

Kida's formula for graphs with ramifications

by

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Abstract. Recently Iwasawa theory for graphs is being developed. A significant achievement includes an analogue of Iwasawa's class number formula, which describes the asymptotic growth of the numbers of spanning trees for \mathbb{Z}_p -coverings of graphs. Moreover, an analogue of Kida's formula concerning the behavior of the λ - and μ -invariants is obtained for unramified coverings. In this paper, we establish Kida's formula for possibly ramified coverings.

1. Introduction

1.1. Classical Iwasawa theory. We begin with a brief review of classical Iwasawa theory. Let p be a prime number. In classical Iwasawa theory, we mainly deal with a \mathbb{Z}_p -extension K_∞/K of number fields, that is,

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots,$$

where K_n/K is a $\mathbb{Z}/p^n\mathbb{Z}$ -extension. Iwasawa's class number formula [5, Theorem 11] (see also Washington [16, Theorem 13.13]) states that

$$\text{ord}_p(\#\text{Cl}(K_n)) = \lambda n + \mu p^n + \nu, \quad n \gg 0,$$

for some integers $\lambda = \lambda(K_\infty/K) \geq 0$, $\mu = \mu(K_\infty/K) \geq 0$, and ν , which are called the *Iwasawa invariants*. Here, $\text{Cl}(K_n)$ denotes the class group of K_n and $\text{ord}_p(-)$ denotes the p -adic valuation that is normalized by $\text{ord}_p(p) = 1$.

Kida [9] obtained a formula that describes the behavior of the Iwasawa invariants when K_∞/K varies. To be concrete, let G be a finite p -group and $\tilde{K}_\infty/\tilde{K}$ be another \mathbb{Z}_p -extension such that \tilde{K}_n/K is a $\mathbb{Z}/p^n\mathbb{Z} \times G$ -extension for any $n \geq 0$. Then conceptually the formula states that $\mu(K_\infty/K) = 0$ is equivalent to $\mu(\tilde{K}_\infty/\tilde{K}) = 0$, in which case $\lambda(\tilde{K}_\infty/\tilde{K})$ can be described by using $\lambda(K_\infty/K)$ together with the ramification information in $\tilde{K}_\infty/K_\infty$. (To

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be precise, Kida’s original formula only concerns the minus components for the cyclotomic \mathbb{Z}_p -extensions of CM-fields.)

Kida’s formula is of interest since it serves as an analogue of the Riemann–Hurwitz formula. Analogues of Kida’s formula have also been established in various branches of Iwasawa theory. A typical example is the one for elliptic curves by Hachimori–Matsuno [4]. For other previous works and various proof methods, we refer the reader to the author’s paper [8, §1.2].

1.2. Iwasawa theory for graphs. The main objects of this paper are graphs, which we assume to be finite and connected in this introduction. For a graph X , let $\kappa(X)$ be the number of spanning trees of X . By Kirchhoff’s theorem, $\kappa(X)$ is equal to the order of the Jacobian group $\text{Jac}(X)$ of X (see §2.1). Note that $\text{Jac}(X)$ is also known as the Picard group of degree 0 divisors, the sandpile group, or the critical group. This interpretation enables us to study $\kappa(X)$ by an algebraic investigation of $\text{Jac}(X)$.

Let X_∞/X be an *unramified* (or *unbranched*) \mathbb{Z}_p -covering of graphs, that is, we are given a family of graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots,$$

where X_n/X is an unramified $\mathbb{Z}/p^n\mathbb{Z}$ -covering. This X_n is called the *n*th layer of X_∞/X . Note that we do not regard X_∞ as a single graph but rather as a family of graphs. Then an analogue of Iwasawa’s class number formula, due to Gonet [2, Theorem 1.1] or McGown–Vallières [10, Theorem 6.1] (see also the author’s article [7, §8.1]), states that

$$\text{ord}_p(\kappa(X_n)) = \lambda n + \mu p^n + \nu, \quad n \gg 0,$$

for some integers $\lambda = \lambda(X_\infty/X) \geq 0$, $\mu = \mu(X_\infty/X) \geq 0$, and ν .

To discuss Kida’s formula, let G be a finite p -group. Let $\tilde{X}_\infty/\tilde{X}$ be another unramified \mathbb{Z}_p -covering such that the *n*th layer \tilde{X}_n is an *unramified* $\mathbb{Z}/p^n\mathbb{Z} \times G$ -covering of X for any $n \geq 0$. The situation is described by a diagram

$$\begin{array}{ccccccc} \tilde{X}_0 & \longleftarrow & \tilde{X}_1 & \longleftarrow & \tilde{X}_2 & \longleftarrow & \cdots \\ G \downarrow & & G \downarrow & & G \downarrow & & \\ X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \end{array}$$

Then the analogue of Kida’s formula, due to Ray–Vallières [12, Theorem 4.1] (see also [7, §8.2]), says that $\mu(X_\infty/X) = 0$ is equivalent to $\mu(\tilde{X}_\infty/\tilde{X}) = 0$, in which case we have

$$\lambda(\tilde{X}_\infty/\tilde{X}) + 1 = [\tilde{X}_\infty : X_\infty](\lambda(X_\infty/X) + 1)$$

(see Remark 1.2 below for the meaning of $[\tilde{X}_\infty : X_\infty]$). Compared to the original formula of Kida, the ramification terms disappear naturally because $\tilde{X}_\infty/X_\infty$ is assumed to be unramified.

1.3. Main theorem. As we have emphasized, Iwasawa theory for graphs has been developed mainly for unramified coverings so far. In a recent paper [1], Gambheera–Vallières initiated an Iwasawa-theoretic study of *ramified* (or *branched*) coverings of graphs (see §2.2 below). As a fundamental result, for a possibly ramified \mathbb{Z}_p -covering X_∞/X , they proved that the aforementioned analogue of Iwasawa's class number formula is still valid literally (see Theorem 4.1).

In this paper, we establish Kida's formula for graphs with ramifications. As above, we let G be a finite p -group and consider \mathbb{Z}_p -coverings X_∞/X and $\tilde{X}_\infty/\tilde{X}$ such that \tilde{X}_n/X is a $\mathbb{Z}/p^n\mathbb{Z} \times G$ -covering for any $n \geq 0$. Let us stress that we allow the ramifications both in $\tilde{X}_\infty/X_\infty$ and in X_∞/X . The main theorem is the following:

THEOREM 1.1. *We have $\mu(\tilde{X}_\infty/\tilde{X}) = 0$ if and only if $\mu(X_\infty/X) = 0$ and (\star) holds, where*

(\star) *any vertex of X that is unramified in X_∞/X remains unramified in \tilde{X}_∞/X .*

If these equivalent conditions hold, then

$$\begin{aligned} \lambda(\tilde{X}_\infty/\tilde{X}) + 1 &= [\tilde{X}_\infty : X_\infty](\lambda(X_\infty/X) + 1) \\ &\quad - \sum_{v \in V_X} n_v(\tilde{X}_\infty/X)(m_v(\tilde{X}_\infty/X_\infty) - 1). \end{aligned}$$

Here, V_X denotes the set of vertices of X , $m_v(\tilde{X}_\infty/X_\infty)$ denotes the ramification index of v in $\tilde{X}_\infty/X_\infty$, and $n_v(\tilde{X}_\infty/X)$ denotes the number of vertices of \tilde{X}_∞ lying over v .

REMARK 1.2. Theorem 1.1 is formulated symbolically, and to be precise it is better to rephrase it by using only finite graphs \tilde{X}_n and X_n instead of \tilde{X}_∞ and X_∞ . The degree $[\tilde{X}_\infty : X_\infty]$ is understood to be $\#G$ (which equals the degree $[\tilde{X}_n : X_n]$ for any n). Let v be a vertex of X .

- Using the ramification indices $m_v(X_n/X) \geq 1$, we say v is *ramified* in X_∞/X if $m_v(X_n/X) > 1$ for some n .
- Using the ramification indices $m_v(\tilde{X}_n/X_n) \geq 1$, which are decreasing with respect to n , we define $m_v(\tilde{X}_\infty/X_\infty)$ as their limit. We say v is *unramified* in $\tilde{X}_\infty/X_\infty$ if $m_v(\tilde{X}_\infty/X_\infty) = 1$.
- Using the numbers $n_v(\tilde{X}_n/X)$ of vertices of \tilde{X}_n lying above v , which are increasing with respect to n , we define $n_v(\tilde{X}_\infty/X)$ as their limit.

In general, $n_v(\tilde{X}_\infty/X)$ can be infinite, but then v is unramified in X_∞/X , so (\star) implies $m_v(\tilde{X}_\infty/X_\infty) = 1$ and we set $n_v(\tilde{X}_\infty/X)(m_v(\tilde{X}_\infty/X_\infty) - 1) = 0$. Consequently, the formula makes sense.

REMARK 1.3. (1) If $\tilde{X}_\infty/X_\infty$ is unramified, then condition (\star) holds, no matter how X_∞/X is ramified. Therefore, we have the equivalence between $\mu = 0$ for $\tilde{X}_\infty/\tilde{X}$ and for X_∞/X , in which case

$$\lambda(\tilde{X}_\infty/\tilde{X}) + 1 = [\tilde{X}_\infty : X_\infty](\lambda(X_\infty/X) + 1).$$

This formula is of the same form as the previous one for the unramified case.

(2) In Kida’s original formula concerning class groups, the terms corresponding to $n_v(\tilde{X}_\infty/X)$ are automatically finite since we deal with cyclotomic \mathbb{Z}_p -extensions of number fields, in which all finite primes are finitely split. By contrast, Iwasawa [6, Theorem 1] constructed \mathbb{Z}_p -extensions of number fields with $\mu > 0$ by using the anti-cyclotomic \mathbb{Z}_p -extensions of imaginary quadratic fields, in which finite primes often split completely. Moreover, Iwasawa [6, Theorem 2] also showed that $\mu = 0$ is retained as long as a condition like (\star) holds. These observations imply that condition (\star) is natural.

1.4. Organization of this paper. In §2, we introduce basic notations concerning graphs. In §3, we study precise functoriality properties of Picard groups, which is essential to prove the main theorem. Then in §4, the main theorem is proved. Finally, in §5, we record very simple examples.

2. Preliminaries. In §2.1, we fix our convention concerning graphs and then define the Jacobian groups. In §2.2, we define Galois coverings of graphs with ramifications. In §§2.3–2.4, we show that Galois coverings arise as the derived graphs of voltage graphs.

2.1. Graphs and Jacobian groups. We first recall the notions of graphs and their Jacobian groups. See [1, §§2.1–2.2] or [7, §2] for the details. We use Serre’s formalism of (undirected) graphs [13, Chapter I, §2.1].

DEFINITION 2.1. A *finite graph* X consists of a finite set V_X of vertices, a finite set \mathbb{E}_X of edges, an involution on \mathbb{E}_X without fixed points, denoted by $e \mapsto \bar{e}$, and two maps $s, t : \mathbb{E}_X \rightarrow V_X$ satisfying

$$s(\bar{e}) = t(e) \quad \text{and} \quad t(\bar{e}) = s(e).$$

Each edge $e \in \mathbb{E}_X$ is regarded as an edge that connects $s(e)$ to $t(e)$, and \bar{e} is the reverse of e . Although each edge has a direction in this formalism, conceptually we often identify an edge with its reverse, treating the structure as an undirected graph. For each $v \in V_X$, we write $\mathbb{E}_{X,v}$ for the set of $e \in \mathbb{E}_X$ such that $s(e) = v$. Note that we allow multi-edges and loops, so our graphs may be called multi-graphs. We will usually study connected graphs, which means that the vertex set is non-empty and any two vertices can be connected by a path of edges.

DEFINITION 2.2. Let X be a finite connected graph. We define the *divisor group* of X as the free \mathbb{Z} -module on the set V_X , i.e.,

$$\text{Div}(X) = \bigoplus_{v \in V_X} \mathbb{Z}[v].$$

We define a \mathbb{Z} -homomorphism $\mathcal{L}_X : \text{Div}(X) \rightarrow \text{Div}(X)$, called the *Laplacian operator*, by

$$\mathcal{L}_X([v]) = \sum_{e \in \mathbb{E}_{X,v}} ([v] - [t(e)]), \quad v \in V_X.$$

We define the *Picard group* $\text{Pic}(X)$ of X as the cokernel of \mathcal{L}_X .

Let $\text{deg}_X : \text{Div}(X) \rightarrow \mathbb{Z}$ be the *degree map*, i.e., deg_X is the \mathbb{Z} -homomorphism that sends any $[v]$ to 1. We clearly have $\text{deg}_X \circ \mathcal{L}_X = 0$, so deg_X factors through $\text{Pic}(X)$. Then we define the *Jacobian group* $\text{Jac}(X)$ as the kernel of the induced map $\text{deg}_X : \text{Pic}(X) \rightarrow \mathbb{Z}$.

It is known that $\text{Jac}(X)$ is a finite abelian group. Indeed, by Kirchhoff's theorem, the order of $\text{Jac}(X)$ is equal to the number $\kappa(X)$ of spanning trees of X (see, e.g., Sunada [15, Theorem 10.8]). It also follows that the kernel of \mathcal{L}_X is a free \mathbb{Z} -module of rank 1 with a basis $\sum_{v \in V_X} [v]$. Therefore, we have two fundamental exact sequences:

$$(2.1) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\iota_X} \text{Div}(X) \xrightarrow{\mathcal{L}_X} \text{Div}(X) \rightarrow \text{Pic}(X) \rightarrow 0,$$

where ι_X sends 1 to $\sum_{v \in V_X} [v]$, and

$$(2.2) \quad 0 \rightarrow \text{Jac}(X) \rightarrow \text{Pic}(X) \xrightarrow{\text{deg}_X} \mathbb{Z} \rightarrow 0.$$

2.2. Galois coverings of graphs. We introduce the notion of (possibly ramified) Galois coverings, basically following [1, §3.1]. We begin with the definition of morphisms of graphs. Let X and Y be finite graphs.

DEFINITION 2.3 ([1, Definition 2.1]). A morphism $f : Y \rightarrow X$ of graphs consists of maps $f_V : V_Y \rightarrow V_X$ and $f_{\mathbb{E}} : \mathbb{E}_Y \rightarrow \mathbb{E}_X$ such that

$$f_V(s(e)) = s(f_{\mathbb{E}}(e)), \quad f_V(t(e)) = t(f_{\mathbb{E}}(e)), \quad f_{\mathbb{E}}(\bar{e}) = \overline{f_{\mathbb{E}}(e)}$$

for any $e \in \mathbb{E}_Y$.

DEFINITION 2.4 ([1, Definition 3.1] or [15, p. 69]). A (*ramified*) *covering* $f : Y \rightarrow X$ is a morphism of graphs such that $f_V : V_Y \rightarrow V_X$ is surjective and the following condition is satisfied:

For each vertex $w \in V_Y$, there is a positive integer $m_w(f)$ such that the map $f_{\mathbb{E}} : \mathbb{E}_{Y,w} \rightarrow \mathbb{E}_{X,f_V(w)}$ is $m_w(f)$ -to-one.

We often write Y/X for this covering, making f implicit.

A vertex $w \in V_Y$ is said to be *lying above* $v \in V_X$ if $f_V(w) = v$. The number $m_w(Y/X) = m_w(f)$ is referred to as the *ramification index* of w .

Naturally we say w is *unramified* if $m_w(Y/X) = 1$, and a vertex v of X is *unramified* if all vertices of Y lying above v are unramified. An *unramified covering* is by definition a covering $Y \rightarrow X$ such that any vertex of Y is unramified.

Now we turn to the definition of Galois coverings. The general definition was not explicitly written in [1]; instead, that paper only deals with derived graphs associated to voltage graphs (see §2.3 below). In this paper, we introduce a reasonable definition of Galois coverings, which agrees with the one given by Shirane [14, Definition 2.1]. We will see in Proposition 2.11 that actually all of them can be constructed as derived graphs.

DEFINITION 2.5. Let Γ be a finite group. A Γ -covering Y/X (or a *Galois covering* with Galois group Γ) is a morphism $f : Y \rightarrow X$ equipped with an action of Γ on Y satisfying the following:

- The action of Γ on Y respects $f : Y \rightarrow X$, so the fibers $f_V^{-1}(v) \subset V_Y$ and $f_{\mathbb{E}}^{-1}(e) \subset \mathbb{E}_Y$ are Γ -stable for each $v \in V_X$ and $e \in \mathbb{E}_X$.
- For each $v \in V_X$, the action of Γ on $f_V^{-1}(v)$ is transitive.
- For each $e \in \mathbb{E}_X$, the action of Γ on $f_{\mathbb{E}}^{-1}(e)$ is transitive and free.

In other words, the action of Γ on \mathbb{E}_Y is free and f induces an isomorphism between the quotient graph $\Gamma \backslash Y$, defined in an obvious way, and X .

It is easy to see that a Γ -covering Y/X is actually a covering (see [1, Proposition 3.3(1)]). The ramification index $m_w(Y/X)$ coincides with the order of the stabilizer subgroup $\text{Stab}_{\Gamma}(w)$ of w . This depends only on the vertex $v = f_V(w) \in V_X$ since the action of Γ on $f_V^{-1}(v)$ is transitive. We then define the ramification index of $v \in V_X$ as $m_v(Y/X) = m_w(Y/X)$ by using any $w \in V_Y$ lying above v .

Finally, let us introduce the notion of infinite Galois coverings.

DEFINITION 2.6. Let Γ be a profinite group. The set of open normal subgroups of Γ is naturally equipped with a direct set structure: the order \preceq is defined by $U \preceq V$ if $U \supset V$. A Γ -covering of a finite graph X is then an inverse system $(X_U)_U$ of graphs, indexed by the open normal subgroups U of Γ , equipped with an action of Γ that induces a Γ/U -covering structure on X_U/X for any U .

For instance, for $\Gamma = \mathbb{Z}_p$, the open subgroups are precisely $\{\Gamma^{p^n}\}_{n \geq 0}$, so a Γ -covering of X consists of a system of graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$$

such that X_n/X is a $\mathbb{Z}/p^n\mathbb{Z}$ -covering.

2.3. Voltage graphs and derived graphs. To deal with (finite or infinite) Galois coverings, it is convenient to use voltage graphs and derived graphs. In the unramified case, detailed explanation can be found in the

classical work of Gross–Tucker [3, Chapter 2], while [7, §5] provides a brief account of our formalism. For the general (possibly ramified) case, we refer to [1, §4].

Firstly, we introduce the voltage graphs.

DEFINITION 2.7. A *voltage graph* (X, Γ, α) consists of a finite graph X , a group Γ , and a map $\alpha : \mathbb{E}_X \rightarrow \Gamma$ satisfying

$$\alpha(\bar{e}) = \alpha(e)^{-1}$$

for any $e \in \mathbb{E}_X$. To handle the ramifications, we consider an extra structure $\mathcal{I} = (I_v \subset \Gamma)_{v \in V_X}$, which is a family of subgroups. By abuse of notation, we also call $(X, \Gamma, \alpha, \mathcal{I})$ a voltage graph.

DEFINITION 2.8. Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph. Let H be a normal subgroup of Γ . Then the *induced voltage graph* $(X, \Gamma/H, \alpha_H, \mathcal{I}_H)$ is defined as follows. Letting $\pi : \Gamma \rightarrow \Gamma/H$ be the natural projection, we define $\alpha_H = \pi \circ \alpha : \mathbb{E}_X \rightarrow \Gamma/H$ and $\mathcal{I}_H = (I_{v,H})_{v \in V_X}$ with $I_{v,H} = \pi(I_v) = I_v H/H \subset \Gamma/H$.

Now we introduce the derived graphs.

DEFINITION 2.9. Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ finite. We construct a finite graph $X(\Gamma, \mathcal{I})$ (the map α is implicit), called the *derived graph*, as follows. The sets of vertices and edges are defined by

$$V_{X(\Gamma, \mathcal{I})} = \coprod_{v \in V_X} (\Gamma/I_v \times \{v\}), \quad \mathbb{E}_{X(\Gamma, \mathcal{I})} = \Gamma \times \mathbb{E}_X.$$

For an edge (γ, e) , we set

$$s((\gamma, e)) = (\gamma I_{s(e)}, s(e)), \quad t((\gamma, e)) = (\gamma \alpha(e) I_{t(e)}, t(e)), \quad \overline{(\gamma, e)} = (\gamma \alpha(e), \bar{e}).$$

The group Γ acts on the graph $X(\Gamma, \mathcal{I})$ from the left in the natural way. We also have a natural morphism $X(\Gamma, \mathcal{I}) \rightarrow X$ by projections to the second components. It is quite easy to see that this morphism is indeed a Γ -covering. More generally, given a normal subgroup H of Γ , setting $(X, \Gamma/H, \alpha_H, \mathcal{I}_H)$ as in Definition 2.8, we have an H -covering $X(\Gamma, \mathcal{I}) \rightarrow X(\Gamma/H, \mathcal{I}_H)$.

DEFINITION 2.10. Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ profinite. For each open normal subgroup U of Γ , we have a voltage graph $(X, \Gamma/U, \alpha_U, \mathcal{I}_U)$ by Definition 2.8, so we can construct the derived graph $X(\Gamma/U, \mathcal{I}_U)$ by Definition 2.9. We write $X(\Gamma, \mathcal{I}) = (X(\Gamma/U, \mathcal{I}_U))_U$ for the Γ -covering of X obtained in this way.

2.4. Galois coverings from voltage graphs. Now let us show that any Galois covering arises from a voltage graph. We begin with the finite case.

PROPOSITION 2.11. *Let $f : Y \rightarrow X$ be a Γ -covering with Γ finite. Then there is a voltage graph structure $(X, \Gamma, \alpha, \mathcal{I})$ such that $Y \simeq X(\Gamma, \mathcal{I})$ as Γ -coverings of X .*

Proof. We construct a voltage graph $(X, \Gamma, \alpha, \mathcal{I})$ in the following way. The construction depends on two choices:

- For each $v \in V_X$, choose a vertex $\tilde{v} \in V_Y$ such that $f_V(\tilde{v}) = v$.
- For each $e \in \mathbb{E}_X$, choose an edge $\tilde{e} \in \mathbb{E}_Y$ such that $f_{\mathbb{E}}(\tilde{e}) = e$ and $s(\tilde{e}) = \widetilde{s(e)}$.

Note that the number of choices for the edge \tilde{e} is the ramification index $m_{s(e)}(f)$, so it is unique if the covering Y/X is unramified. Then we define α and $\mathcal{I} = (I_v)_{v \in V_X}$ as follows:

- Let $\alpha(e) \in \Gamma$ be the unique element such that $\alpha(e) \cdot \tilde{e} = \bar{\tilde{e}}$. Here, \tilde{e} is the lift of the reverse of e , while $\bar{\tilde{e}}$ is the reverse of the lift of e .
- Let $I_v \subset \Gamma$ be the stabilizer of \tilde{v} .

Note that $\alpha(e)$ exists uniquely because both \tilde{e} and $\bar{\tilde{e}}$ are in the fiber of \bar{e} with respect to f and the action of Γ on such a fiber is free and transitive. Then we have $t(\tilde{e}) = \alpha(e)t(\bar{\tilde{e}})$ by

$$t(\tilde{e}) = s(\bar{\tilde{e}}) = s(\alpha(e) \cdot \tilde{e}) = \alpha(e)s(\tilde{e}) = \alpha(e)\widetilde{s(e)} = \alpha(e)t(\bar{\tilde{e}}).$$

We construct a morphism $\phi = (\phi_V, \phi_{\mathbb{E}}) : X(\Gamma, \mathcal{I}) \rightarrow Y$ by

$$\phi_V((\gamma I_v, v)) = \gamma \tilde{v}, \quad \phi_{\mathbb{E}}((\gamma, e)) = \gamma \tilde{e}.$$

The well-definedness of ϕ_V follows from the definition of I_v . It is straightforward to see that ϕ is indeed a morphism:

$$\begin{aligned} \phi_V(s(\gamma, e)) &= \phi_V((\gamma I_{s(e)}, s(e))) = \gamma \widetilde{s(e)} = \gamma s(\tilde{e}) = s(\gamma \tilde{e}) = s(\phi_{\mathbb{E}}((\gamma, e))), \\ \phi_V(t(\gamma, e)) &= \phi_V((\gamma \alpha(e) I_{t(e)}, t(e))) \\ &= \gamma \alpha(e) \widetilde{t(e)} = \gamma t(\tilde{e}) = t(\gamma \tilde{e}) = t(\phi_{\mathbb{E}}((\gamma, e))), \\ \overline{\phi_{\mathbb{E}}((\gamma, e))} &= \bar{\gamma \tilde{e}} = \gamma \bar{\tilde{e}} = \gamma \alpha(e) \tilde{e} = \phi_{\mathbb{E}}((\gamma \alpha(e), \bar{e})) = \phi_{\mathbb{E}}(\overline{(\gamma, e)}). \end{aligned}$$

Moreover, by the definition of the Γ -coverings, both ϕ_V and $\phi_{\mathbb{E}}$ are bijective, so ϕ is an isomorphism. ■

PROPOSITION 2.12. *Let Γ be a profinite group and $(X_U)_U$ be a Γ -covering of X . Then there is a voltage graph structure $(X, \Gamma, \alpha, \mathcal{I})$ such that $(X_U)_U$ is isomorphic to $X(\Gamma, \mathcal{I})$ as Γ -coverings of X .*

Proof. As in the proof of Proposition 2.11, we first make two choices:

- For each vertex v of X and $U \subset \Gamma$, choose a vertex \tilde{v}^U of X_U that goes to v . Moreover, we impose the compatibility with respect to U , that is, \tilde{v}^V goes to \tilde{v}^U whenever $U \supset V$.

- For each edge e of X , choose an edge \tilde{e}^U of X_U that goes to e and satisfies $s(\tilde{e}^U) = \widetilde{s(e)}^U$. Moreover, we impose the compatibility with respect to U , that is, \tilde{e}^V goes to \tilde{e}^U whenever $U \supset V$.

Then, for each U , we define $\alpha_U : \mathbb{E}_X \rightarrow \Gamma/U$ and $I_{v,U} \subset \Gamma/U$ as in the proof of Proposition 2.11 applied to X_U/X . It is straightforward to see that α_U and $I_{v,U}$ are compatible with respect to U , so we may define $\alpha : \mathbb{E}_X \rightarrow \Gamma$ and $I_v \subset \Gamma$ as the limits. By the proof of Proposition 2.11, for any U , we have $X_U \simeq X(\Gamma/U, \mathcal{I}_U)$ as Γ/U -coverings. Therefore, $(X_U)_U \simeq X(\Gamma, \mathcal{I})$ as Γ -coverings. ■

3. Functorialities of Picard groups. The behavior of the Picard groups with respect to coverings plays an important role in the proof of Theorem 1.1. We first consider general coverings in §3.1. Then we consider finite and infinite Galois coverings in §3.2 and §3.3, respectively.

3.1. Coverings. Let $f : Y \rightarrow X$ be a covering (Definition 2.4) between finite connected graphs. We define the *degree* of the covering Y/X as the number

$$[Y : X] := \sum_{w \in f_V^{-1}(v)} m_w(f),$$

which is independent of $v \in V_X$ (see [1, §3.1]; here we need the connectedness of X). Note that, for any edge $e \in \mathbb{E}_X$, we have

$$[Y : X] = \#(f_{\mathbb{E}}^{-1}(e)),$$

that is, the fiber of an arbitrary edge of X consists of $[Y : X]$ edges of Y .

As in [1, §3.2], let

$$f_*, f_r : \text{Div}(Y) \rightarrow \text{Div}(X)$$

be the \mathbb{Z} -homomorphisms such that

$$f_*([w]) = [f_V(w)], \quad f_r([w]) = m_w(f)[f_V(w)]$$

for each $w \in V_Y$.

PROPOSITION 3.1. *The exact sequences (2.1) for X and Y fit into a commutative diagram*

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota_Y} & \text{Div}(Y) & \xrightarrow{\mathcal{L}_Y} & \text{Div}(Y) & \longrightarrow & \text{Pic}(Y) & \longrightarrow & 0 \\
 & & \downarrow [Y:X] & & \downarrow f_r & & \downarrow f_* & & \downarrow & & \\
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota_X} & \text{Div}(X) & \xrightarrow{\mathcal{L}_X} & \text{Div}(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0
 \end{array}$$

where the rightmost vertical map between the Picard groups is induced by f_* .

Proof. This proposition follows from direct computations. The commutativity of the left square follows from

$$\begin{aligned} f_r \left(\sum_{w \in V_Y} [w] \right) &= \sum_{w \in V_Y} m_w(f)[f(w)] = \sum_{v \in V_X} \left(\sum_{w \in f_V^{-1}(v)} m_w(f) \right) [v] \\ &= [Y : X] \sum_{v \in V_X} [v]. \end{aligned}$$

The commutativity of the middle square is [1, Proposition 3.4]. Indeed, for each $w \in V_Y$, setting $v = f_V(w) \in V_X$, we see that

$$\begin{aligned} f_* \circ \mathcal{L}_Y([w]) &= f_* \left(\sum_{e \in \mathbb{E}_{Y,w}} ([w] - [t(e)]) \right) = \sum_{e \in \mathbb{E}_{Y,w}} ([f_V(w)] - [f_V(t(e))]) \\ &= \sum_{e \in \mathbb{E}_{Y,w}} ([v] - [t(f_{\mathbb{E}}(e))]) \end{aligned}$$

and

$$\mathcal{L}_X \circ f_r([w]) = m_w(f) \mathcal{L}_X([v]) = m_w(f) \sum_{e \in \mathbb{E}_{X,v}} ([v] - [t(e)]).$$

Since $f_{\mathbb{E}} : \mathbb{E}_{Y,w} \rightarrow \mathbb{E}_{X,v}$ is $m_w(f)$ -to-one, these coincide. ■

3.2. Finite Galois coverings. Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ finite.

Set $\mathbb{Z}[\Gamma/I_v] = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_v]} \mathbb{Z}$, which we regard as a left $\mathbb{Z}[\Gamma]$ -module. Then we have an isomorphism

$$(3.1) \quad \text{Div}(X(\Gamma, \mathcal{I})) \simeq \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v]$$

by sending $(\gamma I_v, v)$ to the class of γ in $\mathbb{Z}[\Gamma/I_v]$ at the v -component. To the endomorphism $\mathcal{L}_{X(\Gamma, \mathcal{I})}$ on $\text{Div}(X(\Gamma, \mathcal{I}))$, we associate an endomorphism $\mathcal{L}_{X, \Gamma, \mathcal{I}}$ on $\bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v]$. In other words, $\mathcal{L}_{X, \Gamma, \mathcal{I}}$ is defined by a commutative diagram

$$\begin{array}{ccc} \text{Div}(X(\Gamma, \mathcal{I})) & \xrightarrow{\mathcal{L}_{X(\Gamma, \mathcal{I})}} & \text{Div}(X(\Gamma, \mathcal{I})) \\ \simeq \downarrow & & \downarrow \simeq \\ \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] & \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} & \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] \end{array}$$

REMARK 3.2. It is straightforward to see that $\mathcal{L}_{X, \Gamma, \mathcal{I}}$ may be explicitly defined by

$$\mathcal{L}_{X, \Gamma, \mathcal{I}}([v]) = N_{I_v} \cdot \sum_{e \in \mathbb{E}_{X,v}} ([v] - \alpha(e)[t(e)]), \quad v \in V_X.$$

Here, for each $v \in V_X$, we write $[v]$ for the element of $\bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v]$ whose v -component is 1 and the other components are 0. Also, $N_{I_v} = \sum_{\sigma \in I_v} \sigma \in \mathbb{Z}[I_v]$ denotes the norm element of I_v . However, this explicit formula will not be used.

PROPOSITION 3.3. *Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ finite. Suppose that $X(\Gamma, \mathcal{I})$ is connected (so X is also connected).*

(1) *We have an exact sequence*

$$0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] \rightarrow \text{Pic}(X(\Gamma, \mathcal{I})) \rightarrow 0.$$

(2) *Let H be a normal subgroup of Γ . We introduce the voltage graph $(X, \Gamma/H, \alpha_H, \mathcal{I}_H)$ as in Definition 2.8. Then we have a commutative diagram*

$$\begin{array}{ccccccc} 0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] & \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} & \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] & \longrightarrow & \text{Pic}(X(\Gamma, \mathcal{I})) & \longrightarrow & 0 \\ \#H \downarrow & & \beta \downarrow & & \downarrow \text{proj} & & \downarrow \\ 0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v H] & \xrightarrow{\mathcal{L}_{X, \Gamma/H, \mathcal{I}_H}} & \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v H] & \rightarrow & \text{Pic}(X(\Gamma/H, \mathcal{I}_H)) & \rightarrow & 0 \end{array}$$

Here, the map proj is the natural projection, and the map β sends $1 \in \mathbb{Z}[\Gamma/I_v]$ to $\#(H \cap I_v) \in \mathbb{Z}[\Gamma/I_v H]$.

Proof. (1) This is simply a reformulation of (2.1).

(2) Let $f : X(\Gamma, \mathcal{I}) \rightarrow X(\Gamma/H, \mathcal{I}_H)$ be the H -covering. It is quite easy to see that the following diagrams commute:

$$\begin{array}{ccc} \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] & \xrightarrow{\cong} & \text{Div}(X(\Gamma, \mathcal{I})) \\ \beta \downarrow & & \downarrow f_r \\ \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v H] & \xrightarrow{\cong} & \text{Div}(X(\Gamma/H, \mathcal{I}_H)) \\ \\ \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v] & \xrightarrow{\cong} & \text{Div}(X(\Gamma, \mathcal{I})) \\ \text{proj} \downarrow & & \downarrow f_* \\ \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v H] & \xrightarrow{\cong} & \text{Div}(X(\Gamma/H, \mathcal{I}_H)) \end{array}$$

where the horizontal isomorphisms are (3.1). Then the claimed diagram is simply a reformulation of Proposition 3.1 applied to f . ■

The description of β in this proposition will be crucial in what follows.

3.3. Infinite Galois coverings. As in [7, §5.2], we shall define the Picard group of an infinite Galois covering by using the projective limit. Let Λ be a compact flat \mathbb{Z} -algebra; in applications we will set $\Lambda = \mathbb{Z}_p$. This Λ will be used to make the projective limit functor exact.

DEFINITION 3.4. Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ profinite. Suppose that $X(\Gamma/U, \mathcal{I}_U)$ are connected for all U . By Proposition 3.3(2), the Picard groups $\text{Pic}(X(\Gamma/U, \mathcal{I}_U))$ form a projective system. We define

$$\text{Pic}_\Lambda(X(\Gamma, \mathcal{I})) = \varprojlim_U \Lambda \otimes_{\mathbb{Z}} \text{Pic}(X(\Gamma/U, \mathcal{I}_U)).$$

This is a module over the Iwasawa algebra

$$\Lambda[[\Gamma]] = \varprojlim_U \Lambda[\Gamma/U].$$

For any profinite group Γ , we define an ideal ${}_\Gamma\Lambda$ of Λ by

$${}_\Gamma\Lambda = \bigcap_U [\Gamma : U]\Lambda,$$

where U runs over the open normal subgroups of Γ . For instance, if $\Lambda = \mathbb{Z}_p$, we have ${}_\Gamma\Lambda = 0$ if and only if the order of Γ is divisible by p^∞ (i.e., for any $n \geq 0$, there is an open normal subgroup U of Γ with $p^n \mid |\Gamma : U|$).

PROPOSITION 3.5. *Let $(X, \Gamma, \alpha, \mathcal{I})$ be a voltage graph with Γ profinite. Suppose that $X(\Gamma/U, \mathcal{I}_U)$ are connected for all U .*

(1) *We have an exact sequence*

$$0 \rightarrow {}_\Gamma\Lambda \rightarrow \bigoplus_{v \in V_X} I_v \Lambda[[\Gamma/I_v]] \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} \bigoplus_{v \in V_X} \Lambda[[\Gamma/I_v]] \rightarrow \text{Pic}_\Lambda(X(\Gamma, \mathcal{I})) \rightarrow 0,$$

where $\mathcal{L}_{X, \Gamma, \mathcal{I}}$ is introduced in the proof below.

(2) *Let H be a closed normal subgroup of Γ . Let $(X, \Gamma/H, \alpha_H, \mathcal{I}_H)$ be as in Definition 2.8. Then we have a commutative diagram*

$$\begin{array}{ccccccc} 0 \rightarrow {}_\Gamma\Lambda & \longrightarrow & \bigoplus_{v \in V_X} I_v \Lambda[[\Gamma/I_v]] & \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} & \bigoplus_{v \in V_X} \Lambda[[\Gamma/I_v]] & \longrightarrow & \text{Pic}_\Lambda(X(\Gamma, \mathcal{I})) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow {}_{\Gamma/H}\Lambda & \longrightarrow & \bigoplus_{v \in V_X} I_v H \Lambda[[\Gamma/I_v H]] & \xrightarrow{\mathcal{L}_{X, \Gamma/H, \mathcal{I}_H}} & \bigoplus_{v \in V_X} \Lambda[[\Gamma/I_v H]] & \longrightarrow & \text{Pic}_\Lambda(X(\Gamma/H, \mathcal{I}_H)) \longrightarrow 0 \end{array}$$

Here, all the vertical arrows are the natural ones; the left two are induced by the inclusions

$${}_\Gamma\Lambda \hookrightarrow {}_{\Gamma/H}\Lambda \quad \text{and} \quad I_v \Lambda \hookrightarrow I_v H \Lambda.$$

Proof. By Proposition 3.3(2), for open normal subgroups V, U of Γ with $U \supset V$, we have

$$\begin{array}{ccccccc}
 0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v V] & \xrightarrow{\mathcal{L}_{X, \Gamma/V, \mathcal{I}_V}} & \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v V] & \rightarrow \text{Pic}(X(\Gamma/V, \mathcal{I}_V)) & \rightarrow 0 \\
 \downarrow [U:V] & \beta \downarrow & \downarrow \text{proj} & \downarrow & \\
 0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v U] & \xrightarrow{\mathcal{L}_{X, \Gamma/U, \mathcal{I}_U}} & \bigoplus_{v \in V_X} \mathbb{Z}[\Gamma/I_v U] & \rightarrow \text{Pic}(X(\Gamma/U, \mathcal{I}_U)) & \rightarrow 0
 \end{array}$$

Here, β on the v -component sends 1 to

$$\#((U/V) \cap (I_v V/V)) = [U \cap I_v V : V].$$

By taking the tensor product with Λ over \mathbb{Z} and then taking the projective limit with respect to U , we obtain claim (1). For claim (2), we only have to take the limit of the above diagram (after $\Lambda \otimes_{\mathbb{Z}} (-)$) with the restriction $U \supset H$. ■

4. Kida's formula. In §4.1, we review the analogue of Iwasawa's class number formula in our settings. Then, in §4.2, we prove our main theorem (Theorem 1.1).

4.1. Iwasawa's class number formula. From now on, we fix a prime number p and work over $\Lambda = \mathbb{Z}_p$.

Let Γ be a profinite group that is isomorphic to \mathbb{Z}_p . For a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$ -module M , we have the λ, μ -invariants $\lambda(M), \mu(M)$, which are non-negative integers. They are usually defined by using the structure theorem for $\mathbb{Z}_p[[\Gamma]]$ -modules (see Neukirch–Schmidt–Wingberg [11, Definition (5.3.9)]). Instead of the usual definition, we only recall the following properties (see [11, Exercise 2(c), p. 300 and Remark 3 after Definition (5.3.9)]):

- We have $\mu(M) = 0$ if and only if M is finitely generated over \mathbb{Z}_p .
- We have $\lambda(M) = \dim_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M)$.

Now the analogue of Iwasawa's class number formula is the following:

THEOREM 4.1 (Gambheera–Vallières [1, Theorem A]). *Let X_∞/X be a \mathbb{Z}_p -covering of finite connected graphs and X_n be its n th layer. Then*

$$\text{ord}_p(\kappa(X_n)) = \lambda n + \mu p^n + \nu, \quad n \gg 0,$$

with

$$\lambda = \lambda(X_\infty/X) = \lambda(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) - 1, \quad \mu = \mu(X_\infty/X) = \mu(\text{Pic}_{\mathbb{Z}_p}(X_\infty))$$

and some integer $\nu = \nu(X_\infty/X)$.

Proof. The original result [1, Theorem A] claims the formula with $\lambda = \lambda(\text{Jac}_{\mathbb{Z}_p}(X_\infty))$ and $\mu = \mu(\text{Jac}_{\mathbb{Z}_p}(X_\infty))$, where $\text{Jac}_{\mathbb{Z}_p}(X_\infty)$ is defined as the projective limit of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Jac}(X_n)$ (cf. Definition 3.4). We have an exact sequence

$$0 \rightarrow \text{Jac}_{\mathbb{Z}_p}(X_\infty) \rightarrow \text{Pic}_{\mathbb{Z}_p}(X_\infty) \rightarrow \mathbb{Z}_p \rightarrow 0$$

induced by (2.2). This implies

$$\lambda(\text{Jac}_{\mathbb{Z}_p}(X_\infty)) = \lambda(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) - 1, \quad \mu(\text{Jac}_{\mathbb{Z}_p}(X_\infty)) = \mu(\text{Pic}_{\mathbb{Z}_p}(X_\infty)),$$

so Theorem 4.1 is also valid. ■

4.2. Kida’s formula. A key to prove the main theorem is the following algebraic proposition.

PROPOSITION 4.2 ([7, Proposition 8.5]). *Let Γ be a profinite group that is isomorphic to \mathbb{Z}_p , and G be a finite p -group. Let M be a finitely generated torsion $\mathbb{Z}_p[[\Gamma]][G]$ -module. Write M_G for the G -coinvariant of M (i.e., the maximal quotient on which G acts trivially). Then the following hold:*

- (1) $\mu(M) = 0$ if and only if $\mu(M_G) = 0$.
- (2) Suppose that $\text{pd}_{\mathbb{Z}_p[[\Gamma]][G]}(M) \leq 1$, where pd denotes the projective dimension. If the equivalent conditions in (1) hold, then $\lambda(M) = (\#G)\lambda(M_G)$.

Proof. Because of its importance, we include a rough sketch of the proof. Claim (1) follows from Nakayama’s lemma over the local ring $\mathbb{Z}_p[G]$. Under the assumptions in claim (2), the $\mathbb{Z}_p[G]$ -module M is indeed free of finite rank. Then the equality $\lambda(M) = (\#G)\lambda(M_G)$ is clear. ■

We are in a position to prove Theorem 1.1. By Proposition 2.12, the situation in Theorem 1.1 can be realized by a voltage graph. To be precise, let $\tilde{\Gamma}$ be a profinite group of the form

$$\tilde{\Gamma} = \Gamma \times G$$

with Γ isomorphic to \mathbb{Z}_p and G a finite p -group. We regard G as a closed normal subgroup of $\tilde{\Gamma}$ in the usual way, and we identify $\tilde{\Gamma}/G = \Gamma$. Let $(X, \tilde{\Gamma}, \tilde{\alpha}, \tilde{\mathcal{I}})$ be a voltage graph with $\tilde{\mathcal{I}} = (\tilde{I}_v \subset \tilde{\Gamma})_{v \in V_X}$. Let us assume that $X(\tilde{\Gamma}/U, \tilde{\mathcal{I}}_U)$ is connected for any open normal subgroup U of $\tilde{\Gamma}$. We also obtain $(X, \Gamma, \alpha, \mathcal{I})$ (write $\mathcal{I} = (I_v \subset \Gamma)_{v \in V_X}$) by using Definition 2.8 applied to the subgroup G of $\tilde{\Gamma}$. We set

$$X_\infty = X(\Gamma, \mathcal{I}), \quad \tilde{X}_\infty = X(\tilde{\Gamma}, \tilde{\mathcal{I}}).$$

Now Theorem 1.1 can be rephrased as follows:

THEOREM 4.3. *We have $\mu(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) = 0$ if and only if $\mu(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) = 0$ and (\star) holds, where*

- (\star) for any $v \in V_X$, if I_v is trivial, then \tilde{I}_v is trivial.

If these equivalent conditions hold, then

$$\lambda(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) = (\#G)\lambda(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) - \sum_{v \in V_X} [\tilde{\Gamma} : \tilde{I}_v](\#(G \cap \tilde{I}_v) - 1).$$

As in Remark 1.2, if $[\tilde{\Gamma} : \tilde{I}_v]$ is infinite, condition (\star) implies \tilde{I}_v is trivial, so we set $[\tilde{\Gamma} : \tilde{I}_v](\#(G \cap \tilde{I}_v) - 1) = 0$ in this formula.

Proof of Theorem 4.3. Set $V_X^0 = \{v \in V_X \mid I_v \text{ is trivial}\}$. Note that

$$I_v \text{ is trivial} \iff I_v \text{ is finite} \iff \tilde{I}_v \text{ is finite,}$$

where the first equivalence holds since I_v is p -torsion-free. The motivation for this definition is that, for $\Lambda = \mathbb{Z}_p$, we have $I_v \Lambda = \tilde{I}_v \Lambda = 0$ if (and only if) $v \notin V_X^0$. We also have $\Gamma \Lambda = \tilde{\Gamma} \Lambda = 0$. Then, by applying Proposition 3.5(2) to $G \subset \tilde{\Gamma}$ in place of $H \subset \Gamma$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\tilde{\Gamma}/\tilde{I}_v]] & \xrightarrow{\mathcal{L}_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}} & \bigoplus_{v \in V_X} \mathbb{Z}_p[[\tilde{\Gamma}/\tilde{I}_v]] & \rightarrow & \text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty) & \rightarrow 0 \\ & \beta \downarrow & & \downarrow \text{proj} & & \downarrow & \\ 0 \rightarrow & \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\Gamma]] & \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} & \bigoplus_{v \in V_X} \mathbb{Z}_p[[\Gamma/I_v]] & \rightarrow & \text{Pic}_{\mathbb{Z}_p}(X_\infty) & \rightarrow 0 \end{array}$$

Here, β on the v -component sends 1 to $\#\tilde{I}_v$. By taking the G -coinvariant of the upper sequence, and noting $\mathbb{Z}_p[[\tilde{\Gamma}/\tilde{I}_v]]_G \simeq \mathbb{Z}_p[[\Gamma/I_v]]$, we obtain

$$(4.1) \quad \begin{array}{ccccccc} 0 \rightarrow & \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\Gamma]] & \xrightarrow{\overline{\mathcal{L}_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}}} & \bigoplus_{v \in V_X} \mathbb{Z}_p[[\Gamma/I_v]] & \rightarrow & \text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)_G & \rightarrow 0 \\ & \bar{\beta} \downarrow & & \parallel & & \downarrow & \\ 0 \rightarrow & \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\Gamma]] & \xrightarrow{\mathcal{L}_{X, \Gamma, \mathcal{I}}} & \bigoplus_{v \in V_X} \mathbb{Z}_p[[\Gamma/I_v]] & \rightarrow & \text{Pic}_{\mathbb{Z}_p}(X_\infty) & \rightarrow 0 \end{array}$$

Here, $\bar{\beta}$ is the injective homomorphism that is the multiplication by $\#\tilde{I}_v$ on the v -component. This also explains why the induced homomorphism $\overline{\mathcal{L}_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}}$ is injective. By the snake lemma, we obtain an exact sequence

$$0 \rightarrow \bigoplus_{v \in V_X^0} (\mathbb{Z}_p/\#\tilde{I}_v)[[\Gamma]] \rightarrow \text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)_G \rightarrow \text{Pic}_{\mathbb{Z}_p}(X_\infty) \rightarrow 0.$$

By the property of the μ -invariants recalled at the beginning of §4.1, we have $\mu((\mathbb{Z}_p/\#\tilde{I}_v)[[\Gamma]]) = 0$ if and only if \tilde{I}_v is trivial. Thus, we have $\mu(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)_G) = 0$ if and only if $\mu(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) = 0$ and \tilde{I}_v is trivial for any $v \in V_X^0$. Thanks to Proposition 4.2(1), this shows the first claim on the equivalence concerning $\mu = 0$.

In what follows, we assume the equivalent conditions hold. Then the $\bar{\beta}$ in (4.1) is the identity map, so $\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)_G \simeq \text{Pic}_{\mathbb{Z}_p}(X_\infty)$. However, Proposition 4.2(2) is not applicable to $\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)$, because the condition $\text{pd} \leq 1$ does not hold in general (see Lemma 4.4 below). To remedy this, we use a slightly modified version of the Picard groups that satisfies $\text{pd} \leq 1$.

As we are assuming (\star) , we have $V_X^0 = \{v \in V_X \mid \tilde{I}_v \text{ is trivial}\}$. We define

$$\mathcal{L}'_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}} : \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\tilde{\Gamma}]] \rightarrow \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\tilde{\Gamma}]]$$

as the composition of $\mathcal{L}_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}$ with the projection to the V_X^0 -components. Define $\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)$ as the cokernel of this $\mathcal{L}'_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}$. Clearly we have an exact sequence

$$(4.2) \quad 0 \rightarrow \bigoplus_{v \in V_X \setminus V_X^0} \mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v] \rightarrow \text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty) \rightarrow \text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty) \rightarrow 0.$$

Here, for each $v \in V_X \setminus V_X^0$, the index $[\tilde{\Gamma} : \tilde{I}_v]$ is finite. Therefore,

$$\lambda(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) = \lambda(\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)) + \sum_{v \in V_X \setminus V_X^0} [\tilde{\Gamma} : \tilde{I}_v]$$

and $\mu(\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)) = \mu(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) = 0$. We also define $\text{Pic}'_{\mathbb{Z}_p}(X_\infty)$ in the same way: we introduce $\mathcal{L}'_{X, \Gamma, \mathcal{I}}$ that is the projection of $\mathcal{L}_{X, \Gamma, \mathcal{I}}$ to the V_X^0 -components and then define $\text{Pic}'_{\mathbb{Z}_p}(X_\infty)$ as its cokernel. Then the same reasoning shows

$$\lambda(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) = \lambda(\text{Pic}'_{\mathbb{Z}_p}(X_\infty)) + \sum_{v \in V_X \setminus V_X^0} [\Gamma : I_v].$$

Recall that $\mathcal{L}_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}} : \bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\tilde{\Gamma}]] \rightarrow \bigoplus_{v \in V_X} \mathbb{Z}_p[[\tilde{\Gamma}/\tilde{I}_v]]$ is injective. We observe that the homomorphism $\mathcal{L}'_{X, \tilde{\Gamma}, \tilde{\mathcal{I}}}$ is still injective since $\bigoplus_{v \in V_X^0} \mathbb{Z}_p[[\tilde{\Gamma}]]$ is torsion-free and $\bigoplus_{v \in V_X \setminus V_X^0} \mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]$ is torsion as $\mathbb{Z}_p[[\tilde{\Gamma}]]$ -modules. Therefore,

$$\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)) \leq 1.$$

By the same reasoning as for the original Picard groups, since the homomorphism $\bar{\beta}$ is the identity, we have

$$\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)_G \simeq \text{Pic}'_{\mathbb{Z}_p}(X_\infty).$$

Now we are able to apply Proposition 4.2(2) to $\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)$. As a result,

$$\lambda(\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)) = (\#G)\lambda(\text{Pic}'_{\mathbb{Z}_p}(X_\infty)).$$

By combining these formulas, we obtain

$$\lambda(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) - \sum_{v \in V_X \setminus V_X^0} [\tilde{\Gamma} : \tilde{I}_v] = (\#G) \left(\lambda(\text{Pic}_{\mathbb{Z}_p}(X_\infty)) - \sum_{v \in V_X \setminus V_X^0} [\Gamma : I_v] \right).$$

An elementary observation shows

$$(\#G)[\Gamma : I_v] - [\tilde{\Gamma} : \tilde{I}_v] = [\tilde{\Gamma} : \tilde{I}_v](\#(G \cap \tilde{I}_v) - 1).$$

This completes the proof of Theorem 4.3. ■

To clarify the role of the projective dimension in Kida's formula, we record a lemma.

LEMMA 4.4. *Assume that condition (\star) holds. Then the following are equivalent:*

- (i) $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) \leq 1$.
- (ii) $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\text{Pic}_{\mathbb{Z}_p}(\tilde{X}_\infty)) < \infty$.
- (iii) *For any $v \in V_X$, the group $G \cap \tilde{I}_v$ is trivial.*

Proof. Since $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\text{Pic}'_{\mathbb{Z}_p}(\tilde{X}_\infty)) \leq 1$ as in the proof of Theorem 4.3, the exact sequence (4.2) implies that

- (i) \iff for any $v \in V_X \setminus V_X^0$, we have $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]) \leq 1$,
- (ii) \iff for any $v \in V_X \setminus V_X^0$, we have $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]) < \infty$.

By (\star) , the group \tilde{I}_v is trivial for $v \in V_X^0$, so we may impose the restriction $v \in V_X \setminus V_X^0$ in condition (iii). Therefore, it is enough to show that the following are equivalent for each $v \in V_X \setminus V_X^0$:

- (i') $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]) \leq 1$.
- (ii') $\text{pd}_{\mathbb{Z}_p[[\tilde{\Gamma}]]}(\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]) < \infty$.
- (iii') The group $G \cap \tilde{I}_v$ is trivial.

These equivalences follow from purely algebraic considerations. Indeed, both (i') and (ii') are equivalent to $\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]$ being G -cohomologically trivial, which is then equivalent to (iii') because $\mathbb{Z}_p[\tilde{\Gamma}/\tilde{I}_v]$ is isomorphic to a direct sum of copies of $\mathbb{Z}_p[G/G \cap \tilde{I}_v]$ as a $\mathbb{Z}_p[G]$ -module. We omit the details. ■

5. Examples. Let m be a positive integer. Let X be the cycle graph with m vertices: the vertices

$$V_X = \{v_1, \dots, v_m\}$$

and the edges

$$\mathbb{E}_X = \{e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_m\}$$

are such that $s(e_i) = v_i$ and $t(e_i) = v_{i+1}$ for $1 \leq i \leq m$ (here v_{m+1} is understood to be v_1).

Let $\Gamma \simeq \mathbb{Z}_p$, and G be a cyclic group of order p . Let

$$\alpha : \mathbb{E}_X \rightarrow \tilde{\Gamma} = \Gamma \times G$$

be a voltage assignment such that the components of $\alpha(e_1)$ are generators of Γ and G , and $\alpha(e_2), \dots, \alpha(e_m)$ are all the unit element.

We consider three choices of $\tilde{\mathcal{I}}$:

- (a) $\tilde{\mathcal{I}}_v = G$ for any v .
- (b) $\tilde{\mathcal{I}}_v = \Gamma$ for any v .
- (c) $\tilde{\mathcal{I}}_v = \Gamma \times G$ for any v .

Note that condition (\star) holds for (b) and (c) but does not hold for (a). In each case, we set

$$\begin{aligned} X_\infty &= X(\Gamma, \mathcal{I}), & X_n &= X(\Gamma/\Gamma^{p^n}, \mathcal{I}_{\Gamma^{p^n}}), \\ \tilde{X}_\infty &= X(\tilde{\Gamma}, \tilde{\mathcal{I}}), & \tilde{X}_n &= X(\tilde{\Gamma}/\Gamma^{p^n}, \tilde{\mathcal{I}}_{\Gamma^{p^n}}). \end{aligned}$$

As will be clear, \tilde{X}_n are indeed connected.

To compute the Iwasawa λ, μ -invariants, it is convenient to prepare a lemma. For positive integers m and N , we define a graph $Y_{m,N}$ as follows. We set $Y_{m,1} = X$, which is the cycle graph with m vertices. Then we set $Y_{m,N}$ to be the ‘‘totally ramified’’ covering of $Y_{m,1}$ of degree N . More precisely, $Y_{m,N}$ has m vertices, say v_1, \dots, v_m , and mN unoriented edges, and for each $1 \leq i \leq m$, exactly N edges connect v_i and v_{i+1} .

LEMMA 5.1. *We have $\kappa(Y_{m,N}) = mN^{m-1}$.*

Proof. This is easily proved by counting the number of spanning trees. ■

Using this lemma, we can directly compute the Iwasawa λ, μ -invariants of X_∞/X and of $\tilde{X}_\infty/\tilde{X}$ for cases (a)–(c). The results are summarized in Table 1.

Table 1

	Case (a)	Case (b)	Case (c)
X_n	$\simeq Y_{mp^n, 1}$	$\simeq Y_{m, p^n}$	$\simeq Y_{m, p^n}$
$\kappa(X_n)$	mp^n	$mp^{n(m-1)}$	$mp^{n(m-1)}$
$\mu(X_\infty/X)$	0	0	0
$\lambda(X_\infty/X)$	1	$m - 1$	$m - 1$
\tilde{X}_n	$\simeq Y_{mp^n, p}$	$\simeq Y_{mp, p^n}$	$\simeq Y_{m, p^{n+1}}$
$\kappa(\tilde{X}_n)$	$mp^n \cdot p^{mp^n - 1}$	$mp \cdot p^{n(mp-1)}$	$m \cdot p^{(m-1)(n+1)}$
$\mu(\tilde{X}_\infty/\tilde{X})$	m	0	0
$\lambda(\tilde{X}_\infty/\tilde{X})$	1	$pm - 1$	$m - 1$

For example, in case (a), we have $X_n \simeq Y_{mp^n,1}$, so $\kappa(X_n) = mp^n$ by Lemma 5.1. This implies $\mu = 0$, $\lambda = 1$ for X_∞/X . The other entries are filled in a similar way (note that X_∞/X for (b) is the same as for (c)). The towers of graphs with $m = 3$ and $p = 2$ are illustrated in Figures 1–3.

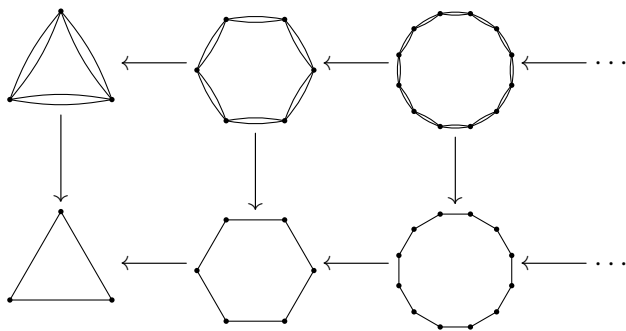


Fig. 1. Case (a)

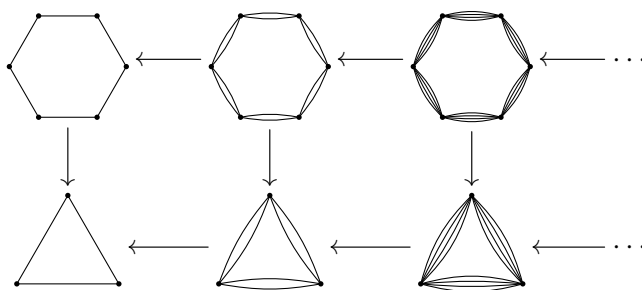


Fig. 2. Case (b)

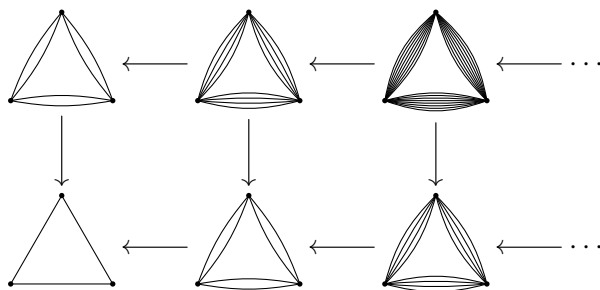


Fig. 3. Case (c)

Let us discuss the validity of Theorem 1.1. In case (a), $\mu = 0$ is not retained because condition (\star) does not hold, as the theorem predicts. In case (b), we have

$$(pm - 1) + 1 = p \cdot ((m - 1) + 1)$$

as predicted. In case (c), we have

$$(m - 1) + 1 = p \cdot ((m - 1) + 1) - \sum_{i=1}^m 1 \cdot (p - 1)$$

as predicted.

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References

- [1] R. Gambheera and D. Vallières, *Iwasawa theory for branched \mathbb{Z}_p -towers of finite graphs*, Doc. Math. 29 (2024), 1435–1468.
- [2] S. R. Gonet, *Iwasawa Theory of Jacobians of Graphs*, Algebr. Combin. 5 (2022), 827–848.
- [3] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York, 1987.
- [4] Y. Hachimori and K. Matsuno, *An analogue of Kida’s formula for the Selmer groups of elliptic curves*, J. Algebr. Geom. 8 (1999), 581–601.
- [5] K. Iwasawa, *On Γ -extensions of algebraic number fields*, Bull. Amer. Math. Soc. 65 (1959), 183–226.
- [6] K. Iwasawa, *On the μ -invariants of \mathbb{Z}_ℓ -extensions*, in: Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki, Kinokuniya Book Store, Tokyo, 1973, 1–11.
- [7] T. Kataoka, *Fitting ideals of Jacobian groups of graphs*, Algebr. Combin. 7 (2024), 597–625.
- [8] T. Kataoka, *Kida’s formula via Selmer complexes*, arXiv:2401.07036v2 (2024).
- [9] Y. Kida, *l -Extensions of CM-fields and cyclotomic invariants*, J. Number Theory 12 (1980), 519–528.
- [10] K. McGown and D. Vallières, *On abelian ℓ -towers of multigraphs III*, Ann. Math. Québec 48 (2024), no. 1, 1–19.
- [11] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, 2nd ed., Grundlehren Math. Wiss. 323, Springer, Berlin, 2008.
- [12] A. Ray and D. Vallières, *An analogue of Kida’s formula in graph theory*, Pure Appl. Math. Quart. 21 (2025), 1853–1891.
- [13] J.-P. Serre, *Trees*, Springer, Berlin, 1980.
- [14] T. Shirane, *Galois covers of graphs and embedded topology of plane curves*, Topology Appl. 257 (2019), 122–143.

- [15] T. Sunada, *Topological Crystallography: With a View Towards Discrete Geometric Analysis*, Surveys Tutorials Appl. Math. Sci. 6, Springer, Tokyo, 2013.
- [16] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. 83, Springer, New York, 1997.

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