal{W}_{\mathbb{Q}_p}$ denote the elements of $\mathcal{W}_p$ with coefficients in $\mathbb{Q}_p$. Likewise, $\mathcal{W}_{\mathbb{Z}_p}$ is the set of elements of $\mathcal{W}_p$ with coefficients in $\mathbb{Z}_p$. Notice that $\mathbb{Z}_p[[z]] \subset \mathcal{W}_{\mathbb{Z}_p}$. The following definition is due to Dwork [15].

Definition 2.1 (Strong Frobenius structure). Let L be a monic differential operator of order n in $\mathbb{Q}(z)[\delta]$ and let A be the companion matrix of L. We say that L has a *Frobenius structure* if there exists $\Phi \in \mathrm{GL}_n(E_p)$ such that

$$\delta\Phi = A\Phi - \Phi pA(z^p).$$

According to [19, Theorem 22.2.1] or [2, p. 111], if L is a Picard–Fuchs operator then L is equipped with a strong Frobenius structure for almost every prime number.

DEFINITION 2.2 (MUM differential operator). Let L be a monic differential operator in $\mathbb{Q}_p[[z]][\delta]$. We say that L is *MUM* if its coefficients belong to $\mathbb{Q}_p(z) \cap z\mathbb{Q}_p[[z]]$.

REMARK 2.3. (1) Let $A(z) \in M_n(\mathbb{Q}_p[[z]])$ and let $N = A(0)$. If the eigenvalues of N are all zero then, from [18, Chap. III, Proposition 8.5], the differential system $\delta X = A(z)X$ has a fundamental matrix of solutions of the shape $Y_A z^N$, where $Y_A \in \mathrm{GL}_n(\mathbb{Q}_p[[z]])$, $Y_A(0) = I$ and $z^N = \sum_{j\geq 0} N^j \frac{(\log z)^j}{j!}$. The matrix $Y_A z^N$ will be called *the fundamental matrix of solutions* of $\delta X = A(z)X$, and the matrix Y_A will be called the *uniform part* of solutions of the system $\delta X = AX$.

(2) Let L be a MUM differential operator of order n with coefficients in $\mathbb{Q}_p[[z]]$, let $A(z)$ be the companion matrix of L and let $N = A(0)$. Since L is MUM, the eigenvalues of N are all zero. We denote by $X_L = Y_L z^N$ the fundamental matrix of solutions of $\delta X = A(z)X$, where the matrix Y_L is the uniform part of this system. Since the eigenvalues of N are all zero, we have $N^n = 0$. For this reason, $z^N = \sum_{j=0}^{n-1} N^j \frac{(\log z)^j}{j!}$. Consequently, there are unique power series $\mathfrak{f}(z) \in 1 + z\mathbb{Q}_p[[z]]$ and $\mathfrak{g} \in z\mathbb{Q}_p[[z]]$ such that $\mathfrak{f}(z)$ and $\mathfrak{f}(z)\log z + \mathfrak{g}(z)$ are solutions of L. Moreover, since $\log z$ is transcendental over $\mathbb{Q}_p[[z]]$, it follows that if $\mathfrak{t}(z) \in \mathbb{Q}_p[[z]]$ is a solution of L then $\mathfrak{t}(z) = c\mathfrak{f}(z)$, where $c = \mathfrak{t}(0)$.

REMARK 2.4. We can now show that if $L \in \mathbb{Q}_p(z)[\delta]$ is a MUM differential operator of order n equipped with a p-integral Frobenius structure given by $\Phi = (\phi_{i,j})_{1\leq i,j\leq n}$ then $\Phi = Y_L\Phi(0)Y_L(z^p)^{-1}$ and $\Phi(0)$ is an upper triangular matrix with diagonal entries $\phi_{1,1}(0), p\phi_{1,1}(0), \ldots, p^{n-1}\phi_{1,1}(0)$. In fact, let $Y_L z^N$ be the fundamental matrix of solutions of $\delta X = A(z)X$, where $N = A(0)$ and $A(z)$ is the companion matrix of L. Then $Y_L(z^p)z^{pN}$ is the fundamental matrix of solutions of $\delta X = pA(z^p)X$. We know that

$$\delta\Phi = A\Phi - p\Phi A(z^p),$$

and thus there exists $C \in M_n(\mathbb{C}_p)$ such that $\Phi = Y_L z^N C(z^{pN})^{-1} Y_L(z^p)^{-1}$. Since

$$(z^{pN})^{-1} = \mathrm{diag}(1, 1/p, \ldots, 1/p^{n-1})(z^N)^{-1}\,\mathrm{diag}(1, p, \ldots, p^{n-1}),$$

and $\log z$ is transcendental over $\mathcal{W}_p$, and since moreover $\Phi \in M_n(\mathcal{W}_{\mathbb{Z}_p})$, it follows that $z^N C(z^{pN})^{-1} = C$. In addition, from this equality it is not hard to see that C is an upper triangular matrix with diagonal entries

$\mu, p\mu, \ldots, p^{n-1}\mu$ for some $\mu \in \mathbb{C}_p$. Consequently, $\Phi = Y_L C Y_L(z^p)^{-1}$ and $C = \Phi(0)$. Therefore, the matrix $\Phi(0)$ is as expected.

3. p-Integrality of y_0 and radius of convergence. In this section we describe our strategy to prove that $y_0 \in 1 + z\mathbb{Z}_p[[z]]$. This strategy relies on Proposition 3.3 and Theorem 3.4. In order to state these results, we recall the definition of *radius of convergence of a differential operator with coefficients in* $\mathcal{W}_p \cap \mathbb{C}_p[[z]]$. For a real number $r > 0$ we have the following ring of analytic functions:

$$\mathcal{A}(z,r) := \Big\{\sum_{j\geq 0} a_j(x-z)^j \in \mathcal{W}_p[[x-z]] : \text{for all } s < r,\ \lim_{j\to\infty} |a_j|_{\mathcal{G}} s^j = 0\Big\}.$$

In other words, $\mathcal{A}(z,r)$ is the ring of power series with coefficients in $\mathcal{W}_p$ that converge in the open disk $D(z,r) := \{x \in \mathcal{W}_p : |x-z| < r\}$.

Now, let us consider $\boldsymbol{\tau} : \mathcal{W}_p \cap \mathbb{C}_p[[z]] \to \mathcal{A}(z,1)$ given by

$$\boldsymbol{\tau}(f) = \sum_{j\geq 0} \frac{(d/dz)^j(f)}{j!}(x-z)^j.$$

The map $\boldsymbol{\tau}$ is well-defined because $\left|\frac{(d/dz)^j(f)}{j!}\right|_{\mathcal{G}} \leq |f|_{\mathcal{G}}$ for all $f \in \mathcal{W}_p \cap \mathbb{C}_p[[z]]$ and all $j \geq 0$. It is clear that $\boldsymbol{\tau}$ is a homomorphism of rings.

Remark 3.1. (1) According to [10, Proposition 1.2], a non-zero element $f = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathcal{W}_p$ is a unit of $\mathcal{W}_p$ if and only if there is $n_0 \in \mathbb{Z}$ such that $|f|_{\mathcal{G}} = |a_{n_0}|$. In particular, every non-zero element of $\mathbb{Z}_p[[z]]$ is a unit of $\mathcal{W}_p$.

(2) Let f be in $\mathcal{W}_p \cap \mathbb{C}_p[[z]]$. If f is a unit of $\mathcal{W}_p$ then $\boldsymbol{\tau}(f)$ is a unit of $\mathcal{A}(z,1)$. Indeed, write

$$\boldsymbol{\tau}(f) = f(1+g), \quad \text{where} \quad g = \sum_{j\geq 1} \frac{(d/dz)^j(f)}{j!f}(x-z)^j.$$

We have $g \in \mathcal{A}(z,1)$ because $\left|\frac{(d/dz)^j(f)}{j!}\right|_{\mathcal{G}} \leq |f|_{\mathcal{G}}$ for all $j \geq 1$ and, by assumption, $1/f \in \mathcal{W}_p$. Further, $1+g$ is a unit element of $\mathcal{A}(z,1)$ because

$$(1+g)\Big(\sum_{k\geq 0}(-1)^k g^k\Big) = 1 \quad \text{and} \quad \sum_{k\geq 0}(-1)^k g^k \in \mathcal{A}(z,1).$$

Thus,

$$\frac{1}{f}\sum_{k\geq 0}(-1)^k g^k \in \mathcal{A}(z,1).$$

Finally, it is clear that

$$\boldsymbol{\tau}(f)\Big(\frac{1}{f}\sum_{k\geq 0}(-1)^k g^k\Big) = 1.$$

For all $f \in \mathcal{W}_p \cap \mathbb{C}_p[[z]]$, we have

$$\boldsymbol{\tau}\left(\frac{d}{dz}f\right) = \frac{d}{dx}(\boldsymbol{\tau} f). \tag{3.1}$$

The ring $(\mathcal{W}_p \cap \mathbb{C}_p[[z]])[[x-z]]$ is equipped with the endomorphism

$$\boldsymbol{F}\Big(\sum_{j\geq 0} a_j(z)(x-z)^j\Big) = \sum_{j\geq 0} a_j(z^p)(x^p - z^p)^j.$$

Since

$$x^p - z^p = \boldsymbol{\tau}(z^p) - z^p = \sum_{i=1}^{p} \frac{(d/dz)^i(z^p)}{i!}(x-z)^i,$$

we have

$$\boldsymbol{\tau} \circ \boldsymbol{F} = \boldsymbol{F} \circ \boldsymbol{\tau}. \tag{3.2}$$

Let L be a monic differential operator of order n with coefficients in $\mathcal{W}_p \cap \mathbb{C}_p[[z]]$ and let A be the companion matrix of L. Then the differential system $\delta_x X = \boldsymbol{\tau}(A)X$ ($\delta_x = x\frac{d}{dx}$) has a unique solution $\mathcal{U} \in \mathrm{GL}_n(\mathcal{W}_p[[x-z]])$ such that $\mathcal{U}(z) = I$. Moreover,

$$\mathcal{U} = \sum_{j\geq 0} \frac{A_j}{j!z^j}(x-z)^j,$$

where $A_0 = I$ and $A_{j+1} = \delta A_j + A_j(A - jI)$ for $j \geq 0$. Following [18, Chap. III, p. 94], the *radius of convergence of* L is the radius of convergence of the matrix $\mathcal{U}$ in $\mathcal{W}_p$. We let $\boldsymbol{r}(L)$ denote the radius of convergence of L. So, $\mathcal{U} \in M_n(\mathcal{A}(z, \boldsymbol{r}(L)))$. For a matrix $C = (c_{i,j})_{1\leq i,j\leq n}$ with coefficients in $\mathcal{W}_p$, we set $\|C\| = \max\{|c_{i,j}|_{\mathcal{G}}\}_{1\leq i,j\leq n}$. So

$$\frac{1}{\boldsymbol{r}(L)} = \limsup_{j\to\infty}\left\|\frac{A_j}{j!}\right\|^{1/j}.$$

Remark 3.2. Let L be a differential operator with coefficients in $\mathbb{Z}_p[[z]]$. If $\boldsymbol{r}(L) \geq 1$ then $\lim_{j\to\infty}\|A_j\| = 0$. Indeed, as $\boldsymbol{r}(L) \geq 1$, we have $\limsup_{j\to\infty}\|A_j/j!\|^{1/j} \leq 1$. Therefore, $\lim_{j\to\infty}\|A_j\| = 0$.

Proposition 3.3. *Let L be a differential operator with coefficients in $\mathbb{Z}_p[[z]]$ and having a p-integral Frobenius structure. Then $\boldsymbol{r}(L) \geq 1$.*

Proof. Let A be the companion matrix of L. We put $A_0 = I$ and $A_{j+1} = \delta A_j + A_j(A - jI)$ for $j \geq 0$. We want to see that $\boldsymbol{r}(L) \geq 1$. Since L belongs to $\mathbb{Z}_p[[z]][\delta]$, we have $\|A\| = 1$, and thus $\|A_j\| \leq 1$ for all $j \geq 0$. So, $\boldsymbol{r}(L) \geq |p|^{1/p-1}$. We know that $\delta_x(\mathcal{U}) = \boldsymbol{\tau}(A)\mathcal{U}$, and thus

$$\boldsymbol{F}(\mathcal{U}) = \sum_{j\geq 0} \frac{A_j(z^p)}{j!z^{jp}}(x^p - z^p)^j$$

is a solution of the system $\delta_x X = p\boldsymbol{F}(\boldsymbol{\tau}(A))X$.

We now prove that $\boldsymbol{F}(\mathcal{U}) \in M_n(\mathcal{A}(z, \boldsymbol{r}(L)^{1/p}))$. As the Gauss norm is non-Archimedean, this is equivalent to showing that if $|x_0 - z|_{\mathcal{G}} < \boldsymbol{r}(L)^{1/p}$ then $\lim_{j\to\infty}\|\frac{A_j(z^p)}{j!z^{jp}}\|\,|x_0^p - z^p|_{\mathcal{G}}^j = 0$. Indeed, if $|x_0 - z|_{\mathcal{G}} < \boldsymbol{r}(L)^{1/p}$ then $|x_0 - z|_{\mathcal{G}}^p < \boldsymbol{r}(L)$ but $(x_0 - z)^p = x_0^p - z^p + \sum_{i=1}^{p-1}(-1)^i\binom{p}{i}x_0^{p-i}z^i$ and

$$\left|\sum_{i=1}^{p-1}(-1)^i\binom{p}{i}x_0^{p-i}z^i\right|_{\mathcal{G}} \leq \frac{1}{p}$$

because the Gauss norm is non-Archimedean and, by Lucas' Theorem, $\left|\binom{p}{i}\right| \leq 1/p$ for all $1 \leq i < p$. Furthermore, $1/p < |p|^{1/p-1} \leq \boldsymbol{r}(L)$. Thus,

$$|x_0^p - z^p|_{\mathcal{G}} \leq \max\{|x_0 - z|_{\mathcal{G}}^p, 1/p\} < \boldsymbol{r}(L).$$

As the radius of convergence of $\mathcal{U}$ is $\boldsymbol{r}(L)$, we have $\lim_{j\to\infty}\|\frac{A_j}{j!z^j}\|\,|x_0^p - z^p|_{\mathcal{G}}^j = 0$. As $\|\frac{A_j}{j!z^j}\| = \|\frac{A_j(z^p)}{j!z^{jp}}\|$, we obtain $\lim_{j\to\infty}\|\frac{A_j(z^p)}{j!z^{jp}}\|\,|x_0^p - z^p|_{\mathcal{G}}^j = 0$. Thus $\boldsymbol{F}(\mathcal{U}) \in M_n(\mathcal{A}(z, \boldsymbol{r}(L)^{1/p}))$, where n is the order of L. Now, by assumption, there is $\Phi \in M_n(\mathbb{Z}_p[[z]])$ such that $\det(\Phi) \neq 0$ and $\delta\Phi = A\Phi - p\Phi A(z^p)$. Hence, by (3.1) and (3.2), we obtain

$$\delta_x(\boldsymbol{\tau}(\Phi)) = \boldsymbol{\tau}(A)\boldsymbol{\tau}(\Phi) - p\boldsymbol{\tau}(\Phi)\boldsymbol{F}(\boldsymbol{\tau}(A)).$$

So, $\boldsymbol{\tau}(\Phi)\boldsymbol{F}(\mathcal{U})$ is a solution of the system $\delta_x X = \boldsymbol{\tau}(A)X$. Since $\det(\Phi) \neq 0$ and $\det(\Phi) \in \mathbb{Z}_p[[z]]$, it follows from Remark 3.1(1) that $\det(\Phi)$ is a unit of $\mathcal{W}_p$. So, by Remark 3.1(2), we conclude that $\boldsymbol{\tau}(\Phi) \in \mathrm{GL}_n(\mathcal{A}(z, 1))$. Consequently, $\boldsymbol{\tau}(\Phi)\boldsymbol{F}(\mathcal{U}) \in M_n(\mathcal{A}(z, r))$, where $r = \min\{1, \boldsymbol{r}(L)^{1/p}\}$. Since $\mathcal{U} \in \mathrm{GL}_n(\mathcal{W}_p[[x - z]])$ is a solution of $\delta_X = \boldsymbol{\tau}(A)X$, there is $C \in \mathrm{GL}_n(\mathcal{W}_p)$ such that $\boldsymbol{\tau}(\Phi)\boldsymbol{F}(\mathcal{U}) = \mathcal{U}C$. As the radius of convergence of $\mathcal{U}C$ is still $\boldsymbol{r}(L)$, we have $\boldsymbol{r}(L) \geq r$. Consequently, $\boldsymbol{r}(L) \geq 1$. ∎

It follows from Remark 2.3 that for a MUM differential operator $L \in \mathbb{Q}_p[[z]][\delta]$ of order $n \geq 2$ there are unique power series $\mathfrak{f}(z) \in 1 + z\mathbb{Q}_p[[z]]$ and $\mathfrak{g}(z) \in z\mathbb{Q}_p[[z]]$ such that $y_0 = \mathfrak{f}$ and $y_1 = \mathfrak{f}\log z + \mathfrak{g}$ are solutions of L. The main ingredient to prove Theorem 1.3(1) is the following result.

THEOREM 3.4. *Let $L \in \mathbb{Z}_p[[z]][\delta]$ be a MUM differential operator. If $\boldsymbol{r}(L) \geq 1$ then $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$.*

4. p-Adic differential equations. The crucial ingredient in the proof of Theorem 3.4 is Proposition 4.4, which is recursively obtained from Lemma 4.1. The proof of this lemma relies on the theory of p-adic differential equations.

We recall that the *Cartier operator* $\Lambda_p : \mathbb{C}_p[[z]] \to \mathbb{C}_p[[z]]$ is given by $\Lambda_p(\sum_{i\geq 0} a(i)z^i) = \sum_{i\geq 0} a(ip)z^i$. From the definition it is clear that if $f(z)$ and $g(z)$ belong to $\mathbb{C}_p[[z]]$ then $\Lambda_p(f(z)g(z^p)) = \Lambda_p(f)g(z)$. Given a matrix

$B \in M_n(\mathbb{C}_p[[z]])$, $\Lambda_p(B)$ is the matrix obtained by applying Λ_p to each entry of B.

LEMMA 4.1. *Let L be a MUM differential operator of order n in $\mathbb{Z}_p[[z]][\delta]$, let A be the companion matrix of L, and let Y_L be the uniform part of $\delta X = AX$ and let $N = A(0)$. If $\boldsymbol{r}(L) \geq 1$ then there exists a MUM differential operator L_1 of order n in $\mathbb{Z}_p[[z]][\delta]$ such that*

(a′) *the fundamental matrix of solutions of $\delta X = B_1 X$ is given by*

$$\operatorname{diag}(1, 1/p, \ldots, 1/p^{n-1})\Lambda_p(Y_L)\operatorname{diag}(1, p, \ldots, p^{n-1})z^N,$$

where B_1 is the companion matrix of L_1,

(b′) *the matrix*

$$H_1 = Y_L(\Lambda_p(Y_L)(z^p))^{-1}\operatorname{diag}(1, p, p^2, \ldots, p^{n-1})$$

belongs to $M_n(\mathbb{Z}_p[[z]]) \cap \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$, $\|H_1\| = 1$, and

$$\delta H_1 = AH_1 - pH_1 B_1(z^p),$$

(c′) $\boldsymbol{r}(L_1) \geq 1$.

In order to prove Lemma 4.1, we first prove Lemma 4.2, via an argument based upon the ideas found in [9].

LEMMA 4.2. *Let L be a MUM differential operator of order n in $\mathbb{Z}_p[[z]][\delta]$, let A be the companion matrix of L, and let Y_L be the uniform part of $\delta X = AX$ and let $N = A(0)$. If $\boldsymbol{r}(L) \geq 1$ then*

(1) *the matrix $H_0 = (\Lambda_p(Y_L)(z^p))Y_L^{-1}(z)$ belongs to $\mathrm{GL}_n(\mathbb{Z}_p[[z]])$,*

(2) *there is $F \in M_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$ such that*

$$\delta H_0 = pF(z^p)H_0 - H_0 A,$$

(3) *the fundamental matrix of solutions of $\delta X = FX$ is given by $\Lambda_p(Y_L)z^{N/p}$.*

Proof. (1) Consider the sequence $\{A_j(z)\}_{j\geq 0}$, where $A_0(z) = I$ is the identity matrix and $A_{j+1}(z) = \delta A_j(z) + A_j(z)(A(z) - jI)$. As $A \in M_n(\mathbb{Z}_p[[z]])$, we have $A_j(z) \in M_n(\mathbb{Z}_p[[z]])$ for all $j \geq 0$. Since $X^p - 1 = (X-1)((X-1)^{p-1} + pt(X))$ with $t(X) \in \mathbb{Z}[X]$, it follows that if $\xi^p = 1$ then $|\xi - 1| \leq |p|^{1/(p-1)}$. Furthermore, for all integers $j \geq 1$, we have $|p|^{j/(p-1)} < |j!|$ and thus $|(\xi-1)^j/j!| < 1$. Since the norm is non-Archimedean and $\sum_{\xi^p=1}(\xi-1)^j/j$ is a rational number, we conclude that, for all $j \geq 0$, $\left|\sum_{\xi^p=1}\frac{(\xi-1)^j}{j!}\right| \leq 1/p$, thus $\left|\sum_{\xi^p=1}\frac{(\xi-1)^j}{pj!}\right| \leq 1$ and so $\sum_{\xi^p=1}\frac{(\xi-1)^j}{pj!} \in \mathbb{Z}_p$. Therefore, for all $j \geq 0$, the matrix

$$A_j\left(\sum_{\xi^p=1}\frac{(\xi-1)^j}{pj!}\right)$$

belongs to $M_n(\mathbb{Z}_p[[z]])$. We set

$$H_0 = \sum_{j\geq 0} A_j(z)\Big(\sum_{\xi^p=1} \frac{(\xi-1)^j}{pj!}\Big).$$

Let us show that $H_0 \in M_n(\mathbb{Z}_p[[z]])$. This is equivalent to

$$\lim_{j\to\infty} \|A_j\| \Big|\sum_{\xi^p=1} \frac{(\xi-1)^j}{pj!}\Big| = 0 \tag{4.1}$$

because $\mathbb{Z}_p[[z]]$ is complete with respect to the Gauss norm and the latter is non-Archimedean. Indeed, as $\boldsymbol{r}(L) \geq 1$, by Remark 3.2, $\lim_{j\to\infty} \|A_j\| = 0$. Therefore, (4.1) follows immediately since $\big|\sum_{\xi^p=1} \frac{(\xi-1)^j}{pj!}\big| \leq 1$ for all $j \geq 0$.

Now, we are going to prove that $H_0 = \Lambda_p(Y_L)(z^p)Y_L^{-1}$. Write $Y_L = \sum_{j\geq 0} Y_j z^j$. We have $N^n = 0$ because L is MUM, and thus it follows from [9, p. 165] that $H_0Y_L = \sum_{j\geq 0} Y_{jp}z^{jp}$. So, $H_0Y_L = \Lambda_p(Y_L)(z^p)$. Therefore, $H_0 = \Lambda_p(Y_L)(z^p)Y_L^{-1}$. Finally, $H_0 \in \mathrm{GL}_n(\mathbb{Z}_p[[z]])$ because $H_0(0)$ is the identity matrix.

(2) Now, we set

$$F(z) = \Big[\delta(\Lambda_p(Y_L)) + \frac{1}{p}\Lambda_p(Y_L)N\Big](\Lambda_p(Y_L))^{-1}. \tag{4.2}$$

As $H_0Y_L = \Lambda_p(Y_L)(z^p)$, from (4.2) we obtain

$$pF(z^p) = \big[p\big(\delta(\Lambda_p(Y_L))\big)(z^p) + H_0Y_LN\big][Y_L^{-1}H_0^{-1}].$$

We also have

$$\delta(H_0)Y_L + H_0(\delta Y_L) = \delta(H_0Y_L) = \delta(\Lambda_p(Y_L)(z^p)) = p\big(\delta(\Lambda_p(Y_L))\big)(z^p).$$

Since $\delta(Y_Lz^N) = AY_Lz^N$, it follows that $\delta Y_L = AY_L - Y_LN$. Thus,

$$p\big(\delta(\Lambda_p(Y_L))\big)(z^p) = \delta(H_0)Y_L + H_0[AY_L - Y_LN].$$

Therefore,

$$\begin{aligned} pF(z^p) &= \big[p\big(\delta(\Lambda_p(Y_L))\big)(z^p) + H_0Y_LN\big][Y_L^{-1}H_0^{-1}] \\ &= [(\delta H_0)Y_L + H_0AY_L][Y_L^{-1}H_0^{-1}] = (\delta H_0)H_0^{-1} + H_0AH_0^{-1}. \end{aligned}$$

Consequently,

$$\delta H_0 = pF(z^p)H_0 - H_0A.$$

Since $pF(z^p) = (\delta H_0 + H_0A)H_0^{-1} \in M_n(\mathbb{Z}_p[[z]])$, we deduce that $F(z) \in M_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$.

(3) Finally, we show that the matrix $\Lambda_p(Y_L)X^{N/p}$ is the fundamental matrix of solutions of $\delta X = FX$. It is clear that $\Lambda_p(Y_L)(0) = I$, and from (4.2)

we obtain $F(z)\Lambda_p(Y_L) = \delta(\Lambda_p(Y_L)) + \Lambda_p(Y_L)\frac{1}{p}N$. Thus,

$$\begin{aligned}\delta(\Lambda_p(Y_L)X^{N/p}) &= \delta(\Lambda_p(Y_L))X^{N/p} + \Lambda_p(Y_L)\frac{1}{p}NX^{N/p}\\ &= \left(\delta(\Lambda_p(Y_L)) + \Lambda_p(Y_L)\frac{1}{p}N\right)X^{N/p} = F(z)\Lambda_p(Y_L)X^{N/p}.\end{aligned}$$

This completes the proof of Lemma 4.2. ∎

The following result is known as the "Dwork–Frobenius" Theorem and it is usually proven for differential operators with coefficients in E_p (see [7, Proposition 8.1]). For completeness we prove it following the same lines.

PROPOSITION 4.3. *Let $L = \delta^n + a_{n-1}(z)\delta^{n-1} + \cdots + a_1(z)\delta + a_0(z)$ be a differential operator with $a_i(z) \in \mathcal{W}_p \cap \mathbb{C}_p[[z]]$ for all $0 \le i < n$. If $\boldsymbol{r}(L) \ge 1$ then $|a_i|_{\mathcal{G}} \le 1$ for all $0 \le i < n$.*

Proof. For every $r < 1$, the ring $\mathcal{A}(z,1)$ is equipped with the absolute value

$$\Big|\sum_{n\ge 0} f_n(x-z)^n\Big|_r = \sup\{|f_n|_{\mathcal{G}}r^n\}_{n\ge 0}.$$

This absolute value extends in a natural way to $\mathcal{M}(z,1) := \mathrm{Frac}(\mathcal{A}(z,1))$ and, for all $g \in \mathcal{M}(z,1)$, $|\delta_x g/g|_r \le 1/r$. Let us now introduce the ring

$$\mathbb{W} = \Big\{f \in \mathcal{M}(z,1) : \limsup_{r\to 1} |f|_r \le 1\Big\}.$$

Thus, for all $g \in \mathcal{M}(z,1)$, $\delta_x g/g \in \mathbb{W}$. Further, the ring $\mathbb{W}$ is closed under derivation because for any $f \in \mathbb{W}$, $|\delta_x f|_r \le \frac{1}{r}|f|_r$. Consider the differential operator $\mathcal{L} = \delta_x^n + \boldsymbol{\tau}(a_{n-1})\delta_x^{n-1} + \cdots + \boldsymbol{\tau}(a_1)\delta_x + \boldsymbol{\tau}(a_0)$. By assumption, there are $f_1,\ldots,f_n \in \mathcal{A}(z,1)$, linearly independent over $\mathrm{Frac}(\mathcal{W}_p)$, such that $\mathcal{L}(f_i) = 0$ for every $1 \le i \le n$. By induction, we set $g_1 = f_1$ and $g_i = \mathcal{L}_{i-1}\circ\cdots\circ\mathcal{L}_1(f_i)$ with $\mathcal{L}_i = \delta_x - \delta_x g_i/g_i$. So, $g_1,\ldots,g_n$ belong to $\mathcal{M}(z,1)$ and $f_1,\ldots,f_n$ are solutions of $\mathcal{L}_n\circ\cdots\circ\mathcal{L}_1$. Since $\mathcal{L}_n\circ\cdots\circ\mathcal{L}_1 = \delta_x^n + \cdots$, the differential operator $\mathcal{L} - \mathcal{L}_n\circ\cdots\circ\mathcal{L}_1$ has order at least $n-1$ and has $f_1,\ldots,f_n$ as solutions. Thus, $\mathcal{L} = \mathcal{L}_n\circ\cdots\circ\mathcal{L}_1$. As $\delta_x g_i/g_i \in \mathbb{W}$ and $\mathbb{W}$ is closed under derivation, we obtain $\boldsymbol{\tau}(a_i) \in \mathbb{W}$ for every $0 \le i < n$. Finally, since $a_i \in \mathcal{W}_p \cap \mathbb{C}_p[[z]]$, we have $\big|\frac{(d/dz)^j(a_i)}{j!}\big|_{\mathcal{G}} \le |a_i|_{\mathcal{G}}$ for all $j \ge 0$, and therefore, $|\boldsymbol{\tau}(a_i)|_r = |a_i|_{\mathcal{G}}$. Consequently, $\limsup_{r\to 1}|\boldsymbol{\tau}(a_i)|_r = |a_i|_{\mathcal{G}}$. Nevertheless, since $\boldsymbol{\tau}(a_i) \in \mathbb{W}$, we have $\limsup_{r\to 1}|\boldsymbol{\tau}(a_i)|_r \le 1$. Thus, $|a_i|_{\mathcal{G}} \le 1$ for all $0 \le i < n$. ∎

Proof of Lemma 4.1. By Lemma 4.2, there is an $F \in M_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$ such that

$$\delta H_0 = pF(z^p)H_0 - H_0A, \tag{4.3}$$

with $H_0 = \Lambda_p(Y_L)(z^p)Y_L^{-1} \in \mathrm{GL}_n(\mathbb{Z}_p[[z]])$. Write $F = (b_{i,j})_{1\le i,j\le n}$. We set

$$L_1 := \delta^n - b_{n,n}\delta^{n-1} - \frac{1}{p}b_{n,n-1}\delta^{n-2} - \cdots - \frac{1}{p^{n-i}}b_{n,i}\delta^{i-1} - \cdots - \frac{1}{p^{n-2}}b_{n,2}\delta - \frac{1}{p^{n-1}}b_{n,1}.$$

It is clear that the coefficients of L_1 belong to $\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]]$. Actually, we are going to see at the end of this proof that $L_1 \in \mathbb{Z}_p[[z]][\delta]$. From (4.3), we deduce that $pF(0) = N$. Since L is MUM, the last row of N is equal to zero. So $b_{n,i}(0) = 0$ for all $i \in \{1,\dots,n-1\}$, and L_1 is MUM.

(a′) Let B_1 be the companion matrix of L_1. We are going to see that

$$T = \mathrm{diag}(1, 1/p, \dots, 1/p^{n-1})\Lambda_p(Y_L)\,\mathrm{diag}(1, p, \dots, p^{n-1})z^N$$

is the fundamental matrix of solutions of $\delta X = B_1X$. We split the proof into three steps.

First step. Write $Y_L = (f_{i,j})_{1\le i,j\le n}$. Then, for all $i \in \{1,\dots,n-1\}$ and $k \in \{1,\dots,n\}$, $f_{i,k-1} + \delta f_{i,k} = f_{i+1,k}$. Indeed, for all $i,j \in \{1,\dots,n\}$, we set

$$\phi_{i,j} = \sum_{k=1}^{j} f_{i,k}\frac{(\log z)^{j-k}}{(j-k)!}.$$

Notice that $Y_Lz^N = (\phi_{i,j})_{1\le i,j\le n}$. Since Y_Lz^N is the fundamental matrix of $\delta X = AX$ and A is the companion matrix of L, it follows that, for all $i \in \{1,\dots,n-1\}$ and $j \in \{1,\dots,n\}$, $\phi_{i+1,j} = \delta\phi_{i,j}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{j} f_{i+1,k}\frac{(\log z)^{j-k}}{(j-k)!} &= \phi_{i+1,j} = \delta\phi_{i,j}\\ &= \delta(f_{i,1})\frac{(\log z)^{j-1}}{(j-1)!} + \sum_{k=2}^{j}(f_{i,k-1} + \delta(f_{i,k}))\frac{(\log z)^{j-k}}{(j-k)!}.\end{aligned}$$

As $\log z$ is transcendental over $\mathbb{Q}_p[[z]]$, it follows from the previous equality that, for all $k \in \{1,\dots,n\}$, $f_{i,k-1} + \delta f_{i,k} = f_{i+1,k}$.

Second step. For every $j \in \{1,\dots,n\}$, we set

$$\theta_j = \sum_{k=1}^{j} p^{k-1}\Lambda_p(f_{1,k})\frac{(\log z)^{j-k}}{(j-k)!}.$$

We are going to show that θ_j is a solution of L_1 for every $1 \le j \le n$.

Write $\Lambda_p(Y_L)X^{N/p} = (\eta_{i,j})_{1\le i,j\le n}$. Then, for all $i,j \in \{1,\dots,n\}$,

$$\eta_{i,j} = \sum_{k=1}^{j} \Lambda_p(f_{i,k})\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!}.$$

On the one hand, we are going to see that, for all $j, l \in \{1, \dots, n\}$,

$$\frac{1}{p^{n-l}}\delta^{l-1}\theta_j = \frac{1}{p^{n-j}}\eta_{l,j}.$$

To prove this, we use induction on $l \in \{1, \dots, n\}$. For $l = 1$, it is clear that $\frac{1}{p^{n-1}}\theta_j = \frac{1}{p^{n-j}}\eta_{1,j}$. Now, suppose that for some $l \in \{1, \dots, n-1\}$, we have $\frac{1}{p^{n-l}}\delta^{l-1}\theta_j = \frac{1}{p^{n-j}}\eta_{l,j}$. So, $\frac{1}{p^{n-l}}\delta^{l}\theta_j = \frac{1}{p^{n-j}}\delta(\eta_{l,j})$. We prove now that $\delta(\eta_{l,j}) = \frac{1}{p}\eta_{l+1,j}$. In fact, as $\Lambda_p \circ \delta = p\delta \circ \Lambda_p$ and, according to the first step, $f_{l,k-1} + \delta(f_{l,k}) = f_{l+1,k}$ for all $k \in \{1, \dots, n\}$, we have

$$\begin{aligned}
\delta(\eta_{l,j}) &= \sum_{k=1}^{j} \delta(\Lambda_p(f_{l,k}))\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!} + \Lambda_p(f_{l,k})\frac{(\log z)^{j-k-1}}{p^{j-k}(j-k-1)!} \\
&= \sum_{k=1}^{j} \frac{1}{p}\Lambda_p(\delta(f_{l,k}))\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!} + \frac{1}{p}\Lambda_p(f_{l,k})\frac{(\log z)^{j-k-1}}{p^{j-k-1}(j-k-1)!} \\
&= \frac{1}{p}\left[\Lambda_p(\delta(f_{l,1}))\frac{(\log z)^{j-1}}{p^{j-1}(j-1)!} + \sum_{k=2}^{j}\Lambda_p(f_{l,k-1} + \delta(f_{l,k}))\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!}\right] \\
&= \frac{1}{p}\left[\Lambda_p(f_{l+1,1})\frac{(\log z)^{j-1}}{p^{j-1}(j-1)!} + \sum_{k=2}^{j}\Lambda_p(f_{l+1,k})\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!}\right] = \frac{1}{p}\eta_{l+1,j}.
\end{aligned}$$

Thus,

$$\frac{1}{p^{n-l}}\delta^{l}\theta_j = \frac{1}{p^{n-j}}\delta(\eta_{l,j}) = \frac{1}{p^{n-j}}\left(\frac{1}{p}\eta_{l+1,j}\right).$$

Hence,

$$\frac{1}{p^{n-l-1}}\delta^{l}\theta_j = \frac{1}{p^{n-j}}\eta_{l+1,j}.$$

So, for all $j, l \in \{1, \dots, n\}$, $\frac{1}{p^{n-l}}\delta^{l-1}\theta_j = \frac{1}{p^{n-j}}\eta_{l,j}$.

On the other hand, by Lemma 4.2, we know that $\Lambda_p(Y_L)X^{N/p}$ is the fundamental matrix of $\delta X = FX$. Hence, for every $j \in \{1, \dots, n\}$,

$$F\begin{pmatrix}\eta_{1,j}\\ \eta_{2,j}\\ \vdots\\ \eta_{n,j}\end{pmatrix} = \begin{pmatrix}\delta(\eta_{1,j})\\ \delta(\eta_{2,j})\\ \vdots\\ \delta(\eta_{n,j})\end{pmatrix}.$$

In particular, for every $j \in \{1, \dots, n\}$,

$$b_{n,1}\eta_{1,j} + b_{n,2}\eta_{2,j} + \cdots + b_{n,k}\eta_{k,j} + \cdots + b_{n,n}\eta_{n,j} = \delta(\eta_{n,j}).$$

Multiplying the previous equality by $1/p^{n-j}$ and using the fact that, for every $l \in \{1, \ldots, n\}$, $\frac{1}{p^{n-j}}\eta_{l,j} = \frac{1}{p^{n-l}}\delta^{l-1}\theta_j$, we get

$$\frac{1}{p^{n-1}}b_{n,1}\theta_j + \frac{1}{p^{n-2}}b_{n,2}\delta\theta_j + \cdots + \frac{1}{p^{n-k}}b_{n,k}\delta^{k-1}\theta_j + \cdots + b_{n,n}\delta^{n-1}\theta_j = \delta^n\theta_j.$$

Therefore, θ_j is a solution of L_1.

THIRD STEP. We are going to see that $T = (\delta^{i-1}\theta_j)_{1\le i,j\le n}$. As we have seen already, for all $i, j \in \{1, \ldots, n\}$,

$$\frac{1}{p^{n-i}}\delta^{i-1}\theta_j = \frac{1}{p^{n-j}}\eta_{i,j},$$

and hence

$$\delta^{i-1}\theta_j = \frac{p^j}{p^i}\eta_{i,j} = \frac{p^j}{p^i}\sum_{k=1}^{j}\Lambda_p(f_{i,k})\frac{(\log z)^{j-k}}{p^{j-k}(j-k)!} = \sum_{k=1}^{j}\frac{\Lambda_p(f_{i,k})}{p^{i-k}}\frac{(\log z)^{j-k}}{(j-k)!}.$$

But

$$\operatorname{diag}(1, 1/p, \ldots, 1/p^{n-1})\Lambda_p(Y_L)\operatorname{diag}(1, p, \ldots, p^{n-1}) = \left(\frac{\Lambda_p(f_{i,k})}{p^{i-k}}\right)_{1\le i,k\le n}.$$

By definition

$$T = \operatorname{diag}(1, 1/p, \ldots, 1/p^{n-1})\Lambda_p(Y_L)\operatorname{diag}(1, p, \ldots, p^{n-1})z^N.$$

Thus, T is as expected.

Finally, notice that $\theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Q}_p$ because $\log z$ is transcendental over $\mathbb{Q}_p[[z]]$. Since B_1 is the companion matrix of L_1 and, according to the second step, $\theta_1, \ldots, \theta_n$ are solutions of L_1, it follows that $(\delta^{i-1}\theta_j)_{1\le i,j\le n}$ is a fundamental matrix of solutions of $\delta X = B_1X$. Since $T = (\delta^{i-1}\theta_j)_{1\le i,j\le n}$, we conclude that T is the fundamental matrix of $\delta X = B_1X$.

(b′) From Lemma 4.2, we know that $H_0^{-1} = Y_L(\Lambda_p(Y_L)(z^p))^{-1}$ belongs to $\mathrm{GL}_n(\mathbb{Z}_p[[z]])$. Thus,

$$\begin{aligned} H_1 &= Y_L(\Lambda_p(Y_L)(z^p))^{-1}\operatorname{diag}(1, p, \ldots, p^{n-1}) \\ &\in \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]]) \cap M_n(\mathbb{Z}_p[[z]]). \end{aligned}$$

Since $H_0^{-1} \in M_n(\mathbb{Z}_p[[z]])$ and $H_0^{-1}(0)$ is the identity matrix, we find that $\|H_0^{-1}\| = 1$. Therefore, $\|H_1\| \le 1$. But $H_1(0) = \operatorname{diag}(1, p, \ldots, p^{n-1})$, and thus $\|H_1\| = 1$. Now, we are going to show that

$$\delta(H_1) = AH_1 - pH_1B_1(z^p).$$

We infer from (a′) that

$$T(z^p) = \operatorname{diag}(1, 1/p, \ldots, 1/p^{n-1})\Lambda_p(Y_L)(z^p)\operatorname{diag}(1, p, \ldots, p^{n-1})X^{pN}$$

is the fundamental matrix of solutions of $\delta X = pB_1(z^p)X$. Furthermore,

$$H_1T(z^p) = Y_L\operatorname{diag}(1, p, \ldots, p^{n-1})X^{pN}.$$

But

$$\operatorname{diag}(1,p,\ldots,p^{n-1})X^{pN} = z^N \operatorname{diag}(1,p,\ldots,p^{n-1}).$$

Consequently,

$$H_1T(z^p) = Y_L z^N \operatorname{diag}(1,p,\ldots,p^{n-1}).$$

Thus, $H_1T(z^p)$ is a fundamental matrix of $\delta X = AX$, so

$$\begin{aligned} AH_1T(z^p) &= \delta(H_1T(z^p)) = \delta(H_1)T(z^p) + H_1\delta(T(z^p)) \\ &= \delta(H_1)T(z^p) + H_1(pB_1(z^p))T(z^p). \end{aligned}$$

Hence,

$$\delta(H_1) = AH_1 - pH_1B_1(z^p). \tag{4.4}$$

This completes the proof of (b′).

(c′) From (3.1), (3.2), and (4.4) we get

$$\delta_x(\boldsymbol{\tau}(H_1)) = \boldsymbol{\tau}(A)\boldsymbol{\tau}(H_1) - p\boldsymbol{\tau}(H_1)\boldsymbol{F}(\boldsymbol{\tau}(B_1)). \tag{4.5}$$

We set $C_0 = I$ and $C_{j+1} = \delta C_j + C_j(B_1 - jI)$ for $j \geq 0$. We want to see that $\boldsymbol{r}(L_1) \geq 1$. By definition, this is equivalent to proving that for all $r < 1$, $\lim_{j\to\infty}\|\frac{C_j}{j!}\|r^j = 0$. We know that

$$\widetilde{\mathcal{U}} = \sum_{j\geq 0} \frac{C_j(z)}{j!z^j}(x-z)^j \in \mathrm{GL}_n(\mathcal{W}_p[[x-z]])$$

is a solution of $\delta_x X = \boldsymbol{\tau}(B_1)X$. Therefore, $\delta_x\boldsymbol{F}(\widetilde{\mathcal{U}}) = p\boldsymbol{F}(\boldsymbol{\tau}(B_1))\boldsymbol{F}(\widetilde{\mathcal{U}})$. By (4.5), we find that $\boldsymbol{\tau}(H_1)\boldsymbol{F}(\widetilde{\mathcal{U}})$ is a solution of $\delta_x X = \boldsymbol{\tau}(A)X$. Since $H_1 \in \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$, we have $\boldsymbol{\tau}(H_1) \in \mathrm{GL}_n(\mathcal{A}(z,1)) \cap \mathrm{GL}_n(\mathcal{W}_p[[x-z]])$. As $\boldsymbol{F}(\widetilde{\mathcal{U}}) \in \mathrm{GL}_n(\mathcal{W}_p[[x-z]])$, we deduce that $\boldsymbol{\tau}(H_1)\boldsymbol{F}(\widetilde{\mathcal{U}}) \in \mathrm{GL}_n(\mathcal{W}_p[[x-z]])$ and since $\boldsymbol{\tau}(H_1)\boldsymbol{F}(\widetilde{\mathcal{U}})$ is a solution of $\delta_x X = \boldsymbol{\tau}(A)X$, there is $C \in \mathrm{GL}_n(\mathcal{W}_p)$ such that $\boldsymbol{\tau}(H_1)\boldsymbol{F}(\widetilde{\mathcal{U}}) = \mathcal{U}C$. Thus, $\boldsymbol{F}(\widetilde{\mathcal{U}}) = \mathcal{U}C\boldsymbol{\tau}(H_1)^{-1}$. By assumption, $\mathcal{U}$ belongs to $M_n(\mathcal{A}(z,1))$ and we know that $\boldsymbol{\tau}(H_1)$ belongs to $\mathrm{GL}_n(\mathcal{A}(z,1))$. Hence, $\boldsymbol{F}(\widetilde{\mathcal{U}}) \in M_n(\mathcal{A}(z,1))$. By definition,

$$\boldsymbol{F}(\widetilde{\mathcal{U}}) = \sum_{j\geq 0} \frac{C_j(z^p)}{j!z^{jp}}(x^p - z^p)^j.$$

Thus, for all $r < 1$, $\lim_{j\to\infty}\|\frac{C_j(z^p)}{j!z^{jp}}\|r^j = 0$. So, for all $r < 1$, $\lim_{j\to\infty}\|\frac{C_j}{j!}\|r^j = 0$ since $\|\frac{C_j(z^p)}{j!z^{jp}}\| = \|\frac{C_j}{j!z^j}\|$. Consequently, $\boldsymbol{r}(L_1) \geq 1$.

Finally, we are in a position to apply Proposition 4.3 to L_1 and we conclude that the coefficients of L_1 have norm less than or equal to 1. But we know that these coefficients belong to $\mathbb{Q}_p[[z]]$ and so $L_1 \in \mathbb{Z}_p[[z]][\delta]$. ■

Proposition 4.4. *Let L be a MUM differential operator of order n in $\mathbb{Z}_p[[z]][\delta]$, let A be the companion matrix of L, let $N = A(0)$, and let Y_Lz^N*

be the fundamental matrix of solutions of $\delta X = AX$. *If* $\boldsymbol{r}(L) \geq 1$ *then, for every integer* $m > 0$, *there exists a MUM differential operator* L_m *of order* n *in* $\mathbb{Z}_p[[z]][\delta]$ *such that*

(a) *the fundamental matrix of solutions of* $\delta X = B_m X$ *is*

$$(4.6)\qquad Y_{L_m} z^N = \operatorname{diag}(1, 1/p^m, \ldots, 1/p^{m(n-1)}) \Lambda_p^m(Y_L) \operatorname{diag}(1, p^m, \ldots, p^{m(n-1)}) z^N,$$

where B_m *is the companion matrix of* L_m,

(b) *for* $H_m = Y_L(\Lambda_p^m(Y_L)(z^{p^m}))^{-1} \operatorname{diag}(1, p^m, \ldots, p^{m(n-1)})$ *we have*

$$H_m \in M_n(\mathbb{Z}_p[[z]]) \cap \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]]),$$

$\|H_m\| = 1$, *and*

$$(4.7)\qquad \delta(H_m) = AH_m - p^m H_m B_m(z^{p^m}),$$

(c) $\boldsymbol{r}(L_m) \geq 1$.

Proof. We proceed by induction on $m \in \mathbb{Z}_{>0}$. For $m = 1$ we can apply Lemma 4.1, and thus there is a MUM differential operator L_1 of order n in $\mathbb{Z}_p[[z]][\delta]$ satisfying conditions (a)–(c). Now, suppose that for some integer $m > 0$ there is a MUM differential operator L_m of order n in $\mathbb{Z}_p[[z]][\delta]$ satisfying (a)–(c). Then we can apply Lemma 4.1 to the differential operator L_m, and thus there exists a MUM differential operator L_{m+1} of order n in $\mathbb{Z}_p[[z]][\delta]$ such that

$$\operatorname{diag}(1, 1/p, \ldots, 1/p^{n-1}) \Lambda_p(Y_{L_m}) \operatorname{diag}(1, p, \ldots, p^{n-1}) z^N$$

is the fundamental matrix of solutions of $\delta X = B_{m+1} X$, where B_{m+1} is the companion matrix of L_{m+1}. So, by (4.6), the fundamental matrix of solutions of $\delta X = B_{m+1} X$ can be written as follows:

$$\operatorname{diag}(1, 1/p^{m+1}, \ldots, 1/p^{(m+1)(n-1)}) \Lambda_p^{m+1}(Y_L) \times \operatorname{diag}(1, p^{m+1}, \ldots, p^{(m+1)(n-1)}) z^N.$$

By invoking Lemma 4.1 again, the matrix

$$H_1 = Y_{L_m}(\Lambda_p(Y_{L_m})(z^p))^{-1} \operatorname{diag}(1, p, \ldots, p^{n-1})$$

belongs to $M_n(\mathbb{Z}_p[[z]]) \cap \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$, $\|H_1\| = 1$, and $\delta(H_1) = B_m H_1 - pH_1 B_{m+1}(z^p)$. Thus,

$$(4.8)\qquad \delta(H_1(z^{p^m})) = p^m B_m(z^{p^m}) H_1(z^{p^m}) - H_1(z^{p^m}) p^{m+1} B_{m+1}(z^{p^{m+1}}).$$

We put $H_{m+1} = H_m(z) H_1(z^{p^m})$. Then $H_{m+1} \in M_n(\mathbb{Z}_p[[z]]) \cap \mathrm{GL}_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$ and

$$H_{m+1} = Y_L(z)((\Lambda_p^{m+1} Y_L)(z^{p^{m+1}}))^{-1} \operatorname{diag}(1, p^{m+1}, \ldots, p^{(m+1)(n-1)}).$$

Furthermore, from (4.7) and (4.8), we obtain

$$\delta(H_{m+1}) = AH_{m+1} - p^{m+1}H_{m+1}B_{m+1}(z^{p^{m+1}}).$$

Finally, $\|H_{m+1}\| = 1$ because $\|H_1\| = 1 = \|H_m\|$ and

$$H_{m+1}(0) = \operatorname{diag}(1, p^{m+1}, \ldots, p^{(m+1)(n-1)}).$$

Therefore, conditions (a) and (b) also hold true for $m+1$. Finally, by Lemma 4.1(c′), we obtain $\boldsymbol{r}(L_{m+1}) \geq 1$. For this reason condition (c) also holds true for $m+1$. ■

5. Proof of Theorem 3.4. Write $y_0(z) = \sum_{j\geq 0} f_j z^j$. In order to prove that $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$, it is sufficient to show that, for all integers $m > 0$, $f_0, f_1, \ldots, f_{p^m-1} \in \mathbb{Z}_p$. Let $m > 0$ be an integer and let A be the companion matrix of L. By hypothesis, $\boldsymbol{r}(L) \geq 1$. Therefore, by Proposition 4.4(b), there exist $B_m \in M_n(\mathbb{Z}_p[[z]])$ and $H_m \in M_n(\mathbb{Z}_p[[z]]) \cap GL_n(\mathcal{W}_{\mathbb{Q}_p} \cap \mathbb{Q}_p[[z]])$ such that $\|H_m\| = 1$ and

$$\delta(H_m) = AH_m - p^m H_m B_m(z^{p^m}). \tag{5.1}$$

We set $\widetilde{y}_0 = \Lambda_p^m(y_0(z))$. Thanks to Proposition 4.4(a), the vector $(\widetilde{y}_0, \delta\widetilde{y}_0, \ldots, \delta^{n-1}\widetilde{y}_0)$ is a solution of $\delta\vec{y} = B_m\vec{y}$. Hence, it follows from (5.1) that

$$H_m \begin{pmatrix} \widetilde{y}_0(z^{p^m}) \\ (\delta\widetilde{y}_0)(z^{p^m}) \\ \vdots \\ (\delta^{n-1}\widetilde{y}_0)(z^{p^m}) \end{pmatrix}$$

is a solution of the system $\delta\vec{y} = A\vec{y}$. If we put $H_m = (h_{i,j}(z))_{1\leq i,j\leq n}$ then

$$h_{1,1}(z)\widetilde{y}_0(z^{p^m}) + h_{1,2}(z)(\delta\widetilde{y}_0)(z^{p^m}) + \cdots + h_{1,n}(z)(\delta^{n-1}\widetilde{y}_0)(z^{p^m})$$

is a solution of L because A is the companion matrix of L. Furthermore, this solution belongs to $\mathbb{Q}_p[[z]]$ because $H_m \in M_n(\mathbb{Z}_p[[z]])$ and $y_0(z) \in \mathbb{Q}_p[[z]]$. Thus, according to Remark 2.3, there exists $c \in \mathbb{Q}_p$ such that

$$h_{1,1}(z)\widetilde{y}_0(z^{p^m}) + h_{1,2}(z)(\delta\widetilde{y}_0)(z^{p^m}) + \cdots + h_{1,n}(z)(\delta^{n-1}\widetilde{y}_0)(z^{p^m}) = cy_0(z). \tag{5.2}$$

As $y_0(0)=1$ and, for all integers $j > 0$, $(\delta^j\widetilde{y}_0)(z^{p^m})(0)=0$, we have $h_{1,1}(0)=c$. But, by Proposition 4.4(b), we conclude that $h_{1,1}(0) = 1$, so $c = 1$. By using Proposition 4.4(b) again, we infer that, for all $j \in \{2, \ldots, n\}$, $h_{1,j}(0) = 0$ and it is clear that, for all integers $j > 0$, $(\delta^j\widetilde{y}_0)(z^{p^m}) \in z^{p^m}\mathbb{Q}_p[[z]]$. Hence, from (5.2), we obtain

$$h_{1,1}(z)(\Lambda_p^m(y_0))(z^{p^m}) \equiv y_0(z) \bmod z^{p^m+1}\mathbb{Q}_p[[z]]. \tag{5.3}$$

Write $h_{1,1}(z) = \sum_{j\geq 0} h_j z^j$. Then, by (5.3), we conclude that, for every

$k \in \{1, \ldots, p^m - 1\}$, $h_k = f_k$. But $h_{1,1}(z) \in \mathbb{Z}_p[[z]]$ since $H_m \in M_n(\mathbb{Z}_p[[z]])$. Consequently, $f_k \in \mathbb{Z}_p$ for every $k \in \{1, \ldots, p^m - 1\}$. ∎

We end this section by proving the following observation.

COROLLARY 5.1. *Let L be an irreducible Picard–Fuchs equation with coefficients in $\mathbb{Q}(z)$ and $y_0(z) \in 1 + z\mathbb{Q}[[z]]$ a solution of L. If L is MUM then $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$ for almost every prime p.*

In order to prove this corollary, we need another result:

PROPOSITION 5.2. *Let L be an irreducible MUM differential operator with coefficients in $\mathbb{Q}(z)$. If L has a Frobenius structure for almost every prime p then L has a p-integral Frobenius structure for almost every prime p. Moreover, if n is the order of L then, for almost every prime p, there is a $\Phi_p = (\phi_{i,j})_{1 \le i,j \le n} \in M_n(\mathbb{Z}_p[[z]]) \cap \mathrm{GL}_n(E_p)$ such that Φ_p is a p-integral Frobenius matrix for L and $\|\Phi_p\| = 1$.*

To prove this proposition, we first recall that Ax–Sen–Tate's Theorem ensures that if $\xi \in \mathbb{C}_p$ and $\sigma(\xi) = \xi$ for any $\sigma \in \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$ then $\xi \in \mathbb{Q}_p$.

Proof of Proposition 5.2. Let $\mathcal{P}$ be the set of prime numbers. By assumption, there is a set $\mathcal{S}_1$ of prime numbers such that $\mathcal{P} \setminus \mathcal{S}_1$ is finite and, for all $p \in \mathcal{S}_1$, L has a Frobenius structure for p. Thus, according to [8, Propositions 4.1.2, 4.6.4, 4.7.2], for all $p \in \mathcal{S}_1$,

(a) the radius of convergence of L is greater than or equal to 1.

Hence, following [18, Chap. III, Proposition 5.1, Theorem 6.1], the singularities of L are all regular and the exponents of L are rational numbers. Thus, there is a set $\mathcal{S}_2$ of primes such that $\mathcal{P} \setminus \mathcal{S}_2$ is finite and

(b) if α, β are two different singularities of L then $|\alpha - \beta|_p = 1$ for all $p \in \mathcal{S}_2$,
(c) if γ is an exponent of L then $|\gamma|_p = 1$ for all $p \in \mathcal{S}_2$.

We set $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$. Then $\mathcal{P} \setminus \mathcal{S}$ is finite. We are going to see that, for all $p \in \mathcal{S}$, L has a p-integral Frobenius structure. Let p be in $\mathcal{S}$. Then there exists $G_p \in \mathrm{GL}_n(E_p)$ such that $\delta G_p = AG_p - pG_pA(z^p)$. Write $G_p = (g_{i,j})_{1 \le i,j \le n}$ and suppose that $\|G_p\| = |g_{k,l}|_{\mathcal{G}}$. Since $g_{k,l}$ is a non-zero element of the field E_p and $E_p \subset \mathcal{W}_p$, we deduce that $g_{k,l}$ is a unit element of $\mathcal{W}_p$. So, by Remark 3.1(1), there is $c \in \mathbb{C}_p$ such that c is a coefficient of $g_{k,l}$ and $|g_{k,l}|_{\mathcal{G}} = |c|$. Note that c is not zero because $g_{k,l}$ is not zero. We put $\Phi_p = \frac{1}{c}G_p$. So $\delta\Phi_p = A\Phi_p - p\Phi_pA(z^p)$, $\|\Phi_p\| = 1$, and 1 is a coefficient of $\frac{1}{c}g_{k,l}$. Furthermore, it is clear that $\Phi_p \in \mathrm{GL}_n(E_p)$. Let σ be in $\mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$. Then σ naturally extends to $\mathcal{W}_p$ as an endomorphism of rings, given by $\sigma(\sum_{n\in\mathbb{Z}} a_n z^n) = \sum_{n\in\mathbb{Z}} \sigma(a_n)z^n$. Further, $\sigma(E_p) = E_p$ and $\delta \circ \sigma = \sigma \circ \delta$. For a matrix $T \in M_n(E_p)$, the matrix T^σ is obtained by applying σ to each entry of T. Since $\delta\Phi_p = A\Phi_p - p\Phi_pA(z^p)$ and $A \in M_n(\mathbb{Q}(z))$, we have

$\delta(\Phi_p^\sigma) = A\Phi_p^\sigma - p\Phi_p^\sigma A(z^p)$. As L is irreducible and conditions (a)–(c) are satisfied because $p \in \mathcal{S}$, we can apply [17, Lemma], and therefore $\Phi_p^\sigma = d\Phi_p$ for some $d \in \mathbb{C}_p$. But 1 is a coefficient of $\frac{1}{c}g_{k,l}$, and thus $1 = \sigma(1) = d$. So, $\Phi_p^\sigma = \Phi_p$. Since σ is an arbitrary element of $\mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$, by Ax–Sen–Tate's Theorem, we conclude that Φ_p is a matrix with coefficients in $\mathcal{W}_{\mathbb{Q}_p}$. But we know that $\|\Phi_p\| = 1$, and consequently $\Phi_p \in M_n(\mathcal{W}_{\mathbb{Z}_p})$. We also have $\det(\Phi_p) \neq 0$ because $\det(G_p) \neq 0$ and $c \neq 0$.

Now, we are going to see that $\Phi_p \in M_n(\mathbb{Z}_p[[z]])$. Let $Y_L z^N$ be the fundamental matrix of solutions of $\delta X = AX$, where A is the companion matrix of L and $N = A(0)$. Then, by Remark 2.4, $\Phi_p = Y_L C Y_L(z^p)^{-1} \in M_n(\mathbb{C}_p[[z]])$, where $C \in M_n(\mathbb{C}_p)$. So, $\Phi_p \in M_n(\mathbb{C}_p[[z]]) \cap M_n(\mathcal{W}_{\mathbb{Z}_p})$. Therefore, $\Phi_p \in M_n(\mathbb{Z}_p[[z]])$ and we have already seen that $\Phi_p \in \mathrm{GL}_n(E_p)$. ∎

Proof of Corollary 5.1. Since L is a Picard–Fuchs equation, it follows from [19, Theorem 22.2.1] that L has a strong Frobenius structure for almost every prime p. Thus, according to Proposition 5.2, for almost every prime p, L has a p-integral Frobenius structure. Then, by Proposition 3.3 and Theorem 3.4, $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$ for almost every primer p. ∎

6. Some p-integrality properties. In order to prove (1) and (2) of Theorem 1.3, we need some additional results that deal mainly with the p-integrality of some power series with coefficients in $\mathbb{Q}_p$. In particular, in Theorem 6.7 we prove that $\exp(y_1/y_0)^p \in z\mathbb{Z}_p[[z]]$.

LEMMA 6.1. *Let* $\mathfrak{A} := a_0 + \frac{a_1}{p}\delta + \cdots + \frac{a_{n-1}}{p^{n-1}}\delta^{n-1}$ *with* $a_0(z), \ldots, a_{n-1}(z) \in \mathbb{Z}_p[[z]]$ *and let* $u \in 1 + z\mathbb{Q}_p[[z]]$. *If* $\mathfrak{A}(u(z^p)) = u$ *then* $u \in 1 + z\mathbb{Z}_p[[z]]$.

Proof. We set $\boldsymbol{\omega}_0 = 1$ and $\boldsymbol{\omega}_{i+1} = \mathfrak{A}(\boldsymbol{\omega}_i(z^p))$ for $i \geq 0$. Then, for all integers $m \geq 1$, $\boldsymbol{\omega}_m \in 1 + z\mathbb{Z}_p[[z]]$ because, for all integers $j \geq 1$, $\delta^j(\boldsymbol{\omega}_i(z^p)) = p^j(\delta^j\boldsymbol{\omega}_i)(z^p)$. We are going to see that, for all integers $m \geq 0$, $\boldsymbol{\omega}_m \equiv u \bmod z^{p^m}\mathbb{Z}_p[[z]]$. As $u(0) = 1$, we have $\boldsymbol{\omega}_0 \equiv u \bmod z\mathbb{Z}_p[[z]]$. Now, we suppose that $\boldsymbol{\omega}_m \equiv u \bmod z^{p^m}\mathbb{Z}_p[[z]]$ for some integer $m \geq 0$. Then $\boldsymbol{\omega}_m(z^p) \equiv u(z^p) \bmod z^{p^{m+1}}\mathbb{Z}_p[[z]]$. Further,

$$\boldsymbol{\omega}_{m+1} - u = \mathfrak{A}(\boldsymbol{\omega}_m(z^p)) - \mathfrak{A}(u(z^p)) = \sum_{i=0}^{n-1} a_i[(\delta^i\boldsymbol{\omega}_m)(z^p) - (\delta^i u)(z^p)].$$

Since $\boldsymbol{\omega}_m(z^p) - u(z^p) \in z^{p^{m+1}}\mathbb{Z}_p[[z]]$, for all integers $i \geq 0$ we have

$$\delta^i(\boldsymbol{\omega}_m(z^p) - u(z^p)) \in p^{(m+1)i} z^{p^{m+1}}\mathbb{Z}_p[[z]].$$

Nevertheless, $\delta^i(\boldsymbol{\omega}_m(z^p)) = p^i(\delta^i(\boldsymbol{\omega}_m))(z^p)$, and similarly $\delta^i(u(z^p)) = p^i(\delta^i(u))(z^p)$. Thus,

$$p^i[(\delta^i\boldsymbol{\omega}_m)(z^p) - (\delta^i u)(z^p)] = \delta^i[\boldsymbol{\omega}_m(z^p) - u(z^p)] \in p^{(m+1)i} z^{p^{m+1}}\mathbb{Z}_p[[z]].$$

Hence,

$$(\delta^i \boldsymbol{\omega}_m)(z^p) - (\delta^i u)(z^p) \in p^{mi} z^{p^{m+1}} \mathbb{Z}_p[[z]] \subset z^{p^{m+1}} \mathbb{Z}_p[[z]].$$

Therefore, $\boldsymbol{\omega}_{m+1} - u \in z^{p^{m+1}}\mathbb{Z}_p[[z]]$. We conclude, for all integers $m \geq 0$, that $\boldsymbol{\omega}_m \equiv u \bmod z^{p^m}\mathbb{Z}_p[[z]]$. Consequently, $u \in 1 + z\mathbb{Z}_p[[z]]$ because for all integers $m \geq 1$, $\boldsymbol{\omega}_m \in 1 + z\mathbb{Z}_p[[z]]$. ∎

LEMMA 6.2. *Let L and D be MUM differential operators in $\mathbb{Q}_p[[z]][\delta]$ of order n and let A and B be the respective companion matrices of L and D. Let $y_0 = \mathfrak{f}(z)$, $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ be solutions of L and let $\widetilde{y}_0 = \mathfrak{f}_1(z)$, $\widetilde{y}_1 = \mathfrak{f}_1(z)\log z + \mathfrak{g}_1(z)$ be solutions of D, with $\mathfrak{f}(z), \mathfrak{f}_1(z) \in 1 + z\mathbb{Q}_p[[z]]$ and $\mathfrak{g}(z), \mathfrak{g}_1(z) \in z\mathbb{Q}_p[[z]]$. Suppose that there is an $H = (h_{i,j}(z))_{1\leq i,j\leq n} \in M_n(\mathbb{Q}_p[[z]])$ such that $\delta H = AH - pHB(z^p)$. Then*

$$\widetilde{\omega}(z^p)\alpha_0 y_0 + (\delta\widetilde{\omega})(z^p)r_2 + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)r_n y_0 = y_0(p\alpha_0\omega + \alpha_1),$$

where $\widetilde{\omega} = \widetilde{y}_1/\widetilde{y}_0$, $\omega = y_1/y_0$, $\alpha_0 = h_{1,1}(0)$, $\alpha_1 = h_{1,2}(0)$, and for $2 \leq j \leq n$, $r_j = \sum_{i=j}^{n} \binom{i-1}{i-j} h_{1,i}(z)(\delta^{i-j}\widetilde{y}_0)(z^p)$.

Proof. Since $\widetilde{y}_0$ is a solution of D, it follows from $\delta H = AH - pHB(z^p)$ that $\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p)$ is a solution of L. It is clear that

$$\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p) \in \mathbb{Q}_p[[z]].$$

So, according to Remark 2.3, there is $c \in \mathbb{Q}_p$ such that

$$\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p) = c\mathfrak{f}(z).$$

Therefore, $c = h_{1,1}(0) = \alpha_0$. In a similar way, since $\widetilde{y}_1$ is a solution of D, it follows from $\delta H = AH - pHB(z^p)$ that $\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_1)(z^p)$ is a solution of L. As $\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_1)(z^p) \in \mathbb{Q}_p[[z]] + \mathbb{Q}_p[[z]]\log z$, it follows from Remark 2.3 that there are $c_0, c_1 \in \mathbb{Q}_p$ such that $\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_1)(z^p) = c_0 y_0 + c_1 y_1$. Now, for all integers $m \geq 1$,

$$\delta^m \widetilde{y}_1 = \delta^m(\mathfrak{f}_1(z))\log z + m\delta^{m-1}\mathfrak{f}_1(z) + \delta^m \mathfrak{g}_1(z).$$

So,

$$\begin{aligned}\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_1)(z^p) &= \Big[\sum_{i=1}^{n} p h_{1,i}(z)(\delta^{i-1}\mathfrak{f}_1)(z^p)\Big]\log z \\ &\quad + \sum_{i=0}^{n-1} h_{1,i+1}(z)(i\delta^{i-1}\mathfrak{f}_1 + \delta^i \mathfrak{g}_1)(z^p) \\ &= c_0 y_0 + c_1 y_1 = c_1\mathfrak{f}(z)\log z + c_0\mathfrak{f}(z) + c_1\mathfrak{g}(z).\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} p h_{1,i}(z)(\delta^{i-1}\mathfrak{f}_1)(z^p) = c_1\mathfrak{f}(z),$$

$$\sum_{i=0}^{n-1} h_{1,i+1}(z)(i\delta^{i-1}\mathfrak{f}_1 + \delta^i\mathfrak{g}_1)(z^p) = c_0\mathfrak{f}(z) + c_1\mathfrak{g}(z).$$

From the second equality, we have $c_0 = h_{1,2}(0)$ because $\mathfrak{g}(0) = 0 = \mathfrak{g}_1(0)$ and $\mathfrak{f}(0) = 1 = \mathfrak{f}_1(0)$. So, $c_0 = \alpha_1$. Further, we have already seen that $\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\mathfrak{f}_1)(z^p) = \alpha_0\mathfrak{f}$. Thus $c_1 = p\alpha_0$.

We put $\widetilde{\omega} = \widetilde{y}_1/\widetilde{y}_0$. Then $\widetilde{\omega}\widetilde{y}_0 = \widetilde{y}_1$ and

$$\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}(\widetilde{\omega}\widetilde{y}_0))(z^p) = \alpha_1 y_0 + p\alpha_0 y_1.$$

Since, for all integers $m \geq 1$, $\delta^m(\widetilde{\omega}\widetilde{y}_0) = \sum_{i=0}^{m} \binom{m}{i}\delta^{m-i}\widetilde{\omega}\cdot\delta^i\widetilde{y}_0$, we obtain

$$\begin{aligned}\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}(\widetilde{\omega}\widetilde{y}_0))(z^p) = \widetilde{\omega}(z^p)\Big[\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p)\Big] \\ + (\delta\widetilde{\omega})(z^p)\Big[\sum_{i=2}^{n}\binom{i-1}{i-2} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p)\Big] \\ + (\delta^2\widetilde{\omega})(z^p)\Big[\sum_{i=3}^{n}\binom{i-1}{i-3} h_{1,i}(z)(\delta^{i-2}\widetilde{y}_0)(z^p)\Big] \\ + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)h_{1,n}(z)(\widetilde{y}_0)(z^p).\end{aligned}$$

We know that

$$\sum_{i=1}^{n} h_{1,i}(z)(\delta^{i-1}\widetilde{y}_0)(z^p) = \alpha_0 y_0,$$

and, for every $j \in \{2, \ldots, n\}$, we set $r_j = \sum_{i=j}^{n}\binom{i-1}{i-j} h_{1,i}(z)(\delta^{i-j}\widetilde{y}_0)(z^p)$. So,

$$\sum_{i=1}^{n} h_{1,i}(\delta^{i-1}(\widetilde{\omega}\widetilde{y}_0))(z^p) = \widetilde{\omega}(z^p)\alpha_0 y_0 + (\delta\widetilde{\omega})(z^p)r_2 + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)r_n.$$

Consequently,

$$\begin{aligned}\widetilde{\omega}(z^p)\alpha_0 y_0 + (\delta\widetilde{\omega})(z^p)r_2 + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)r_n \\ = \alpha_1 y_0 + p\alpha_0 y_1 = y_0(p\alpha_0\omega + \alpha_1).\end{aligned}$$

This finishes the proof of Lemma 6.2. ∎

Theorem 3.4 and Lemma 4.1 are essential in the proof of the following two propositions.

Proposition 6.3. *Let L be a MUM differential operator of order n with coefficients in $\mathbb{Z}_p[[z]]$. Let $y_0 = \mathfrak{f}(z)$ and $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ be solutions of L such that $\mathfrak{f}(z) \in 1 + z\mathbb{Q}_p[z]]$ and $\mathfrak{g}(z) \in z\mathbb{Q}_p[[z]]$. Suppose that L has a p-integral Frobenius structure given by the matrix $\Phi = (\phi_{i,j}(z))_{1\le i,j\le n}$. If $|\phi_{1,1}(0)| = 1$ then $\delta\widetilde{\omega} \in 1 + z\mathbb{Z}_p[[z]]$, where $\widetilde{\omega} = \widetilde{y_1}/\widetilde{y_0}$ with $\widetilde{y_0} = \Lambda_p(\mathfrak{f}(z))$ and $\widetilde{y_1} = \Lambda_p(\mathfrak{f}(z))\log z + p\Lambda_p(\mathfrak{g}(z))$.*

Proof. Let $Y_L z^N$ be the fundamental matrix of solutions of $\delta X = AX$, where A is the companion matrix of L. Since L has a p-integral Frobenius structure, according to Proposition 3.3, $\boldsymbol{r}(L) \ge 1$. As $L \in \mathbb{Z}_p[[z]][\delta]$ is MUM and $\boldsymbol{r}(L) \ge 1$, we can apply Lemma 4.1(a′), and thus there is a MUM differential operator $L_1 \in \mathbb{Z}_p[[z]][\delta]$ of order n such that the fundamental matrix of solutions of $\delta X = B_1X$ is given by

$$\operatorname{diag}(1, 1/p, \dots, 1/p^{n-1})\Lambda_p(Y_L)\operatorname{diag}(1, p, \dots, p^{n-1})z^N,$$

where B_1 is the companion matrix of L_1. Therefore,

$$\widetilde{y_0} := \Lambda_p(\mathfrak{f}(z)) \quad \text{and} \quad \widetilde{y_1} := \Lambda_p(\mathfrak{f}(z))\log z + p\Lambda_p(\mathfrak{g}(z))$$

are solutions of L_1. Set

$$T = \operatorname{diag}(1, 1/p, \dots, 1/p^{n-1})\Lambda_p(\Phi)Y_L(z)(\Lambda_p(Y_L)(z^p))^{-1}\operatorname{diag}(1, p, \dots, p^{n-1}).$$

We are going to see that $\delta T = B_1T - pTB_1(z^p)$. In fact, from Lemma 4.1(b′), we deduce that $\delta(H_1^{-1}) = pB_1(z^p)H_1^{-1} - H_1^{-1}A$, where

$$H_1 = Y_L(z)(\Lambda_p(Y_L)(z^p))^{-1}\operatorname{diag}(1, p, \dots, p^{n-1}).$$

By the hypotheses, we know that $\delta\Phi = A\Phi - p\Phi A(z^p)$. Thus,

$$\delta(H_1^{-1}\Phi) = pB_1(z^p)H_1^{-1}\Phi - pH_1^{-1}\Phi A(z^p).$$

As $\Lambda_p \circ \delta = p\delta \circ \Lambda_p$, we get

$$\delta(\Lambda_p(H_1^{-1}\Phi)) = B_1\Lambda_p(H_1^{-1}\Phi) - \Lambda_p(H_1^{-1}\Phi)A.$$

By invoking Lemma 4.1(b′) again, we have $\delta(H_1) = AH_1 - pH_1B_1(z^p)$. Thus,

$$\delta(\Lambda_p(H_1^{-1}\Phi)H_1) = B_1\Lambda_p(H_1^{-1}\Phi)H_1 - p\Lambda_p(H_1^{-1}\Phi)H_1B_1(z^p).$$

From Remark 2.4, we know that $\Phi = Y_L\Phi(0)Y_L(z^p)^{-1}$. Hence,

$$\Lambda_p(\Phi) = \Lambda_p(Y_L)\Phi(0)Y_L(z)^{-1},$$

and consequently

$$\Lambda_p(H_1^{-1}\Phi)H_1 = T.$$

Let us write $T = (t_{i,j}(z))_{1\le i,j\le n}$. We have $T \in M_n(\mathbb{Q}_p[[z]])$ because $Y_L \in \mathrm{GL}_n(\mathbb{Q}_p[[z]])$ and, by assumption, $\Phi \in M_n(\mathbb{Z}_p[[z]])$. As $\delta T = B_1T - pTB_1(z^p)$, by Lemma 6.2 we deduce that

$$\widetilde{\omega}(z^p)\alpha_0\widetilde{y_0} + (\delta\widetilde{\omega})(z^p)r_2 + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)r_n = \widetilde{y_0}(p\alpha_0\widetilde{\omega} + \alpha_1),$$

where $\alpha_0 = t_{1,1}(0)$, $\alpha_1 = t_{1,2}(0)$ and $r_j = \sum_{i=j}^n \binom{i-1}{i-j} t_{1,i}(z)(\delta^{i-j}\widetilde{y_0})(z^p)$.

Since $|\phi_{1,1}(0)| = 1$ and $\|\Phi\| = 1$, we have $\|\Lambda_p(\Phi)\| = 1$. In addition, by Lemma 4.2, we know that $Y_L(z)(\Lambda_p(Y_L)(z^p))^{-1} \in \mathrm{GL}_n(\mathbb{Z}_p[[z]])$. Therefore, $t_{1,2}, \ldots, t_{1,n} \in p\mathbb{Z}_p[[z]]$, and consequently $r_2, \ldots, r_n \in p\mathbb{Z}_p[[z]]$ and $\alpha_1 = p\beta$ with $\beta \in \mathbb{Z}_p$. So, $r_j = ps_j$ with $s_j \in \mathbb{Z}_p[[z]]$ and we obtain

$$\widetilde{\omega}(z^p)\frac{\alpha_0}{p} + (\delta\widetilde{\omega})(z^p)\frac{s_2}{\widetilde{y_0}} + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)\frac{s_n}{\widetilde{y_0}} = \alpha_0\widetilde{\omega} + \beta.$$

As $\boldsymbol{r}(L) \geq 1$, by Theorem 3.4, $y_0 = \mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$. Thus, $\widetilde{y_0} \in 1 + z\mathbb{Z}_p[[z]]$ and, for every $j \in \{2, \ldots, n\}$, $\gamma_j := s_j/\widetilde{y_0} \in \mathbb{Z}_p[[z]]$. We put $\boldsymbol{\omega} = \delta\widetilde{\omega}$. Then

$$\widetilde{\omega}(z^p)\frac{\alpha_0}{p} + \boldsymbol{\omega}(z^p)\gamma_2 + \cdots + (\delta^{n-2}\boldsymbol{\omega})(z^p)\gamma_n = \alpha_0\widetilde{\omega} + \beta.$$

By applying δ to this equality we obtain

$$\delta(\widetilde{\omega}(z^p))\frac{\alpha_0}{p} + \delta(\boldsymbol{\omega}(z^p)\gamma_2) + \cdots + \delta((\delta^{n-2}\boldsymbol{\omega})(z^p)\gamma_n) = \alpha_0\boldsymbol{\omega}.$$

As $\delta(\widetilde{\omega}(z^p)) = p(\delta\widetilde{\omega})(z^p)$, we have $\delta(\widetilde{\omega}(z^p))\frac{1}{p} = \boldsymbol{\omega}(z^p)$. Therefore,

$$\alpha_0\boldsymbol{\omega}(z^p) + \delta(\boldsymbol{\omega}(z^p)\gamma_2) + \cdots + \delta((\delta^{n-2}\boldsymbol{\omega})(z^p)\gamma_n) = \alpha_0\boldsymbol{\omega}.$$

Now, $\delta((\delta^i\boldsymbol{\omega})(z^p)\gamma_{i+2}) = p(\delta^{i+1}\boldsymbol{\omega})(z^p)\gamma_{i+2} + (\delta^i\boldsymbol{\omega})(z^p)\delta(\gamma_{i+2})$. Further, $\alpha_0 \in \mathbb{Z}_p^*$ since $\alpha_0 = t_{1,1}(0) = \phi_{1,1}(0)$ and by hypotheses $|\phi_{1,1}(0)| = 1$. Thus, there are $a_0, \ldots, a_n \in \mathbb{Z}_p[[z]]$ such that

$$a_0\boldsymbol{\omega}(z^p) + a_1(\delta\boldsymbol{\omega})(z^p) + \cdots + a_{n-1}(\delta^{n-1}\boldsymbol{\omega})(z^p) = \boldsymbol{\omega}.$$

Finally, we set $\mathfrak{A} := a_0 + \frac{a_1}{p}\delta + \cdots + \frac{a_{n-1}}{p^{n-1}}\delta^{n-1}$. So, $\mathfrak{A}(\boldsymbol{\omega}(z^p)) = \boldsymbol{\omega}$. Therefore, from Lemma 6.1, we obtain $\delta\widetilde{\omega} = \boldsymbol{\omega} \in 1 + z\mathbb{Z}_p[[z]]$. ■

PROPOSITION 6.4. *Let L be a MUM differential operator of order n with coefficients in $\mathbb{Z}_p[[z]]$. Let $y_0 = \mathfrak{f}(z)$ and $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ be solutions of L with $\mathfrak{f}(z) \in 1 + z\mathbb{Q}_p[[z]]$ and $\mathfrak{g}(z) \in z\mathbb{Q}_p[[z]]$. Suppose that L has a p-integral Frobenius structure given by the matrix $\Phi = (\phi_{i,j})_{1\leq i,j\leq n}$. If $|\phi_{1,1}(0)| = 1$ then $\delta\omega \in 1 + z\mathbb{Z}_p[[z]]$, where $\omega = y_1/y_0$.*

Proof. Let $Y_L z^N$ be the fundamental matrix of solutions of $\delta X = AX$, where A is the companion matrix of L. Since L has a p-integral Frobenius structure, by Proposition 3.3, we get $\boldsymbol{r}(L) \geq 1$. As $L \in \mathbb{Z}_p[[z]][\delta]$ is MUM and $\boldsymbol{r}(L) \geq 1$, we can apply Lemma 4.1, and thus, from Lemma 4.1(a′), there is a MUM differential operator L_1 such that $\widetilde{y}_0 = \Lambda_p(\mathfrak{f})$ and $\widetilde{y}_1 = \Lambda_p(\mathfrak{f})\log z + p\Lambda_p(\mathfrak{g})$ are solutions of L_1. By Lemma 4.1(b′), we know that $H_1 = Y_L(z)(\Lambda_p(Y_L)(z^p))^{-1}\operatorname{diag}(1, p, p^2, \ldots, p^{n-1})$ belongs to $M_n(\mathbb{Z}_p[[z]])$ and $\delta H_1 = AH_1 - pH_1B_1(z^p)$, where B_1 is the companion matrix of L_1. Additionally, from Lemma 4.2, we have $Y_L(z)(\Lambda_p(Y_L)(z^p))^{-1} \in GL_n(\mathbb{Z}_p[[z]])$. So, if we put $H_1 = (w_{i,j}(z))_{1\leq i,j\leq n}$ then $w_{1,j}(z) \in p^{j-1}\mathbb{Z}_p[[z]]$ for all $1 \leq j \leq n$. Since $H_1 \in M_n(\mathbb{Z}_p[[z]])$ and $H_1(0) = \operatorname{diag}(1, p, \ldots, p^{n-1})$, from Lemma 6.2 we get

$$\widetilde{\omega}(z^p)y_0 + (\delta\widetilde{\omega})(z^p)r_2 + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)r_n = py_0\omega,$$

where $\widetilde{\omega} = \widetilde{y}_1/\widetilde{y}_0$ and $r_j = \sum_{i=j}^{n} \binom{i-1}{i-j} w_{1,i}(z)(\delta^{i-j}\widetilde{y}_0)(z^p)$. As $w_{1,2}, \dots, w_{1,n} \in p\mathbb{Z}_p[[z]]$, we obtain $r_2, \dots, r_n \in p\mathbb{Z}_p[[z]]$. So, $r_j = ps_j$ with $s_j \in \mathbb{Z}_p[[z]]$. For this reason,

$$\widetilde{\omega}(z^p)\frac{1}{p} + (\delta\widetilde{\omega})(z^p)\frac{s_2}{y_0} + \cdots + (\delta^{n-1}\widetilde{\omega})(z^p)\frac{s_n}{y_0} = \omega.$$

We know that $\boldsymbol{r}(L) \geq 1$, and thus, by Theorem 3.4, $y_0 = \mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$. So, $1/y_0 \in 1 + z\mathbb{Z}_p[[z]]$ and, for every $j \in \{1, \dots, n-1\}$, $t_j := s_j/y_0$ belongs to $\mathbb{Z}_p[[z]]$. We put $\boldsymbol{\omega} = \delta\widetilde{\omega}$. Then

$$\widetilde{\omega}(z^p)\frac{1}{p} + (\boldsymbol{\omega})(z^p)t_2 + \cdots + (\delta^{n-2}\boldsymbol{\omega})(z^p)t_n = \omega.$$

By applying δ to this equality we obtain

$$\delta(\widetilde{\omega}(z^p))\frac{1}{p} + \delta((\boldsymbol{\omega})(z^p)t_2) + \cdots + \delta((\delta^{n-2}\boldsymbol{\omega})(z^p)t_n) = \delta\omega.$$

As $\delta(\widetilde{\omega}(z^p)) = p(\delta\widetilde{\omega})(z^p)$, we have $\delta(\widetilde{\omega}(z^p))\frac{1}{p} = \boldsymbol{\omega}(z^p)$. Therefore,

$$\boldsymbol{\omega}(z^p) + \delta((\boldsymbol{\omega})(z^p)t_1) + \cdots + \delta((\delta^{n-2}\boldsymbol{\omega})(z^p)t_{n-1}) = \delta\omega.$$

According to Proposition 6.3, $\delta\widetilde{\omega} = \boldsymbol{\omega}$ belongs to $1 + z\mathbb{Z}_p[[z]]$. Therefore, $\delta\omega$ belongs to $1 + z\mathbb{Z}_p[[z]]$. ∎

To prove Theorem 6.7, we also recall the following classical lemma.

LEMMA 6.5 (Dieudonné–Dwork's Lemma). *Let p be a prime number and let $f(z)$ be in $1 + z\mathbb{Q}_p[[z]]$. Then $f(z) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if $\frac{f(z)^p}{f(z^p)} \in 1 + pz\mathbb{Z}_p[[z]]$.*

For a proof of this lemma we refer the reader to [18, Chap. II, Theorem 5.2]. As a corollary we have

COROLLARY 6.6. *Let p be a prime number and let $f(z)$ be in $z\mathbb{Q}_p[[z]]$. Then $\exp(f(z)) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if $\frac{1}{p}f(z^p) - f(z) \in z\mathbb{Z}_p[[z]]$.*

THEOREM 6.7. *Let L be a MUM differential operator of order n with coefficients in $\mathbb{Z}_p[[z]]$. Let $y_0 = \mathfrak{f}(z)$ and $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ be solutions of L with $\mathfrak{f}(z) \in 1 + z\mathbb{Q}_p[[z]]$ and $\mathfrak{g}(z) \in z\mathbb{Q}_p[[z]]$. Suppose that L has a p-integral Frobenius structure given by the matrix $\Phi = (\phi_{i,j})_{1\leq i,j\leq n}$. If $|\phi_{1,1}(0)| = 1$ then $\exp(y_1/y_0)^p \in \mathbb{Z}_p[[z]]$.*

Proof. We set $\mathfrak{h} = p\frac{\mathfrak{g}(z)}{\mathfrak{f}(z)}$. Since $\exp(y_1/y_0)^p = z\exp(\mathfrak{h})$, according to Corollary 6.6, it is sufficient to show that $\frac{1}{p}\mathfrak{h}(z^p) - \mathfrak{h}(z) \in \mathbb{Z}_p[[z]]$. By assumption, there is $\Phi = (\phi_{i,j}(z))_{1\leq i,j\leq n} \in M_n(\mathbb{Z}_p[[z]])$ such that $\delta\Phi = A\Phi - p\Phi A(z^p)$. We put $\omega = \frac{y_1}{y_0}$, where $y_1 = \mathfrak{f}(z)\log z + \mathfrak{g}(z)$ and $y_0 = \mathfrak{f}(z)$. It follows from Lemma 6.2 that

$$\omega(z^p)\alpha_0 y_0 + (\delta\omega)(z^p)r_2 + \cdots + (\delta^{n-1}\omega)(z^p)r_n = y_0(p\alpha_0\omega + \alpha_1),$$

where $r_j = \sum_{i=j}^{n} \binom{i-1}{i-j}\phi_{1,i}(z)(\delta^{i-j}y_0)(z^p)$, $\alpha_0 = \phi_{1,1}(0)$, and $\alpha_1 = \phi_{1,2}(0)$.

Consequently,

$$\omega(z^p)\alpha_0 + (\delta\omega)(z^p)\frac{r_2}{y_0} + \cdots + (\delta^{n-1}\omega)(z^p)\frac{r_n}{y_0} = p\alpha_0\omega + \alpha_1.$$

By Proposition 3.3 and Theorem 3.4, we know that $y_0 = \mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$. So, $1/y_0 \in 1 + z\mathbb{Z}_p[[z]]$. Then, for every $j \in \{2,\dots,n\}$, r_j/y_0 belongs to $\mathbb{Z}_p[[z]]$. As $|\phi_{1,1}(0)| = 1$, by Proposition 6.4, we have $\delta\omega \in 1+z\mathbb{Z}_p[[z]]$. Thus, we obtain $\omega(z^p)\alpha_0 - p\alpha_0\omega \in z\mathbb{Z}_p[[z]]$. Further, $\alpha_0 \in \mathbb{Z}_p^*$ because $\alpha_0 = \phi_{1,1}(0)$ and $|\phi_{1,1}(0)| = 1$. Hence, $\omega(z^p) - p\omega \in z\mathbb{Z}_p[[z]]$. But

$$\begin{aligned}\omega(z^p) - p\omega &= \left[\frac{p\mathfrak{f}(z^p)\log z + \mathfrak{g}(z^p)}{\mathfrak{f}(z^p)}\right] - \left[\frac{p\mathfrak{f}(z)\log z + p\mathfrak{g}(z)}{\mathfrak{f}(z)}\right] \\ &= \frac{\mathfrak{g}(z^p)}{\mathfrak{f}(z^p)} - p\frac{\mathfrak{g}(z)}{\mathfrak{f}(z)} = \frac{1}{p}\mathfrak{h}(z^p) - \mathfrak{h}(z). \ \blacksquare\end{aligned}$$

7. Proof of Theorem 1.3. Given that, by assumption, L has a p-integral Frobenius structure, it follows from Proposition 3.3 that $\boldsymbol{r}(L) \geq 1$. Thus, by Theorem 3.4, we get $y_0(z) \in 1 + z\mathbb{Z}_p[[z]]$. Let us now prove (1) and (2).

(1) By definition, $L^{(2)} = (\delta - t_2)(\delta - t_1)$, where

$$t_1 = \frac{\delta\mathfrak{f}(z)}{\mathfrak{f}(z)}, \quad t_2 = \frac{\delta\mathfrak{h}(z)}{\mathfrak{h}(z)} \quad \text{with } \mathfrak{h}(z) = \mathfrak{f}(z) + \delta(\mathfrak{g}(z)) - t_1\mathfrak{g}(z).$$

As $|\phi_{1,1}(0)| = 1$, by Proposition 6.4, $\delta(y_1/y_0) \in 1+z\mathbb{Z}_p[[z]]$. In particular, $\delta(\mathfrak{g}(z)/\mathfrak{f}(z))$ belongs to $\mathbb{Z}_p[[z]]$. Since

$$\delta\left(\frac{\mathfrak{g}(z)}{\mathfrak{f}(z)}\right) = \frac{(\delta\mathfrak{g}(z))\mathfrak{f}(z) - \mathfrak{g}(z)(\delta\mathfrak{f}(z))}{\mathfrak{f}(z)^2}$$

and $\mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$, we obtain $(\delta\mathfrak{g}(z))\mathfrak{f}(z) - \mathfrak{g}(z)(\delta\mathfrak{f}(z)) \in \mathbb{Z}_p[[z]]$. Notice that

$$\begin{aligned}\mathfrak{h}(z) &= \mathfrak{f}(z) + \delta(\mathfrak{g}(z)) - t_1\mathfrak{g}(z) = \mathfrak{f}(z) + \delta(\mathfrak{g}(z)) - \frac{\delta\mathfrak{f}(z)}{\mathfrak{f}(z)}\mathfrak{g}(z) \\ &= \frac{\mathfrak{f}(z)^2 + (\delta\mathfrak{g}(z))\mathfrak{f}(z) - (\delta\mathfrak{f}(z))\mathfrak{g}(z)}{\mathfrak{f}(z)}.\end{aligned}$$

So, $\mathfrak{h}(z) \in 1 + z\mathbb{Z}_p[[z]]$. Consequently, t_1 and t_2 belong to $z\mathbb{Z}_p[[z]]$. Hence, $L^{(2)} \in \mathbb{Z}_p[[z]][\delta]$ and it is clear that $L^{(2)}$ is MUM.

We now proceed to show that $L^{(2)}$ has a p-adic Frobenius structure. We set

$$\Theta = J(z)\,\mathrm{diag}(1,p)J(z^p)^{-1}, \quad \text{with} \quad J(z) = \begin{pmatrix}\mathfrak{f}(z) & \mathfrak{g}(z) \\ \delta\mathfrak{f}(z) & \mathfrak{f}(z) + \delta\mathfrak{g}(z)\end{pmatrix}. \tag{7.1}$$

First, we prove that $\delta\Theta = A_2\Theta - p\Theta A_2(z^p)$, where A_2 is the companion matrix of $L^{(2)}$. Notice that

$$\Theta = J(z)\begin{pmatrix}1 & \log z\\ 0 & 1\end{pmatrix}\mathrm{diag}(1,p)\begin{pmatrix}1 & -p\log z\\ 0 & 1\end{pmatrix}J(z^p)^{-1}.$$

So,

$$\Theta J(z^p)\begin{pmatrix}1 & p\log z\\ 0 & 1\end{pmatrix} = J(z)\begin{pmatrix}1 & \log z\\ 0 & 1\end{pmatrix}\mathrm{diag}(1,p). \tag{7.2}$$

It is clear that the matrix $J(z)\left(\begin{smallmatrix}1 & \log z\\ 0 & 1\end{smallmatrix}\right)\mathrm{diag}(1,p)$ is a solution of the system $\delta X = A_2X$ and that $J(z^p)\left(\begin{smallmatrix}1 & p\log z\\ 0 & 1\end{smallmatrix}\right)$ is a solution of the system $\delta X = pA_2(z^p)X$. Consequently, we deduce from (7.2) that

$$\delta\Theta = A_2\Theta - p\Theta A_2(z^p). \tag{7.3}$$

By (7.1), we have

$$\Theta = \begin{pmatrix}\theta_{1,1}(z) & \theta_{1,2}(z)\\ \theta_{2,1}(z) & \theta_{2,2}(z)\end{pmatrix},$$

where

$$\begin{aligned}
\theta_{1,1}(z) &= \frac{\mathfrak{f}(z)((\mathfrak{f}+\delta\mathfrak{g})(z^p)) - p\mathfrak{g}(z)((\delta\mathfrak{f})(z^p))}{\Delta},\\
\theta_{1,2}(z) &= \frac{p\mathfrak{g}(z)(\mathfrak{f}(z^p)) - \mathfrak{f}(z)(\mathfrak{g}(z^p))}{\Delta},\\
\theta_{2,1}(z) &= \frac{(\delta\mathfrak{f}(z))((\mathfrak{f}+\delta\mathfrak{g})(z^p)) - p(\mathfrak{f}(z)+\delta\mathfrak{g}(z))(\delta\mathfrak{f}(z^p))}{\Delta},\\
\theta_{2,2}(z) &= \frac{p(\mathfrak{f}(z)+\delta\mathfrak{g}(z))(\mathfrak{f}(z^p)) - (\delta\mathfrak{f}(z))(\mathfrak{g}(z^p))}{\Delta}.
\end{aligned}$$

with $\Delta = (\mathfrak{f}(\mathfrak{f}+\delta\mathfrak{g}) - \mathfrak{g}(\delta\mathfrak{f}))(z^p)$. We next prove that $\theta_{1,1}(z), \theta_{1,2}(z) \in \mathbb{Z}_p[[z]]$.

Note that $\Delta = (\mathfrak{h}\mathfrak{f})(z^p)$. As $\mathfrak{h}(z), \mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$, we see that $\Delta \in 1 + z\mathbb{Z}_p[[z]]$. From Theorem 6.7, we have $\exp(y_1/y_0)^p \in \mathbb{Z}_p[[z]]$ and, by Corollary 6.6, this is equivalent to saying that $\frac{\mathfrak{g}(z^p)}{\mathfrak{f}(z^p)} - p\frac{\mathfrak{g}(z)}{\mathfrak{f}(z)} \in z\mathbb{Z}_p[[z]]$. Since $\mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$, we obtain $p\mathfrak{g}(z)\mathfrak{f}(z^p) - \mathfrak{f}(z)\mathfrak{g}(z^p) \in \mathbb{Z}_p[[z]]$. Thus, $\theta_{1,2} \in z\mathbb{Z}_p[[z]]$. Furthermore, it is clear from (7.3) that

$$\theta_{1,1}(z)(\mathfrak{f}(z^p)) + \theta_{1,2}(z)((\delta\mathfrak{f})(z^p)) = \mathfrak{f}(z).$$

In particular,

$$\theta_{1,1}(z) = \frac{\mathfrak{f}(z) - \theta_{1,2}(z)((\delta\mathfrak{f})(z^p))}{\mathfrak{f}(z^p)}.$$

We have already seen that $\theta_{1,2}(z) \in \mathbb{Z}_p[[z]]$ and we know that $\mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$. Therefore, $\theta_{1,1}(z) \in 1 + z\mathbb{Z}_p[[z]]$.

Let us write $L^{(2)} = \delta^2 + a_1(z)\delta + a_2(z)$ with $a_1(z), a_2(z) \in z\mathbb{Z}_p[[z]]$. Put

$$\begin{aligned}\Psi &= \begin{pmatrix} \psi_{1,1}(z) & \psi_{1,2}(z) \\ \psi_{2,1}(z) & \psi_{2,2}(z) \end{pmatrix} \\ &= \begin{pmatrix} \theta_{1,1}(z) & \theta_{1,2}(z) \\ \delta\theta_{1,1}(z) - p\theta_{1,2}(z)a_2(z^p) & \delta\theta_{1,2}(z) + p\theta_{1,1}(z) - p\theta_{1,2}(z)a_1(z^p) \end{pmatrix}.\end{aligned}$$

Consequently, $\Psi \in M_2(\mathbb{Z}_p[[z]])$. We now show that

$$\delta\Psi = A_2\Psi - p\Psi A_2(z^p). \tag{7.4}$$

Let y be a solution of $L^{(2)}$ in $\mathbb{Q}_p[[z]] + \mathbb{Q}_p[[z]]\log z$. Then, from (7.3), it follows that

$$r = \theta_{1,1}(z)(y(z^p)) + \theta_{1,2}(z)((\delta y)(z^p))$$

is a solution of $L^{(2)}$. Since, by definition, $\theta_{1,1}(z) = \psi_{1,1}(z)$ and $\theta_{1,2}(z) = \psi_{1,2}(z)$, we deduce that

$$r = \psi_{1,1}(z)(y(z^p)) + \psi_{1,2}(z)((\delta y)(z^p)) \tag{7.5}$$

is a solution of $L^{(2)}$. So, by applying δ to the previous equality and by using the fact that $\delta^2 y = -a_1(z)\delta y - a_2(z)y$, we get

$$\delta r = \psi_{2,1}(z)(y(z^p)) + \psi_{2,2}(z)((\delta y)(z^p)).$$

For this reason,

$$\Psi\begin{pmatrix} y(z^p) \\ (\delta y)(z^p) \end{pmatrix} = \begin{pmatrix} r \\ \delta r \end{pmatrix}.$$

Thus, by applying δ, we get

$$\delta\Psi\begin{pmatrix} y(z^p) \\ (\delta y)(z^p) \end{pmatrix} + \Psi pA_2(z^p)\begin{pmatrix} y(z^p) \\ (\delta y)(z^p) \end{pmatrix} = A_2\begin{pmatrix} r \\ \delta r \end{pmatrix} = A_2\Psi\begin{pmatrix} y(z^p) \\ (\delta y)(z^p) \end{pmatrix}.$$

Since y is an arbitrary solution of $L^{(2)}$, this implies $\delta\Psi = A_2\Psi - p\Psi A_2(z^p)$.

(2) (a)⇒(b). Suppose that $\exp(y_1/y_0) \in z\mathbb{Z}_p[[z]]$. According to Corollary 6.6, this is equivalent to saying that $\frac{1}{p}\frac{\mathfrak{g}(z^p)}{\mathfrak{f}(z^p)} - \frac{\mathfrak{g}(z)}{\mathfrak{f}(z)} \in z\mathbb{Z}_p[[z]]$. By applying Lemma 6.2 to $L^{(2)}$ and Ψ, we deduce that

$$\frac{1}{p}\frac{\mathfrak{g}(z^p)}{\mathfrak{f}(z^p)} - \frac{\mathfrak{g}(z)}{\mathfrak{f}(z)} = -(\delta(y_1/y_0))(z^p)\frac{\psi_{1,2}}{p}\frac{\mathfrak{f}(z^p)}{\mathfrak{f}(z)}. \tag{7.6}$$

From Proposition 6.4, we have $\delta(y_1/y_0) \in 1 + z\mathbb{Z}_p[[z]]$ and we also know that $\mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$. Thus, from (7.6), we get $\psi_{1,2}(z) \in p\mathbb{Z}_p[[z]]$. In addition, from (7.5) it follows that $r = \psi_{1,1}(z)y_0(z^p) + \psi_{1,2}(z)((\delta y_0)(z^p))$ is a solution of $L^{(2)}$. Note that $r \in 1 + z\mathbb{Q}_p[[z]]$. Thus, according to Remark 2.3(2), we get

$$r = \psi_{1,1}(z)y_0(z^p) + \psi_{1,2}(z)((\delta y_0)(z^p)) = y_0(z).$$

Hence, $\psi_{1,1}(z)y_0(z^p) = y_0(z) \bmod p$.

(b)⇒(c). Suppose that $\psi_{1,1}(z)y_0(z^p) = y_0(z) \bmod p$. Then, by applying Λ_p, we get $\Lambda_p(\psi_{1,1}(z))y_0(z) = \Lambda_p(y_0(z)) \bmod p$.

(c)⇒(a). Assume that $\Lambda_p(\psi_{1,1}(z)) = \frac{\Lambda_p(y_0(z))}{y_0(z)} \bmod p$. By construction,

$$\psi_{1,1}(z) = \theta_{1,1}(z) = \frac{y_0(z) - \theta_{1,2}(z)(\delta y_0(z))(z^p)}{y_0(z^p)}.$$

Hence,

$$\Lambda_p(\psi_{1,1}(z)) = \frac{\Lambda_p(y_0(z)) - \big(\Lambda_p(\theta_{1,2}(z))\big)(\delta y_0(z))}{y_0(z)}.$$

Therefore,

$$\frac{\Lambda_p(y_0(z)) - \big(\Lambda_p(\theta_{1,2}(z))\big)(\delta y_0(z))}{y_0(z)} = \frac{\Lambda_p(y_0(z))}{y_0(z)} \bmod p.$$

Hence, we obtain $\Lambda_p(\theta_{1,2}(z)) \in p\mathbb{Z}_p[[z]]$.

It follows from (7.4) that $\delta\Psi = A_2\Psi \bmod p$. Since A_2 is the companion matrix of $L^{(2)}$, we deduce that $\psi_{1,2}(z) \bmod p$ is solution of $L^{(2)} \bmod p$. Given that $L \in \mathbb{Z}_p[[z]][\delta]L^{(2)}$, we then find that $\psi_{1,2}(z) \bmod p$ is a solution of $L \bmod p$. We now want to prove that $\psi_{1,2} \bmod p = 0$. Suppose, for the sake of contradiction, that $\psi_{1,2} \bmod p \neq 0$. Given that L is MUM, the exponents of $L \bmod p$ at zero are equal to zero, and thus [24, Lemma 6.7] implies that $\psi_{1,2}(z) \bmod p = P(z)c(z^p)$, where $c(z) \in \mathbb{F}_p((z))$ and $P(z)$ is a non-zero polynomial with coefficients in $\mathbb{F}_p$. By applying the Cartier operator, we obtain

$$\psi_{1,2}(z) \bmod p = \frac{P(z)}{(\Lambda_p(P))(z^p)}(\Lambda_p(\psi_{1,2} \bmod p))(z^p).$$

But $\Lambda_p(\psi_{1,2}(z)) \bmod p = 0$ because $\theta_{1,2}(z) = \psi_{1,2}(z)$ and we have already seen that $\Lambda_p(\theta_{1,2}(z)) \in p\mathbb{Z}_p[[z]]$. Hence, $\psi_{1,2} \bmod p = 0$, which contradicts our assumption $\psi_{1,2} \neq 0 \bmod p$. Therefore, $\psi_{1,2} \bmod p = 0$.

So $\theta_{1,2} \in p\mathbb{Z}_p[[z]]$ because $\theta_{1,2} = \psi_{1,2}$ and we know that

$$\theta_{1,2}(z) = \frac{p\mathfrak{g}(z)(\mathfrak{f}(z^p)) - \mathfrak{f}(z)(\mathfrak{g}(z^p))}{\Delta},$$

where $\Delta \in 1 + z\mathbb{Z}_p[[z]]$. So $p\mathfrak{g}(z)(\mathfrak{f}(z^p)) - \mathfrak{f}(z)(\mathfrak{g}(z^p)) \in p\mathbb{Z}_p[[z]]$. Thus, since $\mathfrak{f}(z) \in 1 + z\mathbb{Z}_p[[z]]$, we get

$$\frac{1}{p}\frac{\mathfrak{g}(z^p)}{\mathfrak{f}(z^p)} - \frac{\mathfrak{g}(z)}{\mathfrak{f}(z)} \in z\mathbb{Z}_p[[z]].$$

Consequently, Corollary 6.6 implies $\exp(y_1/y_0) \in \mathbb{Z}_p[[z]]$. ∎

Remark 7.1. Assume that L satisfies the assumptions of Theorem 1.3. In this remark we show that $\exp(y_1/y_0) \in z\mathbb{Z}_p[[z]]$ if and only if there exists $\mathcal{B} = B_1(z) + B_2(z)\delta \in \mathbb{Z}_p[[z]]$ with $B_1(0) = 1$ and $B_2(0) = 0$ such that, for every solution y of $L^{(2)}$, $\mathcal{B}(y(z^p))$ is a solution of $L^{(2)}$. Indeed, according to (1) of Theorem 1.3, $L^{(2)}$ has a p-integral Frobenius structure $\Psi = (\psi_{i,j}(z))_{1\le i,j\le 2}$ such that $\Psi(0) = \mathrm{diag}(1,p)$. Suppose that $\exp(y_1/y_0) \in z\mathbb{Z}_p[[z]]$. Thus, by following the proof of (a)⇒(b) in (2) of Theorem 1.3, we know that $\psi_{1,2}(z) = pt_{1,2}(z)$ with $t_{1,2}(z) \in \mathbb{Z}_p[[z]]$. Moreover, $\psi_{1,2}(z)((\delta y)(z^p)) = t_{1,2}(z)\delta(y(z^p))$. Therefore, if we set $\mathcal{B} = \psi_{1,1}(z) + t_{1,2}(z)\delta$ then from (7.5) we deduce, for every y of $L^{(2)}$, that

$$\begin{aligned}\mathcal{B}(y(z^p)) &= \psi_{1,1}(z)(y(z^p)) + t_{1,2}(z)\delta(y(z^p))\\ &= \psi_{1,1}(z)(y(z^p)) + \psi_{1,2}(z)((\delta y)(z^p))\end{aligned}$$

is a solution of $L^{(2)}$. Further, by construction, $\psi_{1,1}(0) = 1$ and $t_{1,2}(0) = 0$ because $\psi_{1,2}(0) = 0$.

We now assume that there exists $\mathcal{B} = B_1(z) + B_2(z)\delta \in \mathbb{Z}_p[[z]]$ with $B_1(0) = 1$ and $B_2(0) = 0$ such that, for every solution y of $L^{(2)}$, $\mathcal{B}(y(z^p))$ is a solution of $L^{(2)}$. Therefore, for every solution y of $L^{(2)}$,

$$B_1(z)y(z^p) + pB_2(z)((\delta y)(z^p))$$

is a solution of $L^{(2)}$. Let us write $L^{(2)} = \delta^2 + a_1(z)\delta + a_2(z)$. According to (1) of Theorem 1.3, $a_1(z), a_2(z) \in z\mathbb{Z}_p[[z]]$. We now put

$$\Gamma = \begin{pmatrix} B_1(z) & pB_2(z)\\ \delta B_1(z) - p^2B_2(z)a_2(z^p) & \delta B_2(z) + pB_1(z) - p^2B_2(z)a_1(z^p)\end{pmatrix}.$$

Then $\Gamma \in M_4(\mathbb{Z}_p[[z]])$ and it is not hard to see that $\Gamma = A_2\Gamma - p\Gamma A_2(z^p)$. Hence, following Remark 2.4, we have $\Gamma = Y_{L^{(2)}}\Gamma(0)(Y_{L^{(2)}}(z^p))^{-1}$. It is clear that $\Gamma(0) = \mathrm{diag}(1,p)$. We already know that Ψ is a p-integral Frobenius structure for $L^{(2)}$, and hence, by Remark 2.4 again, $\Psi = Y_{L^{(2)}}\Psi(0)(Y_{L^{(2)}}(z^p))^{-1}$ and we know that $\Psi(0) = \mathrm{diag}(1,p)$. So $\Gamma = \Psi$, and thus $\psi_{1,2} \in p\mathbb{Z}_p[[z]]$. Further, from (7.5) it follows that

$$r = \psi_{1,1}(z)y_0(z^p) + \psi_{1,2}(z)((\delta y_0)(z^p))$$

is a solution of $L^{(2)}$. Note that $r \in 1 + z\mathbb{Q}_p[[z]]$. Thus, according to (2) of Remark 2.3, we get

$$r = \psi_{1,1}(z)y_0(z^p) + \psi_{1,2}(z)((\delta y_0)(z^p)) = y_0(z).$$

So, $\psi_{1,1}(z)y_0(z^p) = y_0(z) \bmod p$. Consequently, according to Theorem 1.3, $\exp(y_1/y_0) \in z\mathbb{Z}_p[[z]]$.

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