

Characterization of decay rates for discrete operator semigroups

by

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Abstract. Let T be a power-bounded linear operator on a Hilbert space X , and let S be a bounded linear operator from another Hilbert space Y to X . We investigate the non-exponential rate of decay of $\|T^n S\|$ as $n \rightarrow \infty$. First, when $X = Y$ and S commutes with T , we characterize the decay rate of $\|T^n S\|$ in terms of the growth rate of $\|(\lambda I - T)^{-k} S\|$ as $|\lambda| \downarrow 1$ for some $k \in \mathbb{N}$. Next, we provide another characterization by means of an integral estimate of $\|(\lambda I - T)^{-k} S\|$. The second characterization is then applied to asymptotic estimates for perturbed discrete operator semigroups. Finally, we present some results on the relation between the decay rate of $\|T^n S\|$ and the boundedness of the sum $\sum_{n=1}^{\infty} f(n) \|T^n S y\|^p$ for all $y \in Y$ in the Banach space setting, where $f: \mathbb{N} \rightarrow (0, \infty)$ and $p \geq 1$.

1. Introduction. Let X and Y be complex Banach spaces. We denote by $\mathcal{L}(Y, X)$ the space of bounded linear operators from Y to X , and set $\mathcal{L}(X) := \mathcal{L}(X, X)$. Let $T \in \mathcal{L}(X)$ be such that the spectral radius $r(T)$ of T is at most 1, and let $S \in \mathcal{L}(Y, X)$. In this paper, we investigate the asymptotic behavior of $\|T^n S\|$ as $n \rightarrow \infty$ and relate it to the growth of $\|(\lambda I - T)^{-k} S\|$ as $|\lambda| \downarrow 1$ for $k \in \mathbb{N}$.

To motivate our problem, we consider the following sampled-data system with sampling period $\tau > 0$:

$$(1.1) \quad \begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0; \quad z(0) = x \in X; \\ u(t) = Fz(n\tau), & n\tau \leq t < (n+1)\tau, \quad n \in \mathbb{N}_0, \end{cases}$$

where X and U are complex Banach spaces, A is the generator of a C_0 -semigroup $(e^{At})_{t \geq 0}$ on X , $B \in \mathcal{L}(U, X)$, $F \in \mathcal{L}(X, U)$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In the context of control systems, z is the state of the plant and is sampled

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at time $t = n\tau$ for each $n \in \mathbb{N}_0$. The control input u is generated from the sampled state $z(n\tau)$.

Let $z_d(n) := z(n\tau)$ for $n \in \mathbb{N}_0$, and define $T \in \mathcal{L}(X)$ by

$$T\xi := e^{A\tau}\xi + \int_0^\tau e^{At}BF\xi dt, \quad \xi \in X.$$

Then the discrete trajectory z_d satisfies

$$z_d(n+1) = Tz_d(n), \quad n \in \mathbb{N}_0; \quad z_d(0) = x.$$

This discrete-time system can be regarded as a discretized version of the following continuous-time system:

$$(1.2) \quad \dot{z}_c(t) = (A + BF)z_c(t), \quad t \geq 0; \quad z_c(0) = x.$$

Intuitively, the sampled-data system (1.1) approximates the continuous-time system (1.2) when the sampling period τ is sufficiently small. However, the asymptotic behavior of (1.1) may differ from that of (1.2) when X is an infinite-dimensional space. In fact, even exponential stability is not preserved for a certain system; see [39, Remark 3.3] and [57]. This motivates us to find sufficient conditions for preservation of stability, and such conditions were developed for exponential stability in [39, 58], for strong stability in [68], and for polynomial stability in [69]. Here we are interested in the decay rate of the solution z of (1.1) with a smooth initial state $x \in D(A)$. For simplicity, assume that A is invertible. Then the domain $D(A)$ of A coincides with the range of A^{-1} . Since there exists a constant $M > 0$ such that $\|z(t)\| \leq M\|z_d(n)\|$ for all $t \in [n\tau, (n+1)\tau)$ and $n \in \mathbb{N}_0$, uniform rates of decay for the solutions z with initial states $x \in D(A)$ are obtained by investigating the quantitative behavior of $\|T^n A^{-1}\|$.

We begin by reviewing the existing literature in the case $S = I$. It is well known that the following discrete analogue of one implication in the Hille–Yosida theorem for C_0 -semigroups holds: If T is *power-bounded*, i.e., satisfies $\sup_{n \in \mathbb{N}_0} \|T^n\| < \infty$, then $r(T) \leq 1$ and there exists a constant $M > 0$ such that

$$(1.3) \quad \|(\lambda I - T)^{-k}\| \leq \frac{M}{(|\lambda| - 1)^k}$$

for all $k \in \mathbb{N}$ and $|\lambda| > 1$; see, e.g., [40]. However, the converse implication is not true; see [43] for a counterexample.

We call $T \in \mathcal{L}(X)$ a *Kreiss operator* if $r(T) \leq 1$ and there exists $M > 0$ such that (1.3) holds for $k = 1$. If T is a Kreiss operator, then $\|T^n\| = O(n)$ as $n \rightarrow \infty$, i.e., there exist constants $K > 0$ and $n_0 \in \mathbb{N}_0$ such that $\|T^n\| \leq Kn$ for all $n \geq n_0$; see [40]. This growth rate n cannot in general be improved, as shown in [64]. If $T \in \mathcal{L}(X)$ satisfies $r(T) \leq 1$ and if there exist constants

$\alpha \geq 0$ and $M_\alpha > 0$ such that

$$\|(\lambda I - T)^{-1}\| \leq \frac{M_\alpha}{(|\lambda| - 1)^\alpha}$$

for all $1 < |\lambda| < 2$, then $\|T^n\| = O(n^\alpha)$ as $n \rightarrow \infty$. Conversely, if $\|T^n\| = O(n^\alpha)$ as $n \rightarrow \infty$ for some $\alpha \geq 0$, then

$$\|(\lambda I - T)^{-1}\| \leq \frac{M_{\alpha+1}}{(|\lambda| - 1)^{\alpha+1}}$$

for all $1 < |\lambda| < 2$ and some constant $M_{\alpha+1} > 0$; see [23] for these two implications. In the Hilbert space setting, a Kreiss operator T satisfies $\|T^n\| = O(n/\sqrt{\log n})$ as $n \rightarrow \infty$, which was independently proved in [13, 10]. For a UMD space X , the estimate $\|T^n\| = O(n/(\log n)^\beta)$ was obtained in [17], where $\beta \in (0, 1/2]$ is the reciprocal of the minimum of the finite cotypes of X and its dual.

We call $T \in \mathcal{L}(X)$ a *strong Kreiss operator* if $r(T) \leq 1$ and there exists $M > 0$ such that (1.3) holds for all $k \in \mathbb{N}$. If T is a strong Kreiss operator, then $\|T^n\| = O(\sqrt{n})$ as $n \rightarrow \infty$; see [43, 40]. Moreover, the growth rate \sqrt{n} cannot in general be replaced by a better one; see [40]. It was proved in [13] that a strong Kreiss operator T on a Hilbert space satisfies $\|T^n\| = O((\log n)^\beta)$ as $n \rightarrow \infty$, where $\beta > 0$ is a constant dependent on T . Other estimates were obtained in [1] for the case where X is the L^p -space with $1 < p < \infty$ and in [18] for the case where X is a UMD space. A survey on Kreiss operators and strong Kreiss operators can be found in [15].

Next, we review existing studies in the case $S = I - T$. We call $T \in \mathcal{L}(X)$ a *Ritt operator* if $r(T) \leq 1$ and there exists a constant $M > 0$ such that

$$(1.4) \quad \|(\lambda I - T)^{-1}\| \leq \frac{M}{|\lambda - 1|}$$

for all $|\lambda| > 1$. The estimate (1.4) on the sector

$$\{\lambda \in \mathbb{C} \setminus \{1\} : |\arg(\lambda - 1)| < \pi/2 + \delta\}$$

for some $\delta \in (0, \pi/2)$ implies that T is power-bounded and $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$, as shown in [36]. The converse statement was also true; see [45]. It was later proved independently in [44, 41] that T is a Ritt operator if and only if T is power-bounded and $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$. Since the estimate $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$ does not imply that T is power-bounded (see [33]), power-boundedness cannot be omitted from the above characterization of Ritt operators. Estimates for $(\lambda - 1)(\lambda I - T)^{-1}$ were also studied in connection with maximum regularity problems for discrete-time evolution equations; see, e.g., [9, 8, 34, 35]. In [24], a generalization of Ritt operators and Kreiss operators was introduced and applied to derive norm estimates for T^n . See [15] for further information on Ritt operators.

In [46], it was shown that if T is a Kreiss operator and if there exist constants $M > 0$ and $\alpha \in (0, 1]$ such that

$$(1.5) \quad \|(\lambda I - T)^{-1}\| \leq \frac{M}{|\lambda - 1|^{1/\alpha}}$$

for all $1 < |\lambda| < 2$, then

$$(1.6) \quad \|T^n(I - T)\| = O\left(\frac{1}{n^{2\alpha-1}}\right)$$

as $n \rightarrow \infty$. The following characterization for the specific decay rate $n^{-1/2}$ was obtained in [19]: T is power-bounded and satisfies $\|T^n(I - T)\| = O(n^{-1/2})$ as $n \rightarrow \infty$ if and only if $T = cI + (1 - c)S$ for some constant $c \in (0, 1)$ and power-bounded operator S . In [20], the decay estimate $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$ in the case $1/2 < \alpha < 1$ was also characterized in terms of norm estimates for all powers of the resolvent.

Given a power-bounded operator T and a constant $\alpha \in (0, 1]$, the estimate (1.5) holds for all $1 < |\lambda| < 2$ and some constant $M > 0$ if and only if the intersection of the spectrum of T and the unit circle is contained in $\{1\}$ and $\|(e^{i\theta}I - T)^{-1}\| = O(|\theta|^{-1/\alpha})$ as $\theta \rightarrow 0$, which can be obtained from [62, Theorem 3.10 and proof of Lemma 3.9]; see also [53, 14]. Using this equivalence, we see from [61, 62] that the estimate (1.6) can be improved to

$$(1.7) \quad \|T^n(I - T)\| = O\left(\left(\frac{\log n}{n}\right)^\alpha\right)$$

as $n \rightarrow \infty$. It was also shown in [62] that the upper bound given in (1.7) is sharp for $0 < \alpha < 1/2$ in general. The growth rate $|\theta|^{-1/\alpha}$ of $\|(e^{i\theta}I - T)^{-1}\|$ as $\theta \rightarrow 0$ can be generalized to a continuous non-increasing function; see [61, 62] for details. As also shown in [62], when X is a Hilbert space, the estimate (1.7) can be strengthened to $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$. This result in the Hilbert space setting was extended from the polynomial case to the wider class of functions having reciprocally positive increase in [47]. The above-mentioned results of [61, 62, 47] are discrete analogues of those established for C_0 -semigroups in [6, 11, 60]; see also the survey article [12] for the C_0 -semigroup results. For more information on estimates for the rate of decay of $\|T^n(I - T)\|$, we refer to [7], which also includes a survey on the asymptotic behavior of $\|T^n f(T)\|$ for bounded holomorphic functions f on \mathbb{D} .

We characterize the decay rate of $\|T^n S\|$ in terms of the growth rate of $\|(\lambda I - T)^{-k} S\|$ as $|\lambda| \downarrow 1$ for some $k \in \mathbb{N}$ when $X = Y$ and S commutes with T . A special case of this characterization is stated as follows.

THEOREM 1.1. *Let X be a Hilbert space. Let $T \in \mathcal{L}(X)$ be power-bounded, and suppose that $S \in \mathcal{L}(X)$ commutes with T . Then the following statements are equivalent for fixed $\alpha > 0$ and $k \in \mathbb{N}$ satisfying $k > \alpha$:*

- (i) $\|T^n S\| = O(1/n^\alpha)$ as $n \rightarrow \infty$.
(ii) There exists a constant $M > 0$ such that

$$\|(\lambda I - T)^{-k} S\| \leq \frac{M}{(|\lambda| - 1)^{k-\alpha}}$$

for all $|\lambda| > 1$.

We also present a simple example showing that the estimates given in Theorem 1.1 cannot in general be improved. Although the special case above is limited to polynomials, the proposed approach can be extended to a class of functions called α -bounded regularly varying functions for $\alpha \geq 0$, which are variants of regularly varying functions; see Theorem 3.1 for the extended case. Regularly varying functions are products of polynomials and slowly varying functions, and give natural refinements of polynomial scales. This class of functions has been extensively applied in various areas of analysis, including asymptotic estimates for C_0 -semigroups in [5].

We provide another characterization of the decay rate of $\|T^n S\|$ through an integral estimate for resolvents. While here we state the polynomial case, it can also be extended to α -bounded regularly varying functions; see Theorem 4.2 for the extended case.

THEOREM 1.2. *Let X be a Hilbert space, and let Y be a Banach space. Let $T \in \mathcal{L}(X)$ be power-bounded, and let $S \in \mathcal{L}(Y, X)$. Then the following statements are equivalent for fixed $\alpha > 0$ and $k \in \mathbb{N}$ satisfying $k > \alpha + 1/2$:*

- (i) $\|T^n S\| = O(1/n^\alpha)$ as $n \rightarrow \infty$.
(ii) There exists a constant $M > 0$ such that

$$(1.8) \quad \sup_{1 < r < 2} (r - 1)^{2k-2\alpha-1} \int_0^{2\pi} \|R(re^{i\theta}, T)^k S y\|^2 d\theta \leq M \|y\|^2$$

for all $y \in Y$.

The integral estimate (1.8) is rooted in the characterization of bounded C_0 -semigroups established in [27, 63]; see also [16] and [21, Theorem II.1.12] for its discrete analogue. Theorem 1.2 can be viewed as an extension of the case $k = 1$ obtained in [69], which considered the norms $\|T^n S y\|$ of individual orbits rather than the operator norm $\|T^n S\|$ as in statement (i).

We apply the second characterization to robustness analysis for semigroup stability. The robustness problem we study is stated as follows: Assume that a power-bounded operator T on a Hilbert space satisfies $\|T^n S\| = O(1/f(n))$ as $n \rightarrow \infty$ for some α -bounded regularly varying function f . Given a perturbation $D \in \mathcal{L}(X)$, does the perturbed operator $T + D$ remain power-bounded and satisfy $\|T^n S\| = O(1/f(n))$ as $n \rightarrow \infty$? For C_0 -semigroups, several conditions for preservation of non-exponential stability were developed; see [48, 49, 50, 56, 3] for polynomial stability and [51, 52, 56]

for strong stability. For discrete operator semigroups, robustness of strong stability was also explored in [53]. In Corollary 5.2, we present a class of perturbations preserving the power-boundedness of T and the decay property of $\|T^n S\|$.

Finally, we examine the relation between the decay rate of $\|T^n S\|$ and the following summability condition on orbits weighted by a function $f: \mathbb{N} \rightarrow (0, \infty)$: There exist constants $C > 0$ and $p \geq 1$ such that

$$(1.9) \quad \sum_{n=1}^{\infty} f(n) \|T^n S y\|^p \leq C \|y\|^p$$

for all $y \in Y$. While it is assumed in Theorems 1.1 and 1.2 that X is a Hilbert space, here we consider the Banach space setting. It was shown in [76] that when $X = Y$, $S = I$, and $f(n) \equiv 1$, the condition (1.9) implies $r(T) < 1$; see [55] for a different proof. This result was strengthened to the case of weak orbits $|\phi(T^n x)|^p$ in [72], where $x \in X$ and ϕ is a bounded linear functional on X ; see also [67, 26] for further generalizations. We show in Proposition 6.1 that if a power-bounded operator T satisfies the summability condition (1.9), then

$$\|T^n S\| = O\left(\frac{1}{F(n)^{1/p}}\right)$$

as $n \rightarrow \infty$, where

$$F(n) := \sum_{m=1}^n f(m), \quad n \in \mathbb{N}.$$

Moreover, we prove that the converse statement is true in the case when T is a multiplication operator on L^q -space with $1 \leq q \leq p < \infty$, $S = I - T$, and $f(n) = n^\alpha$ for some $\alpha \geq 0$; see Proposition 6.3. This is established by using intermediate properties between Kreiss and Ritt operators investigated in [42]. The summability condition (1.9) with $f(n) \equiv 1$ can be regarded as a discrete counterpart to the C_0 -semigroup case discussed in [70, 54].

This paper is structured as follows. In Section 2, we give preliminaries on Parseval's equality and α -bounded regularly varying functions. Section 3 is devoted to characterizing the decay rate of $\|T^n S\|$ in terms of the growth rate of $\|(\lambda I - T)^{-k} S\|$ for some $k \in \mathbb{N}$. In Section 4, we present an integral estimate for resolvents to characterize the rate of decay of $\|T^n S\|$, and in Section 5, this result is applied to asymptotic estimates for perturbed discrete operator semigroups. Finally, in Section 6, we connect the decay rate of $\|T^n S\|$ to the summability condition (1.9).

NOTATION. Let \mathbb{Z} and \mathbb{N} denote the set of integers and the set of positive integers, respectively. Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write

$$\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \quad \mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \quad \mathbb{E} := \{\lambda \in \mathbb{C} : |\lambda| > 1\}.$$

The closure of a subset Ω of \mathbb{C} is denoted by $\overline{\Omega}$. Given functions $f, g: [a, \infty) \rightarrow (0, \infty)$ for some $a \geq 0$, the notation $f(t) = O(g(t))$ as $t \rightarrow \infty$ means that there exist constants $M > 0$ and $t_0 \geq a$ such that $f(t) \leq Mg(t)$ for all $t \geq t_0$. Similar notation is employed for other types of asymptotic behavior. We denote by Γ the gamma function.

Throughout this paper, we assume that all Banach spaces are complex. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y , and write $\mathcal{L}(X) := \mathcal{L}(X, X)$. Let $T \in \mathcal{L}(X)$. We denote the spectrum of T by $\sigma(T)$, the resolvent set of T by $\varrho(T)$, and the spectral radius of T by $r(T)$. For $\lambda \in \varrho(T)$, we write $R(\lambda, T) := (\lambda I - T)^{-1}$.

2. Preliminaries. In this section, we first recall the vector-valued version of Parseval’s equality. Next, we introduce the notion of α -bounded regularly varying functions and give some properties of this class of functions.

2.1. Parseval’s equality. First, we recall the well-known resolvent representation of the power T^n for a bounded linear operator T . This representation in the case $k = 1$ can be obtained from the Dunford–Riesz functional calculus, and then the case $k \geq 2$ follows by applying integration by parts to the case $k = 1$; see also [43, p. 2].

LEMMA 2.1. *Let X be a Banach space, and let $T \in \mathcal{L}(X)$. Then*

$$\binom{n+k-1}{k-1} T^n = \frac{r^{n+k}}{2\pi} \int_0^{2\pi} e^{i\theta(n+k)} R(re^{i\theta}, T)^k d\theta$$

for all $n, k \in \mathbb{N}$ and $r > r(T)$.

Let Y be a Hilbert space. If $f \in L^2((0, 2\pi); Y)$, then the following Parseval’s equality holds:

$$\frac{1}{2\pi} \int_0^{2\pi} \|f(\theta)\|^2 d\theta = \sum_{n=-\infty}^{\infty} \|c_n\|^2,$$

where

$$c_n := \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta, \quad n \in \mathbb{Z}.$$

Let X be a Banach space. Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X, Y)$. Lemma 2.1 and Cauchy’s theorem show that for all $k \in \mathbb{N}$, $r > r(T)$, and $x \in X$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} SR(re^{-i\theta}, T)^k x d\theta = \begin{cases} \binom{n-1}{k-1} \frac{ST^{n-k}x}{r^n}, & n \geq k, \\ 0, & n < k. \end{cases}$$

Applying Parseval's equality to $f(\theta) := SR(re^{-i\theta}, T)^k x$, we obtain the following lemma, which is also called Parseval's equality throughout this paper.

LEMMA 2.2. *Let X be a Banach space, and let Y be a Hilbert space. Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X, Y)$. Then for all $k \in \mathbb{N}$, $r > r(T)$, and $x \in X$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \|SR(re^{i\theta}, T)^k x\|^2 d\theta = \sum_{n=0}^{\infty} \left\| \binom{n+k-1}{k-1} \frac{ST^n x}{r^{n+k}} \right\|^2.$$

2.2. Special class of functions

2.2.1. *Definition of α -bounded regularly varying functions.* A measurable function $f: (0, \infty) \rightarrow (0, \infty)$ is called *regularly varying of index $\alpha \in \mathbb{R}$* if

$$\lim_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\alpha$$

for all $\lambda \geq 1$. For simplicity, let $f: (0, \infty) \rightarrow (0, \infty)$ be continuously differentiable. If

$$(2.1) \quad \lim_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} = \alpha$$

for some $\alpha \in \mathbb{R}$, then f is regularly varying of index α , and conversely if f is regularly varying of index $\alpha \in \mathbb{R}$ and f' is non-decreasing or non-increasing, then (2.1) holds; see, e.g., [59, Proposition 0.7]. Motivated by this property of regularly varying functions, we consider the following variant of regularly varying functions.

DEFINITION 2.3. Let $\alpha \geq 0$. A function $f: (0, \infty) \rightarrow (0, \infty)$ is called *α -bounded regularly varying* if there exists a constant $t_0 > 0$ such that f is non-decreasing on $[t_0, \infty)$, is absolutely continuous on each compact subinterval of $[t_0, \infty)$, and satisfies

$$\frac{tf'(t)}{f(t)} \leq \alpha$$

for a.e. $t \geq t_0$.

Functions of positive increase introduced in [60] are related to α -bounded regularly varying functions. We say that a measurable function $f: (0, \infty) \rightarrow (0, \infty)$ has *positive increase* if there exist constants $\alpha, t_0 > 0$ and $c \in (0, 1]$ such that

$$\frac{f(\lambda t)}{f(t)} \geq c\lambda^\alpha$$

for all $\lambda \geq 1$ and $t \geq t_0$. For simplicity, here we again assume that f is continuously differentiable on $(0, \infty)$, and define

$$(2.2) \quad \phi(t) := \frac{tf'(t)}{f(t)}, \quad t > 0.$$

Then

$$(2.3) \quad f(t) = f(1) \exp\left(\int_1^t \frac{\phi(s)}{s} ds\right)$$

for all $t > 0$. Therefore, if

$$\liminf_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} = \liminf_{t \rightarrow \infty} \phi(t) > 0,$$

then there exist $\alpha, t_0 > 0$ such that

$$\frac{f(\lambda t)}{f(t)} = \exp\left(\int_t^{\lambda t} \frac{\phi(s)}{s} ds\right) \geq \lambda^\alpha$$

for all $\lambda \geq 1$ and $t \geq t_0$, and hence f has positive increase.

There exist functions that are α -bounded regularly varying but are neither regularly varying nor of positive increase.

EXAMPLE 2.4. Let $f: (0, \infty) \rightarrow (0, \infty)$ be absolutely continuous on each compact subinterval of $(0, \infty)$, and let ϕ be as in (2.2). Suppose that ϕ can be written as

$$\phi(t) = \begin{cases} 0, & 2^{n^2} \leq t < 2^{(n+1)^2-1}, \\ 1, & 2^{(n+1)^2-1} \leq t < 2^{(n+1)^2}, \end{cases}$$

for $n \in \mathbb{N}_0$. Then f is 1-bounded regularly varying. Let $\lambda \geq 1$ and take $n_0 \in \mathbb{N}$ such that $2^{2n_0} \geq \lambda$. If $t = 2^{n^2}$ for some $n \geq n_0$, then $\lambda t \leq 2^{(n+1)^2-1}$ and hence

$$\exp\left(\int_t^{\lambda t} \frac{\phi(s)}{s} ds\right) = 1.$$

This and (2.3) yield

$$(2.4) \quad \liminf_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} = 1.$$

From [60, Lemma 2.1], we see that f does not have positive increase. On the other hand, if $t = 2^{(n+1)^2-1}$ for some $n \in \mathbb{N}_0$, then for all $\lambda \geq 2$,

$$\exp\left(\int_t^{\lambda t} \frac{\phi(s)}{s} ds\right) \geq 2,$$

and hence by (2.3),

$$(2.5) \quad \limsup_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} \geq 2.$$

From (2.4) and (2.5), we see that the limit of $f(\lambda t)/f(t)$ as $t \rightarrow \infty$ does not exist for $\lambda \geq 2$. Hence, f is not regularly varying.

REMARK 2.5. Let $a \geq 0$ and let $f: (a, \infty) \rightarrow (0, \infty)$ be continuously differentiable such that $f'(t) > 0$ for all $t > a$. Then

$$\limsup_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} < \infty \quad \text{or equivalently} \quad \liminf_{t \rightarrow \infty} \frac{f(t)}{tf'(t)} > 0$$

appears as a necessary condition for the function f to be of class P_1 in [22, Lemma 2.2(iv)]. This condition was used there in the proof of a subsequential ergodic theorem.

2.2.2. Properties of α -bounded regularly varying functions. We begin with simple properties of α -bounded regularly varying functions.

LEMMA 2.6. *Let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying for some $\alpha \geq 0$. Then the following statements hold:*

- (a) $f(t) = O(t^\alpha)$ as $t \rightarrow \infty$.
- (b) Let $\gamma > 0$, and define $h(t) := f(t)^\gamma$ for $t > 0$. Then h is $(\alpha\gamma)$ -bounded regularly varying.

Proof. Since f is α -bounded regularly varying, there exists $t_0 > 0$ such that

$$\frac{tf'(t)}{f(t)} \leq \alpha$$

for a.e. $t \geq t_0$.

- (a) Define $g(t) := f(t)/t^\alpha$ for $t > 0$. Then

$$g'(t) = \frac{tf'(t) - \alpha f(t)}{t^{\alpha+1}} \leq 0$$

for a.e. $t \geq t_0$. Hence, $g(t) \leq g(t_0)$ for all $t \geq t_0$. This implies that $f(t) \leq f(t_0)(t/t_0)^\alpha$ for all $t \geq t_0$.

- (b) Since

$$\frac{th'(t)}{h(t)} = \gamma \frac{tf'(t)}{f(t)} \leq \alpha\gamma$$

for a.e. $t \geq t_0$, we see that h is $(\alpha\gamma)$ -bounded regularly varying. ■

The following proposition will be used to characterize rates of decay for discrete operator semigroups.

PROPOSITION 2.7. *Let $\alpha \geq 0$ and $\beta > \alpha - 1$. Let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying. If $c: \mathbb{N}_0 \rightarrow (0, \infty)$ satisfies*

$$c(n) = O\left(\frac{n^\beta}{f(n)}\right)$$

as $n \rightarrow \infty$, then

$$(2.6) \quad \sum_{n=0}^{\infty} \frac{c(n)}{r^n} = O\left(\frac{1}{(r-1)^{\beta+1} f(1/(r-1))}\right)$$

as $r \downarrow 1$.

The estimate (2.6) in the case when $f(t) \equiv 1$ and $0 \leq \beta \leq 1$ can be obtained from [14, Lemma 5.3]. To derive (2.6) in the general case, we need the following integral estimate. This is an extension of [70, Lemma 4.2.a)], which concerned the case where $\beta = 0$ and $f(t) = t^p(\log t)^q$ with $0 \leq p < 1$ and $q \geq 0$.

LEMMA 2.8. *Let $\alpha \geq 0$ and $\beta > \alpha - 1$. If $f: (0, \infty) \rightarrow (0, \infty)$ is α -bounded regularly varying, then there exist constants $M, t_0 > 0$ such that*

$$\int_{t_0}^{\infty} \frac{t^\beta e^{-st}}{f(t)} dt \leq \frac{M}{s^{\beta+1} f(1/s)}$$

for all $s \in (0, 1/t_0)$.

Proof. Define $g(t) := f(t)/t^\beta$ for $t > 0$. By assumption, there exists $t_0 > 0$ such that for a.e. $t \geq t_0$,

$$\frac{tf'(t)}{f(t)} \leq \alpha.$$

Hence,

$$(2.7) \quad \frac{tg'(t)}{g(t)} = \frac{tf'(t)}{f(t)} - \beta \leq \alpha - \beta =: \delta < 1$$

for a.e. $t \geq t_0$. Let $s \in (0, 1/t_0)$. Since f is non-decreasing, it follows that

$$(2.8) \quad \begin{aligned} \int_{1/s}^{\infty} \frac{t^\beta e^{-st}}{f(t)} dt &\leq \frac{1}{f(1/s)} \int_{1/s}^{\infty} t^\beta e^{-st} dt \\ &= \frac{1}{s^{\beta+1} f(1/s)} \int_1^{\infty} t^\beta e^{-t} dt \leq \frac{\Gamma(\beta + 1)}{s^{\beta+1} f(1/s)}. \end{aligned}$$

We also have

$$(2.9) \quad \int_{t_0}^{1/s} \frac{t^\beta e^{-st}}{f(t)} dt \leq \int_{t_0}^{1/s} \frac{1}{g(t)} dt.$$

Integration by parts yields

$$\begin{aligned} \int_{t_0}^{1/s} \frac{1}{g(t)} dt &= \int_{t_0}^{1/s} \frac{1}{t^\delta (g(t)/t^\delta)} dt \\ &= \frac{1}{1 - \delta} \left(\frac{1/s}{g(1/s)} - \frac{t_0}{g(t_0)} \right) + \frac{1}{1 - \delta} \int_{t_0}^{1/s} \frac{t^{1-\delta} (t^\delta g'(t) - \delta t^{\delta-1} g(t))}{g(t)^2} dt \\ &\leq \frac{1}{(1 - \delta)sg(1/s)} + \frac{1}{1 - \delta} \int_{t_0}^{1/s} \left(\frac{tg'(t)}{g(t)^2} - \frac{\delta}{g(t)} \right) dt. \end{aligned}$$

Since (2.7) gives

$$\frac{tg'(t)}{g(t)^2} - \frac{\delta}{g(t)} \leq \frac{\delta}{g(t)} - \frac{\delta}{g(t)} = 0$$

for a.e. $t \geq t_0$, it follows that

$$(2.10) \quad \int_{t_0}^{1/s} \frac{1}{g(t)} dt \leq \frac{1}{(1-\delta)sg(1/s)}.$$

Combining the estimates (2.9) and (2.10), we derive

$$(2.11) \quad \int_{t_0}^{1/s} \frac{t^\beta e^{-st}}{f(t)} dt \leq \frac{1}{(1-\delta)s^{\beta+1}f(1/s)}.$$

By the estimates (2.8) and (2.11),

$$\int_{t_0}^{\infty} \frac{t^\beta e^{-st}}{f(t)} dt \leq \frac{M}{s^{\beta+1}f(1/s)}, \quad \text{where } M := \Gamma(\beta+1) + \frac{1}{1-\delta}.$$

Note that the constant M does not depend on s . ■

Proof of Proposition 2.7. By assumption, there exist M_0 and $n_0 \in \mathbb{N}$ such that

$$(2.12) \quad c(n) \leq \frac{M_0 n^\beta}{f(n)}$$

for all $n \geq n_0$. By Lemma 2.8, there exist $M_1, t_0 > 0$ such that

$$(2.13) \quad \int_{t_0}^{\infty} \frac{t^\beta e^{-st}}{f(t)} dt \leq \frac{M_1}{s^{\beta+1}f(1/s)}$$

for all $s \in (0, 1/t_0)$. Let $n_1 \in \mathbb{N}$ satisfy $n_1 \geq \max\{n_0, t_0\}$. We have

$$(2.14) \quad \sum_{n=0}^{n_1} \frac{c(n)}{r^n} \leq (n_1 + 1) \max_{0 \leq n \leq n_1} c(n)$$

for all $r > 1$. If $n \leq t \leq n+1$ for some $n \in \mathbb{N}$, then

$$\frac{(n+1)^\beta}{r^{n+1}f(n+1)} \leq \frac{(t+1)^\beta}{r^t f(t)}$$

for all $r > 1$. From this and the estimate (2.12), we obtain

$$(2.15) \quad \sum_{n=n_1+1}^{\infty} \frac{c(n)}{r^n} \leq M_0 \sum_{n=n_1}^{\infty} \frac{(n+1)^\beta}{r^{n+1}f(n+1)} \leq M_0 \int_{n_1}^{\infty} \frac{(t+1)^\beta}{r^t f(t)} dt$$

for all $r > 1$. Moreover, if $0 < \log r < 1/n_1$, then the estimate (2.13) gives

$$(2.16) \quad \int_{n_1}^{\infty} \frac{(t+1)^\beta}{r^t f(t)} dt \leq \left(\frac{n_1+1}{n_1}\right)^\beta \int_{n_1}^{\infty} \frac{t^\beta e^{-(\log r)t}}{f(t)} dt$$

$$\leq M_1 \left(\frac{n_1+1}{n_1}\right)^\beta \frac{1}{(\log r)^{\beta+1} f(1/\log r)}.$$

Since

$$\frac{1}{2} \leq \frac{\log r}{r-1} \leq 1$$

for all $r \in (1, 3)$, we deduce from the estimates (2.15) and (2.16) that

$$(2.17) \quad \sum_{n=n_1+1}^{\infty} \frac{c(n)}{r^n} \leq M_0 M_1 \left(\frac{n_1+1}{n_1}\right)^\beta \frac{2^{\beta+1}}{(r-1)^{\beta+1} f(1/(r-1))}$$

whenever $r \in (1, e^{1/n_1})$. By Lemma 2.6(a), $f(t) = O(t^\alpha)$ as $t \rightarrow \infty$, and by assumption, $\beta + 1 > \alpha$. Therefore, $s^{\beta+1} f(1/s) \rightarrow 0$ as $s \downarrow 0$. Thus, the estimate (2.6) follows from (2.14) and (2.17). ■

3. Relation between semigroup decay and resolvent growth.

In this section, we study the relation between decay rates of discrete operator semigroups and growth rates of resolvents. First, we state and prove the main theorem of this section. Next, we give some examples showing that the estimates obtained cannot in general be improved. Finally, we investigate the decay rate of $\|T^n(I - T)\|$ as a special case.

3.1. Main result. The following theorem shows that rates of decay of discrete operator semigroups on Hilbert spaces can be characterized in terms of rates of growth of resolvents in a neighborhood of \mathbb{T} .

THEOREM 3.1. *Let X be a Hilbert space. Let $T \in \mathcal{L}(X)$ be power-bounded, and suppose that $S \in \mathcal{L}(X)$ commutes with T . Let $\alpha > 0$ and $k \in \mathbb{N}$ satisfy $k > \alpha$. If $f: (0, \infty) \rightarrow (0, \infty)$ is α -bounded regularly varying, then the following statements are equivalent:*

- (i) $\|T^n S\| = O(1/f(n))$ as $n \rightarrow \infty$.
- (ii) There exists a constant $M > 0$ such that

$$(3.1) \quad \|R(\lambda, T)^k S\| \leq \frac{M}{(|\lambda| - 1)^k f(1/(|\lambda| - 1))}$$

for all $\lambda \in \mathbb{E}$.

Recall that if T is a power-bounded operator on a Banach space, then T is a strong Kreiss operator, which gives a better decay rate of $\|R(\lambda, T)^k S\|$ as $|\lambda| \rightarrow \infty$ than the estimate (3.1); see (1.3). Hence, Theorem 3.1 shows

that the growth rate of $\|R(\lambda, T)^k S\|$ in a neighborhood of \mathbb{T} characterizes the decay rate of $\|T^n S\|$.

Before proceeding with the proof, we recall a series representation of powers of resolvents. Let T be a bounded linear operator on a Banach space. If $r(T) \leq 1$, then

$$(3.2) \quad R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

for all $\lambda \in \mathbb{E}$; see, e.g., [4, Proposition 9.28.c)]. Hence,

$$(3.3) \quad R(\lambda, T)^k = \frac{R(\lambda, T)^{(k-1)}}{(-1)^{k-1}(k-1)!} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \frac{T^n}{\lambda^{n+k}}$$

for all $\lambda \in \mathbb{E}$ and $k \in \mathbb{N}$. Using this representation, we first show that the semigroup estimate is transferred to the resolvent estimate.

LEMMA 3.2. *Let X , Y , and Z be Banach spaces. Let $T \in \mathcal{L}(X)$ satisfy $r(T) \leq 1$, and let $S_1 \in \mathcal{L}(X, Z)$ and $S_2 \in \mathcal{L}(Y, X)$. Suppose that $\alpha \geq 0$ and $k \in \mathbb{N}$ satisfy $k > \alpha$, and let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying. If*

$$(3.4) \quad \|S_1 T^n S_2\| = O\left(\frac{1}{f(n)}\right)$$

as $n \rightarrow \infty$, then there exists a constant $M > 0$ such that

$$(3.5) \quad \|S_1 R(\lambda, T)^k S_2\| \leq \frac{M}{(|\lambda| - 1)^k f(1/(|\lambda| - 1))}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$.

Proof. It follows from (3.3) that

$$S_1 R(\lambda, T)^k S_2 = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \frac{S_1 T^n S_2}{\lambda^{n+k}}.$$

From this, we obtain

$$\|S_1 R(\lambda, T)^k S_2\| \leq \sum_{n=0}^{\infty} \frac{c(n)}{|\lambda|^n}$$

for all $\lambda \in \mathbb{E}$, where

$$c(n) := \binom{n+k-1}{k-1} \|S_1 T^n S_2\|, \quad n \in \mathbb{N}_0.$$

The assumption (3.4) yields

$$c(n) = O\left(\frac{n^{k-1}}{f(n)}\right)$$

as $n \rightarrow \infty$. By Proposition 2.7 with $\beta = k - 1 > \alpha - 1$, there exists $M > 0$ such that the desired estimate (3.5) holds whenever $1 < |\lambda| < 2$. ■

Next, we establish the converse transference. To this end, we work in the Hilbert space setting for Parseval’s equality and use the commutativity of S and T under the assumption that $X = Y$. The technique employed here originates from [5, Theorem 4.7], which concerned rates of decay for C_0 -semigroups.

LEMMA 3.3. *Let X be a Hilbert space. Let $T \in \mathcal{L}(X)$ be power-bounded, and suppose that $S \in \mathcal{L}(X)$ commutes with T . For a fixed $k \in \mathbb{N}$, if $F: (0, \infty) \rightarrow (0, \infty)$ satisfies*

$$(3.6) \quad \|R(\lambda, T)^k S\| \leq F(|\lambda| - 1)$$

for all $\lambda \in \mathbb{E}$, then there exists a constant $M > 0$ such that

$$(3.7) \quad \|T^n S\| \leq \frac{MF(1/n)}{n^k}$$

for all $n \in \mathbb{N}$.

Proof. Let $x \in X$, $k \in \mathbb{N}$, and $K := \sup_{n \in \mathbb{N}_0} \|T^n\|$. By the inequality (3.6) and the commutativity of S ,

$$\|R(\lambda, T)^{k+1} Sx\| \leq \|R(\lambda, T)^k S\| \|R(\lambda, T)x\| \leq F(|\lambda| - 1) \|R(\lambda, T)x\|$$

for all $\lambda \in \mathbb{E}$. This and Parseval’s equality give

$$\begin{aligned} \int_0^{2\pi} \|R(re^{i\theta}, T)^{k+1} Sx\|^2 d\theta &\leq F(r - 1)^2 \int_0^{2\pi} \|R(re^{i\theta}, T)x\|^2 d\theta \\ &\leq 2\pi F(r - 1)^2 \sum_{n=0}^{\infty} \frac{\|T^n x\|^2}{r^{2(n+1)}} = 2\pi K^2 \frac{F(r - 1)^2}{r^2 - 1} \|x\|^2 \end{aligned}$$

for all $r > 1$. Using Parseval’s equality again, we obtain

$$\sum_{m=0}^{\infty} \left\| \binom{m+k}{k} \frac{T^m Sx}{r^{m+k+1}} \right\|^2 \leq K^2 \frac{F(r - 1)^2}{r^2 - 1} \|x\|^2$$

for all $r > 1$. Hence,

$$\begin{aligned} \sum_{m=0}^n (m+1)^{2k} \|T^m Sx\|^2 &\leq (k! r^{(n+k+1)})^2 \sum_{m=0}^n \left\| \binom{m+k}{k} \frac{T^m Sx}{r^{m+k+1}} \right\|^2 \\ &\leq (k! K r^{(n+k+1)})^2 \frac{F(r - 1)^2}{r^2 - 1} \|x\|^2 \end{aligned}$$

for all $r > 1$ and $n \in \mathbb{N}$. If we fix $n \in \mathbb{N}$ and set

$$r = 1 + \frac{1}{n},$$

then

$$\sum_{m=0}^n (m+1)^{2k} \|T^m Sx\|^2 \leq (k!K)^2 \left(1 + \frac{1}{n}\right)^{2(n+k+1)} \frac{n^2 F(1/n)^2}{2n+1} \|x\|^2.$$

Since

$$\left(1 + \frac{1}{n}\right)^{2(n+k+1)} \leq 2^{2(k+2)}$$

for all $n \in \mathbb{N}$, we have

$$(3.8) \quad \sum_{m=0}^n (m+1)^{2k} \|T^m Sx\|^2 \leq M_1 n F(1/n)^2 \|x\|^2$$

for all $n \in \mathbb{N}$, where $M_1 := (k!K)^2 2^{2k+3}$.

Define

$$p_k(n) := \sum_{m=0}^n (m+1)^k, \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $y \in X$. Since

$$T^n Sx = \frac{1}{p_k(n)} \sum_{m=0}^n (m+1)^k T^{n-m} T^m Sx,$$

the Cauchy–Schwarz inequality yields

$$(3.9) \quad \begin{aligned} |\langle T^n Sx, y \rangle| &= \frac{1}{p_k(n)} \left| \sum_{m=0}^n (m+1)^k \langle T^m Sx, (T^*)^{n-m} y \rangle \right| \\ &\leq \frac{1}{p_k(n)} \left(\sum_{m=0}^n (m+1)^{2k} \|T^m Sx\|^2 \right)^{1/2} \left(\sum_{m=0}^n \|(T^*)^{n-m} y\|^2 \right)^{1/2}. \end{aligned}$$

From $\sup_{n \in \mathbb{N}_0} \|(T^*)^n\| = \sup_{n \in \mathbb{N}_0} \|T^n\| = K$, we obtain

$$(3.10) \quad \sum_{m=0}^n \|(T^*)^{n-m} y\|^2 \leq K^2 (n+1) \|y\|^2.$$

By (3.8)–(3.10),

$$|\langle T^n Sx, y \rangle| \leq K \sqrt{M_1} \frac{\sqrt{n(n+1)} F(1/n)}{p_k(n)} \|x\| \|y\|.$$

Since p_k is a polynomial of degree $k+1$ with positive leading coefficient, the desired inequality (3.7) holds for some constant $M > 0$. ■

Proof of Theorem 3.1. The implication (i) \Rightarrow (ii) follows from Lemma 3.2 and the resolvent estimate (1.3) for power-bounded operators. Suppose that statement (ii) holds. Define

$$F(s) := \frac{M}{s^k f(1/s)}, \quad s > 0.$$

Then

$$\frac{F(1/n)}{n^k} = \frac{M}{f(n)}$$

for all $n \in \mathbb{N}$. Hence, Lemma 3.3 shows that statement (i) holds. ■

Lemma 3.2 with $f(n) = n$ yields an estimate for $R(\lambda, T)^k$ for $k \geq 2$. The following proposition gives an estimate for $R(\lambda, T)$ in the case where $f(n) = n(\log n)^\alpha$ for some $\alpha \geq 0$, although Lemma 3.3 does not provide the converse statement.

PROPOSITION 3.4. *Let X, Y , and Z be Banach spaces, and let $\alpha \geq 0$. If $T \in \mathcal{L}(X)$ with $r(T) \leq 1$, $S_1 \in \mathcal{L}(X, Z)$, and $S_2 \in \mathcal{L}(Y, X)$ satisfy*

$$\|S_1 T^n S_2\| = O\left(\frac{1}{n(\log n)^\alpha}\right)$$

as $n \rightarrow \infty$, then there exists a constant $M > 0$ such that

$$(3.11) \quad \|S_1 R(\lambda, T) S_2\| \leq M H_\alpha(|\lambda| - 1)$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 4/3$, where

$$(3.12) \quad H_\alpha(s) := \begin{cases} |\log s|^{1-\alpha}, & 0 \leq \alpha < 1, \\ \log |\log s|, & \alpha = 1, \\ 1, & \alpha > 1, \end{cases}$$

for $0 < s < e^{-1}$.

Proof. By [70, Lemma 4.2.b)], there exist $M > 0$ and $t_0 > e$ such that

$$\int_{t_0}^{\infty} \frac{e^{-st}}{t(\log t)^\alpha} dt \leq M H_\alpha(s)$$

for all $s \in (0, 1/t_0)$, where H_α is defined by (3.12). The same argument as in the proof of Proposition 2.7 shows that if $c: \mathbb{N}_0 \rightarrow (0, \infty)$ satisfies

$$c(n) = O\left(\frac{1}{n(\log n)^\alpha}\right)$$

as $n \rightarrow \infty$, then

$$\sum_{n=0}^{\infty} \frac{c(n)}{r^n} = O(H_\alpha(r - 1))$$

as $r \downarrow 1$. Except that this fact is used instead of Proposition 2.7, the rest of the proof proceeds in the same way as the proof of Lemma 3.2, so we omit the details. ■

3.2. Examples. The first example shows that the decay rate of $\|T^n S\|$ and the growth rate of $\|R(\lambda, T)^k S\|$ in Theorem 3.1 cannot in general be replaced by better ones in the case of polynomial decay and growth.

EXAMPLE 3.5. Let $X := \ell^2(\mathbb{N})$ and $\alpha > 0$. Define $T, S_\alpha \in \mathcal{L}(X)$ by

$$Tx := \left(\left(1 - \frac{1}{j}\right)x_j \right)_{j \in \mathbb{N}} \quad \text{and} \quad S_\alpha x := \left(\frac{x_j}{j^\alpha} \right)_{j \in \mathbb{N}},$$

where $x = (x_j)_{j \in \mathbb{N}} \in X$. Then

$$(3.13) \quad \limsup_{n \rightarrow \infty} n^\alpha \|T^n S_\alpha\| > 0$$

and

$$(3.14) \quad \|T^n S_\alpha\| = O\left(\frac{1}{n^\alpha}\right)$$

as $n \rightarrow \infty$. Indeed, since

$$\|T^n S_\alpha\| = \sup_{j \in \mathbb{N}} \left| \frac{1}{j^\alpha} \left(1 - \frac{1}{j}\right)^n \right| \geq \frac{1}{n^\alpha} \left(1 - \frac{1}{n}\right)^n$$

for all $n \in \mathbb{N}$, the estimate (3.13) holds. Define $f_{n,\alpha}(s) := s^\alpha(1-s)^n$ for $0 < s \leq 1$. Since

$$f'_{n,\alpha}(s) = s^{\alpha-1}(1-s)^{n-1}(\alpha - (n+\alpha)s),$$

we obtain

$$\|T^n S_\alpha\| \leq \sup_{0 < s \leq 1} f_{n,\alpha}(s) = f_{n,\alpha}\left(\frac{\alpha}{n+\alpha}\right) = \left(\frac{\alpha}{n+\alpha}\right)^\alpha \left(1 - \frac{\alpha}{n+\alpha}\right)^n$$

for all $n \in \mathbb{N}$. This implies that the estimate (3.14) holds.

First, we consider the case $\alpha = 1/2$. We will show that

$$(3.15) \quad \limsup_{r \downarrow 1} \sqrt{r-1} \|R(r, T)S_{1/2}\| > 0$$

and that there exists a constant $M_1 > 0$ such that

$$(3.16) \quad \|R(\lambda, T)S_{1/2}\| \leq \frac{M_1}{\sqrt{|\lambda| - 1}}$$

for all $\lambda \in \mathbb{E}$. From (3.13)–(3.16), we see that the estimates in statements (i) and (ii) of Theorem 3.1 with $k = 1$ cannot in general be improved in the polynomial case.

For all $\lambda \in \mathbb{E}$, we have

$$(3.17) \quad \|R(\lambda, T)S_{1/2}\| = \sup_{j \in \mathbb{N}} \left| \frac{1}{\sqrt{j}(\lambda - (1 - 1/j))} \right| = \sup_{j \in \mathbb{N}} \frac{1}{|g_\lambda(j)|},$$

where

$$g_\lambda(s) := \frac{\lambda s - s + 1}{\sqrt{s}}, \quad s \geq 1.$$

If $\lambda = r \in (1, 2)$, then

$$\inf_{s \geq 1} |g_r(s)| = g_r\left(\frac{1}{r-1}\right) = 2\sqrt{r-1}.$$

For $j \in \mathbb{N}$, define $r_j > 1$ by

$$r_j := \frac{j+1}{j}.$$

Then

$$\frac{1}{r_j - 1} = j \quad \text{and} \quad \lim_{j \rightarrow \infty} r_j = 1.$$

Hence, the estimate (3.15) holds.

To prove the estimate (3.16), let $\lambda = re^{i\theta}$ for some $r \in (1, 2)$ and $\theta \in [-\pi, \pi)$. Then

$$(3.18) \quad |g_\lambda(s)|^2 = \frac{(r^2 - 2r \cos \theta + 1)s^2 + 2(r \cos \theta - 1)s + 1}{s} =: h_{r,\theta}(s)$$

for all $s \geq 1$. The derivative $h'_{r,\theta}$ is given by

$$h'_{r,\theta}(s) = \frac{(r^2 - 2r \cos \theta + 1)s^2 - 1}{s^2},$$

and therefore we define

$$s_{r,\theta} := \frac{1}{\sqrt{r^2 - 2r \cos \theta + 1}}.$$

Then, for all $s \geq 1$,

$$(3.19) \quad h_{r,\theta}(s) \geq \begin{cases} h_{r,\theta}(s_{r,\theta}) = \phi_r(\cos \theta), & r \leq 2 \cos \theta, \\ h_{r,\theta}(1) = r^2, & r > 2 \cos \theta, \end{cases}$$

where

$$\phi_r(w) := 2\sqrt{r^2 - 2rw + 1} + 2(rw - 1)$$

for $w \in [r/2, 1]$. Since ϕ_r decreases on $[r/2, 1]$, we obtain

$$(3.20) \quad \phi_r(w) \geq \phi_r(1) = 4(r - 1)$$

for all $w \in [r/2, 1]$. The estimates (3.19) and (3.20) yield

$$h_{r,\theta}(s) \geq \min \{4(r - 1), r^2\}$$

for all $s \geq 1$. Combining this with (3.17) and (3.18), we see that the estimate (3.16) holds whenever $1 < |\lambda| < 2$. The case $|\lambda| \geq 2$ follows from the power-boundedness of T ; see (1.3).

Let $k \in \mathbb{N}$. In the case $\alpha = k/2$, we have

$$\|R(\lambda, T)^k S_{k/2}\| = \sup_{j \in \mathbb{N}} \left| \frac{1}{j^{k/2}(\lambda - (1 - 1/j))^k} \right| = \|R(\lambda, T)S_{1/2}\|^k.$$

Therefore, (3.15) and (3.16) yield, respectively,

$$\limsup_{r \downarrow 1} (r - 1)^{k/2} \|R(r, T)^k S_{k/2}\| > 0$$

and

$$\|R(\lambda, T)^k S_{k/2}\| \leq \frac{M_k}{(|\lambda| - 1)^{k/2}}$$

for all $\lambda \in \mathbb{E}$ and some constant $M_k > 0$. Hence, for each $k \in \mathbb{N}$, the estimates in statements (i) and (ii) of Theorem 3.1 cannot in general be improved in the polynomial case.

Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X)$ be power-bounded, and let $S \in \mathcal{L}(X, Y)$. Assume that $\|ST^n\| = O(n^{-1})$ as $n \rightarrow \infty$. By Lemma 3.2, there exists a constant $M_1 > 0$ such that $\|SR(\lambda, T)^2\| \leq M_1/(|\lambda| - 1)$ for all $\lambda \in \mathbb{E}$. Moreover, we see from Proposition 3.4 that $\|SR(\lambda, T)\| \leq M_2 \log(|\lambda| - 1)$ for all $\lambda \in \mathbb{E}$ and some constant $M_2 > 0$. One may ask whether $\sup_{\lambda \in \mathbb{E}} \|SR(\lambda, T)\| < \infty$ holds in the Hilbert space setting. The second example shows that the answer to this question is negative.

EXAMPLE 3.6. Let $X := \ell^2(\mathbb{N})$. Define $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X, \mathbb{C})$ by

$$Tx := \left(\left(1 - \frac{1}{\sqrt{j}}\right) x_j \right)_{j \in \mathbb{N}} \quad \text{and} \quad Sx := \sum_{j=1}^{\infty} \frac{x_j}{j},$$

where $x = (x_j)_{j \in \mathbb{N}} \in X$. We will prove below that there exist constants $K_1, K_2, M_1, M_2 > 0$ such that

$$(3.21) \quad \frac{K_1}{n} \leq \|ST^n\| \leq \frac{K_2}{n}.$$

for all $n \in \mathbb{N}$ and

$$(3.22) \quad M_1 \sqrt{|\log(r - 1)|} \leq \|SR(r, T)\| \leq M_2 \sqrt{|\log(r - 1)|}$$

for all $r \in (1, 2)$. These estimates show that $\|ST^n\| = O(n^{-1})$ as $n \rightarrow \infty$ does not in general imply $\sup_{\lambda \in \mathbb{E}} \|SR(\lambda, T)\| < \infty$.

Let $n \in \mathbb{N}$. Then

$$\|ST^n\|^2 = \sum_{j=2}^{\infty} \left| \frac{1}{j} \left(1 - \frac{1}{\sqrt{j}}\right)^n \right|^2.$$

If $j \leq s < j + 1$ for some $j \in \mathbb{N}$ with $j \geq 2$, then

$$\frac{1}{2(s - 1)} \left(1 - \frac{1}{\sqrt{s - 1}}\right)^n \leq \frac{1}{j} \left(1 - \frac{1}{\sqrt{j}}\right)^n \leq \frac{2}{s} \left(1 - \frac{1}{\sqrt{s}}\right)^n.$$

Therefore,

$$(3.23) \quad \frac{1}{4} \int_1^{\infty} \frac{1}{s^2} \left(1 - \frac{1}{\sqrt{s}}\right)^{2n} ds \leq \|ST^n\|^2 \leq 4 \int_2^{\infty} \frac{1}{s^2} \left(1 - \frac{1}{\sqrt{s}}\right)^{2n} ds.$$

In addition, we have

$$(3.24) \quad \int_1^\infty \frac{1}{s^2} \left(1 - \frac{1}{\sqrt{s}}\right)^{2n} ds = 2 \int_0^1 \tau(1-\tau)^{2n} d\tau = \frac{1}{(n+1)(2n+1)}.$$

By (3.23) and (3.24), there exist $K_1, K_2 > 0$ such that the estimate (3.21) holds for all $n \in \mathbb{N}$.

Let $r \in (1, 2)$. We have

$$\|SR(r, T)\|^2 = \sum_{j=1}^\infty \frac{1}{((r-1)j + \sqrt{j})^2}.$$

If $j \leq s < j+1$ for some $j \in \mathbb{N}$, then

$$\frac{1}{(r-1)s + \sqrt{s}} \leq \frac{1}{(r-1)j + \sqrt{j}} \leq \frac{2}{(r-1)s + \sqrt{s}}.$$

Hence,

$$\int_1^\infty \frac{1}{((r-1)s + \sqrt{s})^2} ds \leq \|SR(r, T)\|^2 \leq \int_1^\infty \frac{4}{((r-1)s + \sqrt{s})^2} ds.$$

Since

$$\begin{aligned} \int_1^\infty \frac{1}{((r-1)s + \sqrt{s})^2} ds &= \int_1^\infty \frac{2}{((r-1)\tau + 1)^2 \tau} d\tau \\ &= 2 \left(\log r - \log(r-1) - \frac{1}{r} \right), \end{aligned}$$

there exist $M_1, M_2 > 0$ such that the estimate (3.22) holds for all $r \in (1, 2)$.

Finally, we give an example showing that the estimate in Proposition 3.4 cannot in general be improved.

EXAMPLE 3.7. Let $X := c_0(\mathbb{N})$ and let T be the left shift operator on X , that is, $Tx = (x_{j+1})_{j \in \mathbb{N}}$ for $x = (x_j)_{j \in \mathbb{N}} \in X$. Then we have $\|T\| \leq 1$ and $\lim_{n \rightarrow \infty} \|T^n x\| = 0$ for all $x \in X$. Let $b = (b_j)_{j \in \mathbb{N}}$ be a non-increasing sequence of positive real numbers. Define $S \in \mathcal{L}(X)$ by

$$Sx := (b_j x_j)_{j \in \mathbb{N}}, \quad x = (x_j)_{j \in \mathbb{N}} \in X.$$

Since

$$T^n Sx = (b_{j+n} x_{j+n})_{j \in \mathbb{N}}, \quad x = (x_j)_{j \in \mathbb{N}} \in X,$$

for all $n \in \mathbb{N}$, it follows immediately that

$$\|T^n S\| = O(b_n)$$

as $n \rightarrow \infty$. Let $\alpha \geq 0$. When

$$(3.25) \quad b_j = \frac{1}{j(\log(j+1))^\alpha}, \quad j \in \mathbb{N},$$

there exists $c > 0$ such that

$$(3.26) \quad \|R(r, T)S\| \geq cH_\alpha(r - 1)$$

for all $r \in (1, 4/3)$, where H_α is defined by (3.12).

To prove (3.26), let $r \in (1, 4/3)$ and define

$$x^m = (x_j^m)_{j \in \mathbb{N}} := \left(\frac{m}{j + m} \right)_{j \in \mathbb{N}}$$

for $m \in \mathbb{N}$. Then $x^m \in c_0(\mathbb{N})$ and $\|x^m\| \leq 1$ for all $m \in \mathbb{N}$. Moreover,

$$(3.27) \quad \lim_{m \rightarrow \infty} x_j^m = 1$$

for all $j \in \mathbb{N}$. Since (3.2) yields

$$R(r, T)S = \sum_{n=0}^{\infty} \frac{T^n S}{r^{n+1}},$$

the first entry of $R(r, T)Sx^m$ is given by

$$\sum_{n=1}^{\infty} \frac{b_n x_n^m}{r^n}.$$

Using (3.27) and the dominated convergence theorem, we obtain

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \frac{b_n x_n^m}{r^n} = \sum_{n=1}^{\infty} \frac{b_n}{r^n}.$$

Let $b = (b_j)_{j \in \mathbb{N}}$ be given by (3.25) for some $\alpha \geq 0$. Then

$$\sum_{n=1}^{\infty} \frac{b_n}{r^n} \geq \int_1^{\infty} \frac{e^{-(\log r)t}}{t(\log(t + 1))^\alpha} dt \geq \int_2^{\infty} \frac{e^{-(\log r)t}}{t(\log t)^\alpha} dt.$$

Moreover,

$$\int_2^{\infty} \frac{e^{-(\log r)t}}{t(\log t)^\alpha} dt \geq \int_2^{1/\log r} \frac{e^{-(\log r)t}}{t(\log t)^\alpha} dt \geq e^{-1} \int_2^{1/\log r} \frac{1}{t(\log t)^\alpha} dt.$$

We have

$$e^{-1} \int_2^{1/\log r} \frac{1}{t(\log t)^\alpha} dt \geq cH_\alpha(\log r)$$

for some $c > 0$ independent of r . Since $\log r \leq r - 1$, we obtain $H_\alpha(\log r) \geq H_\alpha(r - 1)$. Therefore,

$$\sup_{m \in \mathbb{N}} \|R(r, T)Sx^m\| \geq cH_\alpha(r - 1).$$

Recalling that $\|x^m\| \leq 1$ for all $m \in \mathbb{N}$, we conclude that (3.26) holds for all $r \in (1, 4/3)$.

3.3. The special case when $S = I - T$. Now we focus on the case when $S = I - T$. In this subsection, we often use the following simple relation: Let T be a bounded linear operator on a Banach space. Then

$$(3.28) \quad R(\lambda, T)(I - T) = 1 - (\lambda - 1)R(\lambda, T)$$

for all $\lambda \in \rho(T)$.

We recall the following result on Ritt operators; see [44, 41] for the proof.

THEOREM 3.8. *Let X be a Banach space, and let $T \in \mathcal{L}(X)$. Then the following statements are equivalent:*

- (i) T is a Ritt operator.
- (ii) T is power-bounded, and $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$.

We characterize Ritt operators on Hilbert spaces in terms of the powers of resolvents as a corollary of Theorem 3.1.

COROLLARY 3.9. *Let X be a Hilbert space. Let $T \in \mathcal{L}(X)$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

- (i) T is a Ritt operator.
- (ii) T is power-bounded, and there exists a constant $M > 0$ such that

$$(3.29) \quad \|R(\lambda, T)^{k+1}\| \leq \frac{M}{|\lambda - 1|(|\lambda| - 1)^k}$$

for all $\lambda \in \mathbb{E}$.

Proof. Suppose that statement (i) holds. By Theorem 3.8, T is power-bounded. The estimate (3.29) follows immediately from the definition of Ritt operators and

$$\frac{1}{|\lambda - 1|^{k+1}} \leq \frac{1}{|\lambda - 1|(|\lambda| - 1)^k}$$

for all $\lambda \in \mathbb{E}$.

Suppose that statement (ii) holds. Since T is a Kreiss operator, there exists $M_1 > 0$ such that

$$(3.30) \quad \|R(\lambda, T)^k\| \leq \frac{M_1}{(|\lambda| - 1)^k}$$

for all $\lambda \in \mathbb{E}$. From (3.28)–(3.30), we obtain

$$\|R(\lambda, T)^{k+1}(I - T)\| \leq \|R(\lambda, T)^k\| + \|(\lambda - 1)R(\lambda, T)^{k+1}\| \leq \frac{M_1 + M}{(|\lambda| - 1)^k}$$

for all $\lambda \in \mathbb{E}$. Theorem 3.1 shows that $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$. Thus, statement (i) holds by Theorem 3.8. ■

For $c, \delta \geq 1$, define

$$(3.31) \quad S_c^\delta := \left\{ \lambda \in \mathbb{D} : \frac{|1 - \lambda|^\delta}{1 - |\lambda|} < c \right\} \cup \{1\},$$

Following [42], we call S_c^δ the δ -Stolz domain. It is worth noting that a closely related concept was previously introduced in [2], where it was referred to as a *generalized Stolz domain with parameter δ* . The (3/2)-Stolz domain is illustrated in [42, Fig. 3]. The 1-Stolz domain coincides with the Stolz domain studied in [28].

By [14, Proposition 2.6], if a power-bounded operator T on a Banach space satisfies $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$ for some $\alpha \in (1/2, 1)$, then $\sigma(T)$ is contained in a certain quasi-Stolz set. Here we show that a similar implication is true for $\alpha \in (0, 1)$, by using $(1/\alpha)$ -Stolz domains instead of quasi-Stolz sets. A short discussion of comparison between quasi-Stolz sets and δ -Stolz domains can be found in [42, Remark 2.7].

LEMMA 3.10. *Let X be a Banach space. Let $T \in \mathcal{L}(X)$ satisfy $r(T) \leq 1$. Then the following statements hold:*

- (a) *If $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$ for some $\alpha \in [0, 1)$, there exists a constant $M > 0$ such that*

$$(3.32) \quad \|R(\lambda, T)\| \leq \frac{M}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$.

- (b) *Let $\alpha \in (0, 1)$. If there exists a constant $M > 0$ such that T satisfies (3.32) for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$ and*

$$(3.33) \quad \|R(\lambda, T)\| \leq \frac{M}{|\lambda| - 1}$$

for all $\lambda \in \mathbb{E}$, then

$$(3.34) \quad \sigma(T) \subset S_c^{1/\alpha}$$

for some constant $c \geq 2$, where $S_c^{1/\alpha}$ is the $(1/\alpha)$ -Stolz domain defined by (3.31).

Proof. (a) By Lemma 3.2, there exists $M_1 > 0$ such that

$$\|R(\lambda, T)(I - T)\| \leq \frac{M_1}{(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$. Combining this with (3.28), we obtain

$$\|R(\lambda, T)\| \leq \frac{1}{|\lambda - 1|} + \frac{M_1}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}} \leq \frac{1 + M_1}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$.

- (b) By the estimates (3.32) and (3.33), there exists $M_1 > 0$ such that

$$(3.35) \quad \|R(\lambda, T)\| \leq \frac{M_1|\lambda|^{1-\alpha}}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{E}$. Hence, [42, Theorem 2.6] yields the desired inclusion (3.34). ■

If T is a power-bounded operator on a Hilbert space, then the converse of Lemma 3.10(a) also holds.

PROPOSITION 3.11. *Let X be a Hilbert space, and let $T \in \mathcal{L}(X)$ be power-bounded. Then the following statements are equivalent for a fixed $\alpha \in (0, 1)$:*

- (i) $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$.
- (ii) *There exists a constant $M > 0$ such that*

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$.

Proof. The implication (i) \Rightarrow (ii) follows from Lemma 3.10(a). Suppose that statement (ii) holds. Then (3.28) yields

$$\|R(\lambda, T)(I - T)\| \leq \frac{1 + M}{(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$. Combining this with the fact that T is a Kreiss operator, we see that there exists $M_1 > 0$ such that

$$\|R(\lambda, T)(I - T)\| \leq \frac{M_1}{(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{E}$. Hence, Theorem 3.1 shows that $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$. ■

REMARK 3.12. The implication (ii) \Rightarrow (i) of Proposition 3.11 can be proved alternatively by using the results in [62, 42]. Let $\alpha, \beta \geq 0$. Following [42, Definition 1.1], we say that $T \in \mathcal{L}(X)$ is an (α, β) -RK operator if T satisfies $r(T) \leq 1$ and

$$\|R(\lambda, T)\| \leq \frac{M|\lambda|^{\alpha+\beta-1}}{|\lambda - 1|^\alpha(|\lambda| - 1)^\beta}$$

for all $\lambda \in \mathbb{E}$ and some constant $M > 0$. As seen in the proof of Lemma 3.10(b), if a power-bounded operator T on a Banach space X satisfies the estimate given in Proposition 3.11(ii), then T is a $(1, 1 - \alpha)$ -RK operator. It follows from [42, Lemma 2.1] that $\sigma(T) \cap \mathbb{T} \subset \{1\}$. Moreover,

$$(3.36) \quad \|R(e^{i\theta}, T)\| = O(1/|\theta|^{1/\alpha})$$

as $\theta \rightarrow 0$ by [42, Proposition 2.2]. Therefore, [62, Corollary 3.1] shows that

$$\|T^n(I - T)\| = O((\log n)^\alpha/n^\alpha)$$

as $n \rightarrow \infty$; see also [42, Theorem 3.5]. When X is a Hilbert space, $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$ by [62, Theorem 3.10].

Let T be a normal contraction on a Hilbert space, and let $\alpha \in (1/2, 1)$. In [14, Corollary 2.7], the decay rate $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$ was characterized in terms of the property that $\sigma(T)$ is contained in some

quasi-Stolz set. Using the δ -Stolz domain and Lemma 3.10, we obtain an extended characterization in the Banach space case with $\alpha \in (0, 1)$.

PROPOSITION 3.13. *Let X be a Banach space, and let $T \in \mathcal{L}(X)$ be a Kreiss operator. Assume that there exist constants $C \geq 1$ and $n_0 \in \mathbb{N}$ such that*

$$(3.37) \quad \|T^n(I - T)\| \leq C \max_{\lambda \in \sigma(T)} |\lambda|^n |1 - \lambda|$$

for all $n \geq n_0$. Then the following statements are equivalent for a fixed $\alpha \in (0, 1)$:

- (i) $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$.
- (ii) There exists a constant $M > 0$ such that

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda - 1|(|\lambda| - 1)^{1-\alpha}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$.

- (iii) $\sigma(T) \subset S_c^{1/\alpha}$ for some constant $c \geq 2$, where $S_c^{1/\alpha}$ is the $(1/\alpha)$ -Stolz domain defined by (3.31).

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) follow from Lemma 3.10. It remains to prove the implication (iii) \Rightarrow (i).

Let $n \geq n_0$. By assumption, $|1 - \lambda| \leq c^\alpha (1 - |\lambda|)^\alpha$ for all $\lambda \in \sigma(T)$. Hence,

$$(3.38) \quad \|T^n(I - T)\| \leq c^\alpha C \max_{\lambda \in \mathbb{D}} |\lambda|^n (1 - |\lambda|)^\alpha.$$

Define $f(s) := s^n(1 - s)^\alpha$ for $0 \leq s \leq 1$. Then

$$f'(s) = s^{n-1}(1 - s)^{\alpha-1}(n - (n + \alpha)s)$$

for all $s \in [0, 1)$. Therefore,

$$(3.39) \quad \max_{0 \leq s \leq 1} f(s) = f\left(\frac{n}{n + \alpha}\right) = \left(\frac{n}{n + \alpha}\right)^n \left(\frac{\alpha}{n + \alpha}\right)^\alpha.$$

From (3.38) and (3.39), we conclude that statement (i) holds. ■

The estimate (3.37) is satisfied for operators similar to quasi-multiplication operators studied in [47, Section 5]. Typical examples of quasi-multiplication operators are normal operators on Hilbert spaces and multiplication operators on L^p -spaces with $1 \leq p < \infty$. An analytic Toeplitz operator on $H^2(\mathbb{T})$, which is not normal in general, is also a quasi-multiplication operator, as shown in [47, Example 5.1].

4. Integral estimate for resolvents. An integral estimate for resolvents that characterizes bounded C_0 -semigroups on Hilbert spaces was given in [27, 63]. This characterization has the following discrete analogue; see [16] and [21, Theorem II.1.12] for the proof.

THEOREM 4.1. *Let X be a Hilbert space, and let $T \in \mathcal{L}(X)$ satisfy $r(T) \leq 1$. Then the following statements are equivalent:*

- (i) *T is power-bounded.*
- (ii) *There exists a constant $M > 0$ such that*

$$\sup_{1 < r < 2} (r - 1) \int_0^{2\pi} (\|R(re^{i\theta}, T)x\|^2 + \|R(re^{i\theta}, T^*)x\|^2) d\theta \leq M\|x\|^2$$

for all $x \in X$.

The following theorem gives an integral estimate to characterize rates of decay for discrete operator semigroups on Hilbert spaces. This result is a generalization of [69, Proposition 4.3], where the case $f(n) = n^\alpha$ with $0 < \alpha < 1/2$ was considered for the rate of decay of an individual orbit.

THEOREM 4.2. *Let X be a Hilbert space, and let Y be a Banach space. Let $T \in \mathcal{L}(X)$ be power-bounded, and let $S \in \mathcal{L}(Y, X)$. Suppose that $\alpha > 0$ and $k \in \mathbb{N}$ satisfy $k > \alpha + 1/2$. If $f: (0, \infty) \rightarrow (0, \infty)$ is α -bounded regularly varying, then the following statements are equivalent:*

- (i) *$\|T^n S\| = O(1/f(n))$ as $n \rightarrow \infty$.*
- (ii) *There exists a constant $M > 0$ such that*

$$\sup_{1 < r < 2} F_k(r^2 - 1) \int_0^{2\pi} \|R(re^{i\theta}, T)^k S y\|^2 d\theta \leq M\|y\|^2$$

for all $y \in Y$, where

$$(4.1) \quad F_k(s) := s^{2k-1} f(1/s)^2, \quad s > 0.$$

The transference from the semigroup estimate to the integral estimate in Theorem 4.2 is obtained from the following lemma.

LEMMA 4.3. *Let X and Y be Banach spaces, and let Z be a Hilbert space. Let $T \in \mathcal{L}(X)$ satisfy $r(T) \leq 1$, and let $S_1 \in \mathcal{L}(X, Z)$ and $S_2 \in \mathcal{L}(Y, X)$. Suppose that $\alpha \geq 0$ and $k \in \mathbb{N}$ satisfy $k > \alpha + 1/2$, and let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying. If $\|S_1 T^n S_2\| = O(1/f(n))$ as $n \rightarrow \infty$, then there exists a constant $M > 0$ such that*

$$\sup_{1 < r < 2} F_k(r^2 - 1) \int_0^{2\pi} \|S_1 R(re^{i\theta}, T)^k S_2 y\|^2 d\theta \leq M\|y\|^2$$

for all $y \in Y$, where F_k is as in Theorem 4.2.

Proof. Parseval's equality yields

$$(4.2) \quad \frac{1}{2\pi} \int_0^{2\pi} \|S_1 R(re^{i\theta}, T)^k S_2 y\|^2 d\theta = \sum_{n=0}^{\infty} \left\| \binom{n+k-1}{k-1} \frac{S_1 T^n S_2 y}{r^{n+k}} \right\|^2 \leq \sum_{n=0}^{\infty} \frac{c(n)}{r^{2n}} \|y\|^2$$

for all $r > 1$ and $y \in Y$, where

$$c(n) := \left(\binom{n+k-1}{k-1} \|S_1 T^n S_2\| \right)^2$$

for $n \in \mathbb{N}_0$. Define $h(t) := f(t)^2$ for $t > 0$. Since $\|S_1 T^n S_2\|^2 = O(1/h(n))$ as $n \rightarrow \infty$ by assumption, we have

$$c(n) = O\left(\frac{n^{2(k-1)}}{h(n)}\right)$$

as $n \rightarrow \infty$. Lemma 2.6(b) shows that h is (2α) -bounded regularly varying. By the estimate (4.2) and Proposition 2.7 with $\beta = 2(k-1) > 2\alpha - 1$, there exist $M > 0$ and $r_1 > 1$ such that

$$\int_0^{2\pi} \|S_1 R(re^{i\theta}, T)^k S_2 y\|^2 d\theta \leq \frac{M \|y\|^2}{F_k(r^2 - 1)}$$

for all $r \in (1, r_1)$ and $y \in Y$. Thus, the desired estimate holds. ■

Next, we show that the integral estimate leads to the semigroup estimate.

LEMMA 4.4. *Let X be a Hilbert space, and let Y be a Banach space. Let $T \in \mathcal{L}(X)$ be power-bounded, and let $S \in \mathcal{L}(Y, X)$. If there exist a function $F : (0, \infty) \rightarrow (0, \infty)$ and constants $M > 0$ and $k \in \mathbb{N}$ such that*

$$(4.3) \quad \sup_{1 < r < 2} F(r^2 - 1) \int_0^{2\pi} \|R(re^{i\theta}, T)^k S y\|^2 d\theta \leq M \|y\|^2$$

for all $y \in Y$, then

$$(4.4) \quad \|T^n S\| = O\left(\frac{1}{n^{k-1/2} F(1/n)^{1/2}}\right)$$

as $n \rightarrow \infty$.

Proof. Let $n \in \mathbb{N}$, $x \in X$, and $y \in Y$. Define

$$p_k(n) := \binom{n+k}{k}.$$

By Lemma 2.1,

$$\begin{aligned} \langle T^n S y, x \rangle &= \left\langle \frac{r^{n+k+1}}{2\pi p_k(n)} \int_0^{2\pi} e^{i\theta(n+k+1)} R(re^{i\theta}, T)^{k+1} S y \, d\theta, x \right\rangle \\ &= \frac{r^{n+k+1}}{2\pi p_k(n)} \left\langle \int_0^{2\pi} e^{i\theta(n+k+1)} R(re^{i\theta}, T)^k S y \, d\theta, R(re^{-i\theta}, T^*) x \right\rangle. \end{aligned}$$

Theorem 4.1 shows that

$$(4.5) \quad \sup_{1 < r < 2} (r - 1) \int_0^{2\pi} \|R(re^{-i\theta}, T^*) x\|^2 \, d\theta \leq M_1 \|x\|^2$$

for all $x \in X$ and some constant $M_1 > 0$. Combining the Cauchy–Schwarz inequality with the estimates (4.3) and (4.5), we obtain

$$\begin{aligned} &|\langle T^n S y, x \rangle| \\ &\leq \frac{r^{n+k+1}}{2\pi p_k(n)} \left(\int_0^{2\pi} \|R(re^{i\theta}, T)^k S y\|^2 \, d\theta \right)^{1/2} \left(\int_0^{2\pi} \|R(re^{-i\theta}, T^*) x\|^2 \, d\theta \right)^{1/2} \\ &\leq \frac{r^{n+k+1}}{2\pi p_k(n)} \sqrt{\frac{M M_1}{(r - 1) F(r^2 - 1)}} \|y\| \|x\| \end{aligned}$$

for all $r \in (1, 2)$. Set

$$r = \left(1 + \frac{1}{n} \right)^{1/2}.$$

Then

$$r - 1 \geq \frac{1}{3n}.$$

Since $p_k(n) \geq n^k/k!$, we obtain the desired estimate (4.4). ■

Combining Lemmas 4.3 and 4.4, we prove Theorem 4.2.

Proof of Theorem 4.2. The implication (i)⇒(ii) follows from Lemma 4.3. Since the function F_k defined by (4.1) satisfies

$$\frac{1}{n^{k-1/2} F_k(1/n)^{1/2}} = \frac{1}{f(n)}$$

for $n \in \mathbb{N}$, Lemma 4.4 with $F = F_k$ yields the implication (ii)⇒(i). ■

We conclude this section by presenting a characterization of Ritt operators on Hilbert spaces as a corollary of Theorem 4.2.

COROLLARY 4.5. *Let X be a Hilbert space. Let $T \in \mathcal{L}(X)$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

- (i) T is a Ritt operator.
- (ii) T is power-bounded, and there exists a constant $M > 0$ such that

$$\sup_{1 < r < 2} (r - 1)^{2k-1} \int_0^{2\pi} \|(re^{i\theta} - 1)R(re^{i\theta}, T)^{k+1}x\|^2 d\theta \leq M\|x\|^2$$

for all $x \in X$.

Proof. By Theorems 3.8 and 4.2, it suffices to prove that the following statements are equivalent under the assumption that T is power-bounded:

- (ii-a) There exists $M_1 > 0$ such that

$$(4.6) \quad \sup_{1 < r < 2} (r - 1)^{2k-1} \int_0^{2\pi} \|(re^{i\theta} - 1)R(re^{i\theta}, T)^{k+1}x\|^2 d\theta \leq M_1\|x\|^2$$

for all $x \in X$.

- (ii-b) There exists $M_2 > 0$ such that

$$(4.7) \quad \sup_{1 < r < 2} (r - 1)^{2k-1} \int_0^{2\pi} \|R(re^{i\theta}, T)^{k+1}(I - T)x\|^2 d\theta \leq M_2\|x\|^2$$

for all $x \in X$.

Since $(a + b)^2 \leq 2(a^2 + b^2)$ for all $a, b \geq 0$, we deduce from (3.28) that

$$\begin{aligned} \|R(\lambda, T)^{k+1}(I - T)x\|^2 &\leq 2\|(\lambda - 1)R(\lambda, T)^{k+1}x\|^2 + 2\|R(\lambda, T)^kx\|^2, \\ \|(\lambda - 1)R(\lambda, T)^{k+1}x\|^2 &\leq 2\|R(\lambda, T)^{k+1}(I - T)x\|^2 + 2\|R(\lambda, T)^kx\|^2 \end{aligned}$$

for all $\lambda \in \mathbb{E}$ and $x \in X$. Moreover, since T is a Kreiss operator, it follows that there exists $M_3 > 0$ such that

$$\|R(\lambda, T)^kx\| \leq \frac{M_3\|R(\lambda, T)x\|}{(|\lambda| - 1)^{k-1}}$$

for all $\lambda \in \mathbb{E}$ and $x \in X$. Combining this with Theorem 4.1, we see that

$$\sup_{1 < r < 2} (r - 1)^{2k-1} \int_0^{2\pi} \|R(re^{i\theta}, T)^kx\|^2 d\theta \leq M_4\|x\|^2$$

for all $x \in X$ and some constant $M_4 > 0$. Thus, statements (ii-a) and (ii-b) are equivalent. ■

5. Decay rates of perturbed discrete semigroups. We use the following Sherman–Morrison–Woodbury formula as in the robustness analysis for strong stability of discrete operator semigroups established in [53]. This formula can be obtained from a direct calculation, and hence the proof is omitted.

LEMMA 5.1. *Let X and Y be Banach spaces. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y, X)$, $C \in \mathcal{L}(X, Y)$, and $\lambda \in \varrho(A)$. If $1 \in \varrho(CR(\lambda, A)B)$, then $\lambda \in \varrho(A + BC)$ and*

$$(5.1) \quad R(\lambda, A + BC) = R(\lambda, A) + R(\lambda, A)B(I - CR(\lambda, A)B)^{-1}CR(\lambda, A).$$

The following corollary of Theorem 4.1 provides a condition for the preservation of boundedness and decay properties under perturbations. Similar conditions were obtained for polynomial stability and strong stability of C_0 -semigroups in [56, Theorems 4.1 and 4.2].

COROLLARY 5.2. *Let X be a Hilbert space, and let Y be a Banach space. Let $T \in \mathcal{L}(X)$ be power-bounded, and let $S \in \mathcal{L}(Y, X)$. If $D \in \mathcal{L}(X)$ commutes with T and satisfies $\sup_{\lambda \in \mathbb{E}} \|R(\lambda, T)D\| < 1$, then the following statements hold:*

- (a) $T + D$ is power-bounded.
- (b) Let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying for some $\alpha > 0$. If $\|T^n S\| = O(1/f(n))$ as $n \rightarrow \infty$, then $\|(T + D)^n S\| = O(1/f(n))$ as $n \rightarrow \infty$.

Proof. Define $\delta := \sup_{z \in \mathbb{E}} \|R(z, T)D\| < 1$, and let $\lambda \in \mathbb{E}$. Since $1 \in \varrho(R(\lambda, T)D)$, the Sherman–Morrison–Woodbury formula (5.1) gives

$$(5.2) \quad R(\lambda, T + D) = R(\lambda, T) + R(\lambda, T)D(I - R(\lambda, T)D)^{-1}R(\lambda, T).$$

Moreover,

$$(5.3) \quad \|R(\lambda, T)D\| \|(I - R(\lambda, T)D)^{-1}\| \leq \frac{\delta}{1 - \delta}.$$

(a) By (5.2) and (5.3),

$$(5.4) \quad \|R(\lambda, T + D)x\| \leq \frac{1}{1 - \delta} \|R(\lambda, T)x\|$$

for all $x \in X$. Since D commutes with $R(z, T)$ for all $z \in \mathbb{E}$, we have

$$\sup_{z \in \mathbb{E}} \|R(z, T^*)D^*\| = \sup_{z \in \mathbb{E}} \|DR(z, T)\| = \delta.$$

As in (5.4), it follows that

$$(5.5) \quad \|R(\lambda, T^* + D^*)x\| \leq \frac{1}{1 - \delta} \|R(\lambda, T^*)x\|$$

for all $x \in X$. By Theorem 4.1, there exists $M_1 > 0$ such that

$$\sup_{1 < r < 2} (r - 1) \int_0^{2\pi} (\|R(re^{i\theta}, T)x\|^2 + \|R(re^{i\theta}, T^*)x\|^2) d\theta \leq M_1 \|x\|^2$$

for all $x \in X$. This, together with the estimates (5.4) and (5.5), yields

$$\sup_{1 < r < 2} (r - 1) \int_0^{2\pi} (\|R(re^{i\theta}, T + D)x\|^2 + \|R(re^{i\theta}, T^* + D^*)x\|^2) d\theta \leq \frac{M_1}{(1 - \delta)^2} \|x\|^2$$

for all $x \in X$. Using Theorem 4.1 again, we deduce that $T + D$ is power-bounded.

(b) From the commutativity of D and T , we see that $R(\lambda, T)$ commutes with

$$H(\lambda) := R(\lambda, T)D(I - R(\lambda, T)D)^{-1}.$$

Let $k \in \mathbb{N}$ satisfy $k > \alpha + 1/2$. Since (5.2) yields

$$R(\lambda, T + D)^k = (1 + H(\lambda))^k R(\lambda, T)^k,$$

by (5.3) we have

$$(5.6) \quad \|R(\lambda, T + D)^k S y\| \leq \frac{1}{(1 - \delta)^k} \|R(\lambda, T)^k S y\|$$

for all $y \in Y$. By Theorem 4.2, there exists $M_2 > 0$ such that

$$\sup_{1 < r < 2} F_k(r^2 - 1) \int_0^{2\pi} \|R(re^{i\theta}, T)^k S y\|^2 d\theta \leq M_2 \|y\|^2$$

for all $y \in Y$, where F_k is as in that theorem. Combining this with (5.6), we obtain

$$\sup_{1 < r < 2} F_k(r^2 - 1) \int_0^{2\pi} \|R(re^{i\theta}, T + D)^k S y\|^2 d\theta \leq \frac{M_2}{(1 - \delta)^{2k}} \|y\|^2$$

for all $y \in Y$. Thus, Theorem 4.2 shows that $\|(T + D)^n S\| = O(1/f(n))$ as $n \rightarrow \infty$. ■

6. Relation between semigroup decay and summability conditions. Now we study the relation between the rate of decay of discrete operator semigroups and summability conditions.

PROPOSITION 6.1. *Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y, X)$, and $p > 0$. Then the following statements hold:*

(a) *Assume that T is power-bounded. If there exist functions $f, G: \mathbb{N} \rightarrow (0, \infty)$ such that*

$$(6.1) \quad \sum_{m=1}^n f(m) \|T^m S y\|^p \leq G(n) \|y\|^p$$

for all $n \in \mathbb{N}$ and $y \in Y$, then

$$(6.2) \quad \|T^n S\| \leq \frac{KG(n)^{1/p}}{F(n)^{1/p}},$$

for all $n \in \mathbb{N}$, where $K := \sup_{n \in \mathbb{N}_0} \|T^n\|$ and

$$(6.3) \quad F(n) := \sum_{m=1}^n f(m).$$

(b) Let $f, g: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying for some constant $\alpha \geq 0$. Define

$$F(n) := \sum_{m=1}^n f(m)g(m) \quad \text{and} \quad G(n) := \sum_{m=1}^n \frac{1}{mg(m)}$$

for $n \in \mathbb{N}$. If $\|T^n S\| = O(1/F(n)^{1/p})$ as $n \rightarrow \infty$, then there exists a constant $C > 0$ such that

$$\sum_{m=1}^n f(m) \|T^m S y\|^p \leq CG(n) \|y\|^p$$

for all $n \in \mathbb{N}$ and $y \in Y$.

Proof. (a) For all $n \in \mathbb{N}$ and $y \in Y$,

$$F(n) \|T^n S y\|^p \leq \sum_{m=1}^n f(m) \|T^{n-m}\|^p \|T^m S y\|^p \leq K^p \sum_{m=1}^n f(m) \|T^m S y\|^p.$$

From this and (6.1), we obtain the desired estimate (6.2).

(b) By assumption, there exists $t_0 > 0$ such that

$$(6.4) \quad \frac{tf'(t)}{f(t)} \leq \alpha \quad \text{and} \quad \frac{tg'(t)}{g(t)} \leq \alpha$$

for a.e. $t \geq t_0$. For all $t_1 \geq t_0$, integration by parts yields

$$\begin{aligned} \int_{t_0}^{t_1} f(t)g(t) dt &= t_1 f(t_1)g(t_1) - t_0 f(t_0)g(t_0) - \int_{t_0}^{t_1} t(f'(t)g(t) + f(t)g'(t)) dt \\ &\geq t_1 f(t_1)g(t_1) - t_0 f(t_0)g(t_0) - 2\alpha \int_{t_0}^{t_1} f(t)g(t) dt. \end{aligned}$$

If $t_1 \geq 2t_0$, then

$$t_1 f(t_1)g(t_1) \geq 2t_0 f(t_0)g(t_0),$$

and hence

$$(6.5) \quad \int_{t_0}^{t_1} f(t)g(t) dt \geq \frac{t_1 f(t_1)g(t_1)}{2(1 + 2\alpha)}.$$

By assumption, there exist $M_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$(6.6) \quad \|T^n S\| \leq \frac{M_0}{F(n)^{1/p}}.$$

Let $n_1 \in \mathbb{N}$ satisfy $n_1 \geq \max\{n_0, 2t_0\}$. By (6.5), we have

$$F(m) \geq \int_0^{t_0} f(t)g(t) dt + \int_{t_0}^m f(t)g(t) dt \geq \frac{mf(m)g(m)}{2(1+2\alpha)}$$

for all $m \geq n_1$. This gives

$$(6.7) \quad \sum_{m=n_1+1}^n \frac{f(m)}{F(m)} \leq 2(1+2\alpha) \sum_{m=n_1+1}^n \frac{1}{mg(m)} \leq 2(1+2\alpha)G(n)$$

for all $n \geq n_1 + 1$. Using the estimates (6.6) and (6.7), we obtain

$$\begin{aligned} \sum_{m=1}^n f(m)\|T^m S y\|^p &\leq \left(\sum_{m=1}^{n_1} f(m)\|T^m S\|^p + M_0^p \sum_{m=n_1+1}^n \frac{f(m)}{F(m)} \right) \|y\|^p \\ &\leq \left(\sum_{m=1}^{n_1} f(m)\|T^m S\|^p + 2(1+2\alpha)M_0^p G(n) \right) \|y\|^p \end{aligned}$$

for all $n \geq n_1 + 1$ and $y \in Y$. Thus, the desired conclusion holds. ■

The following result is a special case of Proposition 6.1.

COROLLARY 6.2. *Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y, X)$, and $p > 0$. Let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying for some $\alpha \geq 0$, and define F by (6.3). Then the following statements hold:*

(a) *Assume that T is power-bounded. If there exists a constant $C > 0$ such that*

$$(6.8) \quad \sum_{n=1}^{\infty} f(n)\|T^n S y\|^p \leq C\|y\|^p$$

for all $y \in Y$, then $\|T^n S\| = O(1/F(n)^{1/p})$ as $n \rightarrow \infty$.

(b) *If $\|T^n S\| = O(1/F(n)^{1/p})$ as $n \rightarrow \infty$, then there exists a constant $C > 0$ such that*

$$(6.9) \quad \sum_{m=1}^n f(m)\|T^m S y\|^p \leq C(\log n)\|y\|^p$$

for all $n \in \mathbb{N}$ and $y \in Y$.

Proof. Statement (a) follows from Proposition 6.1(a) with $G(n) \equiv C$. If $g(t) \equiv 1$ in Proposition 6.1(b), then

$$G(n) = \sum_{m=1}^n \frac{1}{m} \leq 2 \log(n+1)$$

for all $n \in \mathbb{N}$. Hence, Proposition 6.1(b) shows that statement (b) is true. ■

There is a logarithmic gap between (6.8) and (6.9). However, in the special case when $S = I - T$ and $f(n) = n^{\alpha p - 1}$, this gap can be bridged for a multiplication operator T on L^q -space with $1 \leq q \leq p < \infty$. An analogous result for C_0 -semigroups was obtained under the condition $\alpha p = 1$ in [70, Theorem 3.8]. For discrete semigroups, we use the spectral property based on δ -Stolz domains.

PROPOSITION 6.3. *Let (Ω, μ) be a σ -finite measure space. Let $\phi: \Omega \rightarrow \mathbb{C}$ be measurable with essential range $\phi_{\text{ess}}(\Omega)$ in $\overline{\mathbb{D}}$. Let $1 \leq q \leq p < \infty$, and let T be the multiplication operator induced by ϕ on $L^q(\Omega, \mu)$, i.e., $Tx = \phi x$ for $x \in L^q(\Omega, \mu)$. Then the following statements are equivalent for a fixed $\alpha \in (0, 1)$:*

- (i) $\|T^n(I - T)\| = O(n^{-\alpha})$ as $n \rightarrow \infty$.
- (ii) *There exists a constant $C > 0$ such that*

$$\sum_{n=1}^{\infty} n^{\alpha p - 1} \|T^n(I - T)x\|^p \leq C \|x\|^p$$

for all $x \in L^q(\Omega, \mu)$.

Proof. Since $\phi(\omega) \in \phi_{\text{ess}}(\Omega)$ for μ -a.e. $\omega \in \Omega$ (see, e.g., [37, Chapter VII, Exercise 19]), T is power-bounded. The implication (ii) \Rightarrow (i) is obtained from Proposition 6.1(a) with $f(n) = n^{\alpha p - 1}$, $G(n) \equiv C$, and $S = I - T$. It remains to prove the implication (i) \Rightarrow (ii).

Recall from [25, Proposition I.4.10] that $\sigma(T) = \phi_{\text{ess}}(\Omega)$. Since $\sigma(T) \subset \overline{S_c^{1/\alpha}}$ by Lemma 3.10, we obtain

$$(6.10) \quad \phi(\omega) \in \overline{S_c^{1/\alpha}} \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

Define $\beta := \alpha p - 1$. For all $x \in L^q(\Omega, \mu) \setminus \{0\}$,

$$(6.11) \quad \frac{1}{\|x\|^p} \sum_{n=1}^{\infty} n^\beta \|T^n(I - T)x\|^p = \sum_{n=1}^{\infty} n^\beta \left(\int_{\Omega} |\phi(\omega)^n (1 - \phi(\omega))|^q \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega) \right)^{p/q}.$$

Since $p/q \geq 1$, Jensen's inequality yields

$$(6.12) \quad \left(\int_{\Omega} |\phi(\omega)^n (1 - \phi(\omega))|^q \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega) \right)^{p/q} \leq \int_{\Omega} |\phi(\omega)|^{np} |1 - \phi(\omega)|^p \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega)$$

for all $n \in \mathbb{N}$ and $x \in L^q(\Omega, \mu) \setminus \{0\}$. From (6.10)–(6.12), we now investigate

$$|1 - \lambda|^p \sum_{n=1}^{\infty} n^\beta |\lambda|^{np}$$

for $\lambda \in \overline{S_c^{1/\alpha}} \setminus \{1\}$. Since $\overline{S_c^{1/\alpha}} \cap \mathbb{T} = \{1\}$, it follows that $\lambda \in \overline{S_c^{1/\alpha}} \setminus \{1\}$ satisfies $|\lambda| < 1$. If $\lambda \in \overline{S_c^{1/\alpha}}$ satisfies $|\lambda| \leq e^{-1}$, then

$$|1 - \lambda|^p \sum_{n=1}^{\infty} n^\beta |\lambda|^{np} \leq (1 + e^{-1})^p \sum_{n=1}^{\infty} n^\beta e^{-np} =: C_1 < \infty.$$

Suppose that $\lambda \in \overline{S_c^{1/\alpha}}$ satisfies $e^{-1} < |\lambda| < 1$, and define $r := 1/|\lambda|$. If $n \leq t < n + 1$ for some $n \in \mathbb{N}$, then

$$\frac{n^\beta}{r^{np}} \leq \frac{t^\beta}{r^{(t-1)p}} = t^\beta e^{-(p \log r)(t-1)}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} n^\beta |\lambda|^{np} &= \sum_{n=1}^{\infty} \frac{n^\beta}{r^{np}} \leq \int_1^{\infty} t^\beta e^{-(p \log r)(t-1)} dt \\ &\leq \frac{e^p}{(p \log r)^{\beta+1}} \int_0^{\infty} t^\beta e^{-t} dt \leq \frac{e^p \Gamma(\beta + 1)}{(p \log r)^{\beta+1}}. \end{aligned}$$

Since $r - 1 \leq 2 \log r$, we have

$$\sum_{n=1}^{\infty} n^\beta |\lambda|^{np} \leq \frac{C_2}{(r - 1)^{\beta+1}}, \quad \text{where } C_2 := e^p \Gamma(\beta + 1) \left(\frac{2}{p}\right)^{\beta+1}.$$

From $\lambda \in \overline{S_c^{1/\alpha}}$, it follows that

$$|1 - \lambda|^{1/\alpha} \leq c(1 - |\lambda|).$$

Recalling that $\beta + 1 = \alpha p$, we obtain

$$|1 - \lambda|^p \sum_{n=1}^{\infty} n^\beta |\lambda|^{np} \leq c^{\alpha p} (1 - |\lambda|)^{\alpha p} \frac{C_2 |\lambda|^{\beta+1}}{(1 - |\lambda|)^{\beta+1}} \leq c^{\alpha p} C_2.$$

We have shown that

$$(6.13) \quad |1 - \lambda|^p \sum_{n=1}^{\infty} n^\beta |\lambda|^{np} \leq C_3 := \max \{C_1, c^{\alpha p} C_2\}$$

for all $\lambda \in \overline{S_c^{1/\alpha}} \setminus \{1\}$. Let $\Omega_0 := \Omega \setminus \{\omega \in \Omega : \phi(\omega) = 1\}$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} n^\beta \int_{\Omega} |\phi(\omega)|^{np} |1 - \phi(\omega)|^p \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega) \\ = \sum_{n=1}^{\infty} n^\beta \int_{\Omega_0} |\phi(\omega)|^{np} |1 - \phi(\omega)|^p \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega) \end{aligned}$$

for all $x \in L^q(\Omega, \mu) \setminus \{0\}$. Using (6.10), (6.13), and the monotone convergence theorem, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} n^\beta \int_{\Omega_0} |\phi(\omega)|^{np} |1 - \phi(\omega)|^p \frac{|x(\omega)|^q}{\|x\|^q} d\mu(\omega) \\ \leq \frac{C_3}{\|x\|^q} \int_{\Omega_0} |x(\omega)|^q d\mu(\omega) \leq C_3 \end{aligned}$$

for all $x \in L^q(\Omega, \mu) \setminus \{0\}$. Thus, statement (ii) with $C = C_3$ holds. ■

We have seen the relation between semigroup decay and summability conditions. On the other hand, the relation between semigroup decay and resolvent growth has been investigated in Section 3. The following proposition shows how summability conditions can be transferred to resolvent estimates.

PROPOSITION 6.4. *Let $X, Y,$ and Z be Banach spaces. Let $T \in \mathcal{L}(X)$ satisfy $r(T) \leq 1$, and $S_1 \in \mathcal{L}(X, Z)$ and $S_2 \in \mathcal{L}(Y, X)$. Let $f: (0, \infty) \rightarrow (0, \infty)$ be α -bounded regularly varying for some $\alpha \geq 0$, and assume that there exist constants $C > 0$ and $p > 1$ such that*

$$(6.14) \quad \sum_{n=1}^{\infty} f(n) \|S_1 T^n S_2 y\|^p \leq C \|y\|^p$$

for all $y \in Y$. Then for all $k \in \mathbb{N}$ with $k > (\alpha + 1)/p$, there exists a constant $M > 0$ such that

$$\|S_1 R(\lambda, T)^k S_2\| \leq \frac{M}{(|\lambda|^q - 1)^{k-1/p} f(1/(|\lambda|^q - 1))^{1/p}}$$

for all $\lambda \in \mathbb{C}$ with $1 < |\lambda| < 2$, where $q > 1$ satisfies $1/p + 1/q = 1$.

Proof. Let $k \in \mathbb{N}$ satisfy $k > (\alpha + 1)/p$. Using (3.3), we obtain

$$S_1 R(\lambda, T)^k S_2 = \frac{S_1 S_2}{\lambda^k} + \frac{1}{\lambda^k} \sum_{n=1}^{\infty} \binom{n+k-1}{k-1} \frac{S_1 T^n S_2}{\lambda^n}.$$

Hölder’s inequality yields

$$(6.15) \quad \left\| \sum_{n=1}^{\infty} \binom{n+k-1}{k-1} \frac{S_1 T^n S_2 y}{\lambda^n} \right\| \leq \left(\sum_{n=1}^{\infty} f(n) \|S_1 T^n S_2 y\|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{c(n)}{|\lambda|^{nq}} \right)^{1/q}$$

for all $\lambda \in \mathbb{E}$, where

$$c(n) := \frac{(n+k)^{(k-1)q}}{f(n)^{q/p}}, \quad n \in \mathbb{N}.$$

By Lemma 2.6(b), $h := f^{q/p}$ is $(\alpha q/p)$ -bounded regularly varying. Since $k > (\alpha + 1)/p$ is equivalent to

$$(k-1)q > \frac{\alpha q}{p} - 1$$

and since

$$c(n) = O\left(\frac{n^{(k-1)q}}{h(n)}\right)$$

as $n \rightarrow \infty$, Proposition 2.7 shows that

$$(6.16) \quad \sum_{n=1}^{\infty} \frac{c(n)}{r^{nq}} = O\left(\frac{1}{(r^q - 1)^{(k-1)q+1} h(1/(r^q - 1))}\right)$$

as $r \downarrow 1$. Applying (6.14) and (6.16) to (6.15), we derive the desired estimate. ■

REMARK 6.5. It is legitimate to ask if the converse of Proposition 6.4 is true. When $p = 2$ and $f(n) = n^\alpha$ for $\alpha \geq 0$, the answer to this question is affirmative for some classes of operators, as shown in the literature on admissible observation operators; see [29, 74, 73, 75, 32, 38, 31]. The systematic study of admissible observation operators can be traced back to the work [71] by Weiss for C_0 -semigroups. We refer to the survey article [30] and the books [65, 66] for more information on admissibility for C_0 -semigroups.

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